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Model Theory and Modules

Mike Prest
University of Manchester

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To my wife Deborah,
my daughter Abigail
and to the memory of my sister Anne-Louise.

PREFACE

Let me begin with a caricature.

At school, "algebra" means equations. Later on, we discover that algebra is really about structures, substructures and morphisms. Thus we leap from the most elementary first-order logic to full-blown second-order logic, with barely a glance at what we are passing over. We may, at some point, be exposed to mathematical logic and discover that it is worthwhile to take some account of the way in which we define things; learning that if we concentrate on what can be defined within a first-order language, then there is available to us a range of ideas and techniques which we would not otherwisehave had. In other words, we can use model theory, which is designed to handle that information which may be expressed in a first-order language.

That is a misrepresentation of modern algebra (and, indeed, of model theory): for instance, we do encounter first-order quantifiers when we consider solvability of systems of linear equations. Nevertheless, there is a rich seam of information which tends to be overlooked because the relatively unfamiliar tools of model theory are required for its extraction.

There are places where algebra and model theory have interacted. Algebra provides much of the raw material for model theory and a number of algebraic problems have been solved by dragging them into the arena of model theory and tackling them there. But also, model theory has the power to enrich algebra by fusing with it at certain points and at certain levels.

In particular, model theory provides tools which are potentially available to anyone working with modules. This is not to suggest that every module-theorist should rush off to learn model theory. For a start, model theory may have nothing to say about a particular problem. And neither would one suggest that a ring-theorist who needs some group theory should become an expert in groups.

There is a theorem which says that, in the context of modules, model theory is not very far from algebra. It says, roughly, that any subset of a module which can be defined using equations, inequations, and, or, ... and quantifiers, may be expressed rather simply in terms of solution sets of systems of linear equations. This makes it quite plausible that model theory can say something significant about modules. It also means that most of the fundamental ideas can be expressed algebraically or model-theoretically, as one prefers. So, an algebraist who is willing to take on board a few basic theorems from model theory should find most chapters of this book, if not entirely plain sailing, then at least not unduly demanding (the excepted chapters are those directly concerned with stability theory).

I have been rather neglecting the model-theorists* in my comments: so, what is in this book for the average model-theorist? Those who are, or could become, interested in modules, will find a great deal of material and many open problems. Even those without any specific concern for modules should find something of interest. For modules seem to be a good source of examples in stability theory: they are amenable, but non-trivial.

This book is meant to be an introduction to, and a reasonably comprehensive account of, the interaction of model theory and modules. It started life as a set of lecture notes for a graduate course at Yale in the autumn of '83. Those notes were enlarged upon and added to: the result was a fairly hefty typed set of notes which enjoyed some limited circulation. A number of chapters and sections have since been added. Some additions reflect new results: others are simply sections which had not, at the time, been in a sufficiently well-worked form for circulation. Almost all sections have been re-worked to some extent and, I hope, improved.

I have tried to make this book readable by algebraists and logicians; by graduate students and established workers. Thus, the first few chapters are rather lengthier than they would otherwise be, since I tend to include details of arguments which would be suppressed in a more narrowly directed text. I introduce a number of examples in the first chapters, in which also the exercises tend to be thicker on the ground. All this combines to swell the fairly elementary Chapter 2 (Chapter 1 contains background material) to such a size that there may well be a danger of losing some readers on the way through. So let me point out that the reader with some knowledge of either models or modules should be able to read through sizable portions of that chapter rather rapidly. The pace does pick up in the later chapters; though I hope that I have managed to avoid being at all obscure in proofs.

Let me describe briefly the nature of the various chapters. Chapter 1 is background. Chapters 2 - 4 contain the core material, with Chapter 2 having an introductory character. After Chapter 4, the reader's path may be guided more by his or her interests. Chapters 5 - 7 are concerned mainly with stability-theory and modules. Chapters 8 - 10 explore the relationship between pp-types and pure-injective modules: these chapters contain many important results, but the reader should find it possible, at least in the first instance, to use the results without reading through the chapters in detail. Chapters 11 - 13 are mostly concerned with modules over artinian rings, especially over Artin algebras. Chapters 14, 15, 16 and 17 are relatively independent; the first three deal with particular aspects and special types of modules; Chapter 17 is concerned with questions of decidability.

In addition to the main chapters and sections there are a few supplementary sections scattered through the notes. At some point, because of limitations on time and space, I had to draw a line around what I would include in these notes: this boundary is indicated by the topics of some of those sections. The other supplementary sections contain material which has no obvious home in terms of the chapter structure of the book.

A substantial part of this book was written, and the book itself was completed, while I held a University Research Fellowship at the University of Liverpool.

University of Manchester,
October 1987.

* My use of terms such as "algebraist" and "model-theorist" is for purpose of contrast only: certainly I do not wish to harden perceptions of boundaries between parts of mathematics.

Contents

Introduction xi
Acknowledgements xi
Notations and conventions xii
Remarks on the development of the area xiii

Section summaries xv

Chapter 1 Some preliminaries 1
1.1 An introduction to model theory 1
1.2 Injective modules and decomposition theorems 8

Chapter 2 Positive primitive formulas and the sets they define 13
2.2 pp formulas 14
2.2 pp-types 22
2.3 Pure embeddings and pure-injective modules 27
2.4 pp-elimination of quantifiers 32
2.5 Immediate corollaries of pp-elimination of quantifiers 39
2.6 Comparison of complete theories of modules 41
2.$\mathbb{Z}$ pp formulas and types in abelian groups 44
2.L Other languages 52

Chapter 3 Stability and totally transcendental modules 54
3.1 Stability for modules 55
3.2 A structure theorem for totally transcendental modules; part I 60
3.A Abelian structures 63

Chapter 4 Hulls 67
4.1 pp-essential embeddings and the construction of hulls 68
4.2 Examples of hulls 76
4.3 Decomposition of injective and pure-injective modules 80
4.4 Irreducible types 83
4.5 Limited and unlimited types 89
4.6 A structure theory for totally transcendental modules; part II 91
4.C Categoricity 100
4.7 The space of indecomposables 102

Chapter 5 Forking and ranks 109
5.1 Forking and independence 109
5.2 Ranks 116
5.3 An algebraic characterisation of independence 123
5.4 Independence when $T = T^{\aleph_0}$ 125

Chapter 6 Stability-theoretic properties of types 132
6.1 Free parts of types and the stratified order 132
6.2 Domination and the RK-order 136
6.3 Orthogonality and the RK-order 141
6.4 Regular types 143
6.I An example: injective modules over noetherian rings 149
6.5 Saturation and pure-injective modules 150
6.6 Multiplicity and strong types 152

Chapter 7 Superstable modules 155
7.1 Superstable modules: the uncountable spectrum 155
7.2 Modules of U-rank 1 157
7.3 Modules of finite U-rank 171

Chapter 8 The lattice of pp-types and free realisations of pp-types 173
8.1 The lattice of pp-types 173
8.2 Finitely generated pp-types 176
8.3 pp-types and matrices 178

8.3 Duality and pure-semisimple rings 184

Chapter 9 Types and the structure of pure-injective modules 188
9.1 Minimal pairs 188
9.2 Associated types 191
9.3 Notions of isolation 194
9.4 Neg-isolated types and elementary cogenerators 198

Chapter 10 Dimension and decomposition 201
10.1 Existence of indecomposable direct summands 202
10.2 Dimensions defined on lattices 203
10.3 Modules with width 208
10.4 Classification for theories with dimension 211
10.5 Krull dimension 216
10.T T^{eq} 220
10.6 Dimension and height 222
10.V Valuation rings 225

Chapter 11 Modules over artinian rings 228
11.1 Pure-semisimple rings 228
11.2 Pure-semisimple rings and rings of finite representation type 233
11.3 Finite hulls over artinian rings 237
11.4 Finite Morley rank and finite representation type 241
11.P "Pathologies" 245

Chapter 12 Functor categories 247
12.1 Functors defined from pp formulas 248
12.2 Simple functors 254
12.3 Embedding into functor categories 258
12.P Pure global dimension and dimensions of functor categories 265

Chapter 13 Modules over Artin algebras 267
13.1 The space of indecomposables 267
13.2 Representation type of quivers 270
13.3 Describing the space of indecomposables 277

Chapter 14 Projective and flat modules 283
14.1 Definable subgroups of flat and projective modules 283
14.2 Projective modules and totally transcendental rings 289

Chapter 15 Torsion and torsionfree classes 294
15.1 Torsion, torsionfree classes and radicals 294
15.2 Universal Horn classes, varieties and torsion classes 298
15.3 Model-companions and model-completions of universal Horn classes 301
15.4 Elementary localisation 313

Chapter 16 Elimination of quantifiers 317
16.1 Complete elimination of quantifiers and its consequences 317
16.2 Modules over regular rings 323
16.C Continuous pure-injectives 330

Chapter 17 Decidability and undecidability 332
17.1 Introduction 332
17.2 Undecidability 337
17.3 Decidability 342
17.4 Summary 348

Problems page 350

Bibliography 351

Examples index 370
Notation index 371
Index 373

INTRODUCTION

I assume that the reader knows what a module is: indeed, I take for granted a certain knowledge of basic module theory. The reader who does not have this or who needs his or her memory refreshed has a wide range of texts to choose from. On the other hand, I have tried to cater for the reader who has no idea what model theory is about. So, the first section of the first chapter is an introduction to the subject. I don't include any proofs there, but I do present the definitions and try to explain the ideas. That section ends with the formal statements of some results which I call on later. Despite the lack of proofs, the reader will not have to take too much on trust and will, in any case, see many of the ideas being developed within the specific context of modules. The second section of the first chapter introduces injective modules and may simply be referred as the need arises.

Chapters 2, 3 and 4 form the core of the book and provide a common foundation for the more specialised topics of later chapters. Especially in the second chapter, the pace is quite leisurely and there are many exercises and illustrative examples. The central result of Chapter 2 is the description of the definable subsets of a module ("pp-elimination of quantifiers"). Chapter 3 characterises modules of the various stability classes in terms of their definable subgroups. Hulls of elements and pp-types are the building blocks of pure-injective modules: they are introduced in the fourth chapter, together with a number of central ideas (irreducible pp-types, unlimited components, the space of indecomposable pure-injectives).

Beyond this core, the reader's path may be guided more by his or her interest. Although the later chapters are all inter-related to some extent, they do break into a number of more or less coherent blocks. I give some general indication of their contents now: the section summaries below are more specific.

Chapters 5, 6 and 7 delve more deeply into the stability-theoretic aspects of modules. Stability theory is based on a very general notion of dependence and Chapter 5 describes that notion in the context of modules (in the right context, it reduces to direct-sum independence of hulls). Chapter 6 is concerned with the way in which pure-injective modules are built up around realisations of types (belonging to orthogonality classes) by fitting hulls around them: also in that chapter we produce irreducible types with certain prescribed properties. Chapter 7 is centered around a detailed analysis of modules of U-rank 1 and Vaught's Conjecture.

Chapters 8, 9 and 10 deal with the relationship between pp-types and their hulls. The idea that pp-types generalise right ideals is taken much further in these chapters. Chapter 8 looks at the lattice of pp-types, with the lattice of right ideals strongly in mind. Chapter 9 explores the relation between types whose hulls have some direct summand in common. In Chapter 10 we prove structure theorems under various finiteness conditions on the lattice of pp-definable subgroups.

Modules over artinian rings, and especially over artin algebras, are considered in Chapters 11, 12 and 13. Rings of finite representation type and pure-semisimple rings are the topic of Chapter 11. Chapter 12 interprets pp formulas and pp-types in terms of functor categories. In Chapter 13 we consider pure-injectives over algebras of tame and wild representation type.

Chapters 14, 15 and 16 all deal with special kinds of modules. Projective and flat modules are considered in Chapter 14. Dually, injective and absolutely pure modules are considered in Chapter 15, along with existentially closed modules in universal Horn classes ("torsionfree" classes). Chapter 16 begins by considering modules which have complete elimination of quantifiers, then goes on to deal with modules over regular rings.

The final Chapter 17 gives some partial answers to the question: over which rings is the theory common to all modules decidable?

I finish by indicating what I believe are currently the main unresolved questions.

Acknowledgements I would like to begin by thanking, for their useful comments, those who attended the lectures at Yale on which these notes are based. Next, I thank Bernadette Highsmith for the typing of the first version of these notes and Philipp Rothmaler for thoroughly reading them and pointing out to me a number of minor, and some major, errors. I would also like to thank, for their comments, for useful discussion and for information that they have provided: John Baldwin, Michael Butler, Sheila Brenner, Ivo Herzog, Wilfrid Hodges, Angus Macintyre, Anand Pillay, Francoise Point and Gabriel Srour.

Notations and conventions

I was a postgraduate student at Leeds and so my modules are usually right modules. Rings always have a "1". Functions (apart from ring multiplications!) usually operate from the left, and I do not normally put parentheses round the argument(s).

An "n-tuple" is an ordered sequence of n objects, and I can only apologise for the ugly word which results when the "n" is dropped.

If X and Y are sets, then X^Y means the set of all functions from Y to X. If we are working in a category $\mathcal{C}$ then the set of morphisms from an object X to the object Y is denoted $\mathcal{C}(X,Y)$ or (X,Y). Beware that I use "$<$", "$\subset$" and their opposites to indicate strict containment.

By, for example, $K[X,Y]$, I denote the ring of polynomials with coefficients in K in the commuting indeterminates X and Y: $K\langle X,Y\rangle$ denotes the ring of non-commuting polynomials. If I am dealing with a factor ring of such a ring, then I use the convention that "x" and "y" are the canonical images of X and Y.

In various contexts, I use the notation "$\langle * \rangle$" to denote "generated by $*$".

For definitions of notations, consult the appropriate index.

Internal referencing The book is composed of chapters, each of which has an introduction followed by a number of sections. The chapter introductions say briefly what is in each section, give some sort of overview of the chapter and indicate how it fits with the rest of the book. The first few paragraphs of a section may say more. Probably the best (and most specific) overview of the book may be obtained from the "Section Summaries" below.

Sections are referred to by "§m.n", meaning Section n of Chapter m, and by "§n", meaning Section n within the same chapter. If "n" is a letter, then the section is one of the supplementary sections.

Numbering of results is consecutive within each chapter: "m.n", where m and n are integers, means result n in Chapter m.

Exercises and examples are numbered afresh within each section. The abbreviation "Ex" is used for "Example"; "Ex n" means Example n within the same section; "Ex.m.n/k" means Example k in Section m.n. Similarly for exercises: "Exercise n" and "Exercise m.n/k".

There is an index of examples: included are those which appear many times (mainly, I list places where they are developed, rather than simply used to illustrate a point), as well as those to which I often refer.

Bibliography I have tried to make the bibliography comprehensive. When a bibliographic reference "X" precedes the statement of a result, this means that the result, some significant part thereof, or some special case, or even just the germ of the idea, is in X. If it is not the first or second, then I normally use a qualifier such as "see" or "also": but these terms have other uses (I use "see" for secondary sources: mainly standard texts). I am sure that I have not managed to be consistent in this, so I can only apologise and urge the interested reader to consult the works cited.

A reference to the bibliography has the form **[abcxy;*]**, where "abc" is a short sequence of letters, and "xy" gives the date. Usually, the letters are the first two or three letters of the author's name, but there are some exceptions (e.g., if there are two or more authors): so if you discover that you are attempting to find one of these exceptions, consult the list which follows the bibliography.

Remarks on the development of the area

With more than a little trepidation, I will try to give the reader some idea of the origins of the material in this book. This is not in any sense a history, but simply is an account of how the area seems to me to have developed.

In its barest outline, the model that I have in mind is: little before 1970; a burst of activity by a coherent group of people, linked through Yale; relative quiescence, though with important developments; a number of people independently picking up the threads again in the late 70's; these new strands gradually being brought together, with a mass of new results in the late 70's and early 80's; consolidation of the core material since then, together with an opening up of new lines of enquiry.

Like any model, that falls short of truth. For instance, the key pp-elimination of quantifiers was proved in what I have characterised as a quiet period (on the other hand, this result was not immediately exploited to the full).

The development has not been smooth and orderly. A number of results have been proved independently (often in contexts which were not obviously related) or have been re-derived. Also, in the mid/late seventies, a fairly small number of people independently began to work in the area but, by and large, did not have an immediate influence upon one another, mainly because of differing aims and backgrounds. For example, I received the (first version of the) preprint [Gar80a] a number of months before I began to look seriously at it: indeed, I had received preprints of [Gar79] and [Gar80] even earlier. My delay in looking at them may be explained by the fact that I was working on the injective case, where one essentially has full elimination of quantifiers: therefore, I needed little model theory and certainly no stability theory (indeed, I did not even have to take account of the pp-elimination of quantifiers).

The earliest relevant result is Szmielew's proof that any formula in the language of abelian groups is (effectively) equivalent to a boolean combination of sentences and formulas of a certain prescribed form (and with clear algebraic meanings). This result was not established for modules over an arbitrary ring until the mid-seventies.

During the 60's there were various algebraic developments, concerning the notions of purity and algebraic compactness, which were to be useful later.

At the beginning of the 70's, a fairly coherent group of people, many of whom visited Yale around then, worked on the model theory of modules. Probably the most influential paper to emerge was that of Eklof and Sabbagh [ES71]: it was the starting point for my own work in the area, and I am not alone in this. Other results from this period include Eklof and Fisher's re-proof (and extension) of Szmielew's result, by use of structure theory for pure-injective abelian groups; the attempt to extend the pp-elimination of quantifiers to arbitrary rings, by Fisher and by Sabbagh; (a little later) Baur's work on (un)decidability and Fisher's work on "abelian structures".

Let me now say a little about the latter work of Fisher. This was written up in [Fis75], but that was never published, although the first part did appear in print somewhat later. It contains many fundamental results, including the existence of hulls and the general decomposition theorem for pure-injective modules. But, partly because it was not widely circulated and partly because it was set in a very general context, this did not have the influence that it should have had. For instance, the (importance of the) existence of hulls was generally not appreciated by those who started later to work in the area. Also, the decomposition for pure-injectives and the fact that they have local endomorphism rings had to be re-proved by algebraists (more than once).

One aim of Fisher's work was to establish pp-elimination of quantifiers, but he was able to obtain this, using "structural" arguments, only in very restricted circumstances. Baur tried a more syntactic approach and succeeded in proving it. Independently, L. Monk proved the result for abelian groups, and his argument works over any ring: there were also partial results by Mart'yanov. With the pp-elimination of quantifiers to hand, the proofs of many of the earlier results were much simplified, and other theorems which had been proved under special hypotheses could now be proved in general. Nevertheless, the vigorous exploitation of this result was not immediate.

Garavaglia, in a series of papers, was first to make full use of pp-elimination of quantifiers. These papers have been the key to the further development of the subject.

Meanwhile, a number of other people had begun to work independently on the area. In particular, there were independent contributions from Bouscaren, Kucera, myself and Rothmaler (at least the first three either drawing their initial inspiration from, or being significantly influenced by, Eklof and Sabbagh's paper [ES71]). Initially working in rather different directions, as the aforesaid gradually took account of each others' approaches, the central lines of thought became clearer. By now, others had joined in: Pillay in contact with those already mentioned and, rather independently, some of those visiting Jerusalem during the model theory year held there in 1980/81 - in particular, there was another approach by Srour and a major contribution by Ziegler, the latter having been inspired by Garavaglia's paper [Gar80a]. By now, the model theory of modules had become a unified area.

During this period there was other work either within, or relevant to, the area (see below!), but that which I have mentioned above has had the most influence on the shape of the "core material" of the subject. I should, finally, point to the more algebraic work of Gruson, Jensen, Lenzing, Simson, Zimmermann and Zimmermann-Huisgen on pure-injective modules. This work often overlapped what was being done by those already mentioned, but I think that it has not yet been properly assimilated into the model theory of modules.

The core material of the area seems to be fairly settled now. The main current (1987) interest lies in two directions: more detailed analysis of what goes on within pure-injective modules, especially as it related to Vaught's Conjecture; connections with representation type of algebras. But there are many relatively unexplored avenues and I would not hazard a guess as to what the subject will look like in ten, or even five, years' time (nevertheless, I have, after Chapter 17, indicated what I see as the main current open problems).

Section summaries

§1.1 Background from model theory.

§1.2 Injectives, injective hulls and Λ-atomic types; finiteness conditions on the lattice of right ideals and decomposition of injective modules (1.11, 1.12).

§2.1 pp-definable subgroups; examples.

§2.2 pp-types; group associated to a pp-type.

§2.3 Pure-injectives (2.8); Σ-pure-injectives (2.11).

§2.4 Neumann's Lemma (2.12); invariants; pp-elimination of quantifiers (2.13); criterion for elementary equivalence (2.18).

§2.5 Arithmetic of the invariants (2.23); pure embeddings between elementarily equivalent modules are elementary (2.26); every module elementarily equivalent to its pure-injective hull (2.27).

§2.6 Partial ordering on complete theories of modules; largest complete theory of modules (2.32).

§2.$\mathbb{Z}$ Simpler pp formulas over principal ideal domains (2.$\mathbb{Z}$1); indecomposable pure-injectives over discrete rank 1 valuation domains (2.$\mathbb{Z}$3); localisation (2.$\mathbb{Z}$5, 2.$\mathbb{Z}$8); indecomposable pure-injectives over Dedekind domains (2.$\mathbb{Z}$11).

§2.L Other languages

§3.1 All modules are stable, characterisation of superstable and totally transcendental modules (3.1); totally transcendental $\equiv$ Σ-pure-injective (3.2).

§3.2 Totally transcendental modules are direct sums of indecomposable submodules (3.14).

§3.A Abelian structures

§4.1 Construction of hulls by realising maximal pp-types; characterisations of hull (4.10); transferability of hulls between theories (4.12); uniqueness of hulls (4.15).

§4.2 1. Injective hulls; 2. pure-injective hulls; other points and examples.

§4.3 Decomposition in spectral categories (4.A3); Uniqueness of decomposition when have local endomorphism rings (4.A7); structure theorem for pure-injective modules (4.A14).

§4.4 Endomorphism ring of indecomposable pure-injective (4.27); syntactic characterisation of irreducible types (4.29); construction of irreducible types (4.33); every module is equivalent to a discrete pure-injective (4.36); no new indecomposable pure-injectives in $T^{\aleph_0}$ (4.39).

§4.5 Characterisations of unlimited types (4.41-4.43).

§4.6 *t.t. theories* minimal index in indecomposable pure-injective equals number of automorphisms modulo endomorphisms (4.53, 4.55); irreducible type limited implies hull always occurs a fixed finite number of times (4.60); description of prime model (4.62); description of arbitrary model (4.63).

§4.C Categoricity.

§4.7 Topology is compact and neighbourhood bases given by irreducible types (4.66); closed sets = component theories (4.67); conditions for prime and/or minimal discrete pure-injective models (4.73).

§5.1 Non-forking extensions characterised (5.3); independent sets characterised "syntactically" (5.5).

§5.2 Description of U-rank (5.12, 5.13); rank of a type is rank of some pp formula in it (5.15); ranks coincide (5.18).

§5.3 Unlimited types (are stationary, and realisations of) (5.26, 5.27, 5.28); independent sets characterised "algebraically" (5.29).
§5.4 Simplifications when $T = T^{\aleph_0}$; independence as pushout (5.40).

§6.1 Prime-pure-injective extensions (6.2); the stratified order; free part of a type; over a pure-injective model, the free part splits off (6.4); types with the same free part (6.6).
§6.2 A set dominates its hull (6.9, 6.11, also 6.15); algebraic characterisation of domination (6.13); types RK-equivalent iff free parts have isomorphic hulls (6.16); characterisation of RK-minimal types (6.17).
§6.3 Characterisation of orthogonality (6.20); all modules non-multidimensional (6.21).
§6.4 Type regular iff free part critical (6.23); regular type realised between superstable modules (6.26); strongly regular types (6.27); irreducible types with specified properties (6.28, 6.29, 6.30); T countable and non-$\aleph_0$-categorical implies infinitely many countable models (6.32).
§6.I Injective modules over noetherian rings
§6.5 Superstable pure-injectives are $\aleph_0$-homogeneous (6.35); (superstable) $F^a_{\aleph_0}$-saturated equivalent to pure-injective plus weakly (or $\aleph_0$-) saturated (6.37).
§6.6 Characterisation of strong types (6.43); count of multiplicity of a type (6.44); realisations of strong types (6.45, 6.46).

§7.1 Relative saturation and pure-injectivity; possible uncountable spectra (7.4).
§7.2 Unidimensional modules are superstable (7.9); structure theorem for modules of U-rank 1 (7.14); characterisation of small theories (7.15); re. Vaught's Conjecture for modules of U-rank 1 (7.19, 7.20); examples of small theories.
§7.3 U-rank n implies $\leqslant n$-dimensional (7.23).

§8.1 Irreducible pp-types are ∩-irreducible (8.2); neg-isolated types (8.3).
§8.2 pp-type finitely generated iff realised in finitely presented module (8.4); irreducible in P^f iff irreducible in P (8.7); neg-isolated in P^f iff neg-isolated in P (8.8).
§8.3 Closed sets of matrices; matrix criterion for one pp formula to imply another (8.10); free realisation of pp formula (8.12); pp-types ≡ closed sets of matrices (8.16).
§8.4 Duality between right and left pp-types (8.21); applications to pure-semisimple rings (8.23-25); right pss rings have only finitely many modules of each finite length (8.27).

§9.1 Types which share a minimal pair have isomorphic hulls (9.3); multiplicity of a limited indecomposable is fixed (9.5).
§9.2 Linked pp-types have isomorphic intervals (9.9); irreducible components are associated *via* large formulas (9.16).
§9.3 Neg-isolation and weight (9.17); isolated ≡ finitely generated + neg-isolated (9.20); implications between notions of isolation (9.26).
§9.4 Elementary cogenerators and neg-isolated types.

§10.1 Conditions which guarantee an indecomposable summand (10.1, 10.2).
§10.2 Dimension with respect to a class of lattices; m-dimension (10.6); breadth (10.8); width ≡ breadth (10.7).
§10.3 Breadth $<\infty$ implies no continuous pure-injectives (10.9); converse for countable theories (10.13).
§10.4 T countable plus dense set of pp-definable subgroups implies $2^{\aleph_0}$ indecomposables (10.15); T countable or with continuous part zero implies isolated points have minimal

pairs (10.15, 10.16); CB-rank = m-dimension if enough minimal pairs (10.19); structure theorem for theories with m-dimension (10.24).
§10.5 (Elementary) Krull dimension.
§10.T T^{eq}; more regular types in T^{eq}.
§10.6 Krull dimension and foundation rank (10.41, 10.42); relative foundation rank; Pillay's and Lascar's ranks.
§10.V Valuation rings.

§11.1 Right pure-semisimple iff every module is t.t. (11.6); consequences of rt. pss (11.10).
§11.2 Right pure semisimple iff every direct sum of f.g. modules is t.t. (11.13); artin algebra with only finitely many indecomposables is of finite representation type (11.15); R, countable, is of FRT iff there are fewer than $2^{\aleph_0}$ countable modules (11.18).
§11.3 Finitely presented versions of hulls (11.21); (R rt. artinian) p.i. hulls of indecomposable f.g. modules are indecomposable (11.24); (R rt. artinian) every f.g. type over 0 has finite weight (11.26).
§11.4 Finite representation type iff every module has finite Morley rank iff every irreducible pp-type is neg-isolated (11.31 (artin algebras), 11.38 (in general)); neg-isolation and products (11.32, 11.34)
§11.P Pathologies displayed by "large" modules.

§12.1 Every subfunctor of the forgetful functor is a sum of pp-functors (12.2); pp-functors form a generating set of finitely presented objects (12.3); functor category which sees infinitely generated pp-types.
§12.2 Simple functors and irreducible types (12.7, 12.8, 12.9); simple functor finitely presented iff corresponding irreducible pp-type isolated (12.10); every non-zero functor has simple subfunctor iff ring right pure-semisimple (12.18); functorial characterisation of finite representation type (12.19).
§12.3 How to turn pure-injectives into injectives; localisation.
§12.P Pure global dimension; dimensions of functor categories.

§13.1 (R an artin algebra) The finite points of $\mathcal{I}(T)$ are precisely the isolated points and they are dense (13.3, 13.4).
§13.2 Path algebras; representation type of quivers - tame and wild (13.A, 13.B); the AR-quiver of an extended Dynkin diagram; string and band modules; refinements of tame type.
§13.3 (R the path algebra of an extended Dynkin diagram) The closure of the regular indecomposables includes all infinite-dimensional indecomposables (13.6); classification of the infinite-dimensional points; the infinite-dimensional points in relation to the AR-quiver; wild algebras have continuous pure-injectives (13.7).

§14.1 The pp-definable subgroups of a flat module (14.9); pure-projectives; left coherent iff pp-definable subgroups are exactly the f.g. left ideals (14.16); left coherent iff flat modules axiomatisable (14.18).
§14.2 If left coherent, then totally transcendental iff right perfect (14.19), hence left perfect (14.23); right perfect and left coherent iff projectives axiomatisable (14.25); characterisation of when free modules axiomatisable (14.28).

§15.1 Preradicals, torsion and torsionfree classes; cogenerators.
§15.2 Torsionfree class elementary iff of finite type (15.9); torsion class elementary iff a TTF class (15.15).
§15.3 Relative absolute purity and injectivity; existentially complete modules; universal Horn class $\mathcal{K}$ has model-companion iff indecomposable $\mathcal{K}$-injectives are closed in $\mathcal{I}(\mathcal{K})$

iff $\mathcal{K}$-absolutely pures are axiomatisable (15.26, 15.27); universal Horn class of abelian groups has a model-companion (15.30); (with amalgamation) $\mathcal{K}$ has a model-completion iff $\mathcal{K}$ is a coherent class (15.34); pp-definable subsets of injective modules are annihilators (15.40); right coherent iff every injective has elimination of quantifiers (15.35, 15.42).

§15.4 Class of localised modules is elementary iff radical is elementary (15.43); locally finitely presented abelian Grothendieck category iff elementary localisation of a Module category (15.46).

§16.1 Elim-Q and Elim-Q^+ (16.5); Elim-Q^+ implies indecomposable pure-injectives are uniform (16.7); forking and ranks in terms of associated right ideals (16.9, 16.10, 16.11).

§16.2 Ring regular iff every module has elim-Q (16.16); canonical form for invariants over regular rings (16.18); characterisation of t.t. (16.21), superstable (16.22) modules over regular rings; (R commutative regular) m-dim(T^*)$<\infty$ iff B(R) superatomic (16.25) iff w(T^*)$<\infty$ (16.26); T^* has prime model iff B(R) atomic (16.27).

§16.C Possibilities for decomposition of continuous pure-injectives.

§17.1 Context in which to ask about decidablity of theory of modules; decidability invariant under effective Morita equivalence (17.1, 17.2).

§17.2 Undecidable theories: $K\langle X,Y\rangle$-modules (17.3); five-subspace problem (17.5); pairs of abelian groups (17.6); vectorspace + subspace + endomorphism (17.7); $K[X,Y]$-modules (17.8); wild quivers; wild posets; certain local algebras; certain commutative algebras.

§17.3 Criteria for decidability in terms of $\mathcal{I}(T)$ (17.10-12); decidable theories: modules over Dedekind domains (17.13); pairs of torsionfree abelian groups (17.15); representations of extended Dynkin quivers (17.16); certain local artinian rings (17.19-17.21).

§17.4 Summary.

CHAPTER 1 SOME PRELIMINARIES

The two sections of this chapter are rather different in nature.

The first section is a brief review of some basic model theory. Those who are already acquainted with this material will need only to glance at it for some conventions. On the other hand, those who know nothing of model theory may find that I have been too concise (although I hope not). The section does at least contain the essential definitions and results and I have tried to explain the main points. A number of texts may be recommended to the reader who desires more detail. The standard reference in model theory was, for some time, the book [CK73] of Chang and Keisler; the book [Sac72] by Sacks is quite readable, although less comprehensive. More specifically on model-theoretic algebra is Cherlin's [Che76]. All these are to be recommended, as are the very readable articles by Barwise, Keisler, and Eklof (and, relevant later, that of Macintyre) in the Handbook of Mathematical Logic [Bar77].

I would not recommend texts on logic in general, since these tend to begin with a very careful treatment of first order logic, which is tedious and probably off-putting to most mathematical readers.

The situation has changed recently, and I am now in the happy position of being able to recommend some more up-to-date texts which either have appeared or are soon to appear. In particular, as introductions (and a good deal more) to model theory, I recommend the books of Poizat [Poi85] and of Hodges [Hod8?]. The first of these also goes some distance into stability theory. There are also a number of recent texts specifically on stability theory: for these see the introduction to Chapter 3.

The second section introduces injective modules and describes some structure theorems for them. I decided to place this section at this point, partly because it was with injective modules that my own work began, and this has influenced my approach since (certainly it has influenced my presentation in this book). But also, I feel that a reading of this section will allow one to discern the shape of some of the main concerns of the book. It will be of use to some readers because it gives a foretaste, in "purely algebraic" surroundings, of ideas seen later: in particular, for those familiar with injective modules, it may illustrate and motivate some of what is to come. Moreover, it will help one to appreciate both the similarities and the contrasts between the purely algebraic (quantifier-free) and the general cases, and to understand the effect of taking account of quantifiers. But the section certainly may be skipped and then referred back to for definitions and results as the need arises.

1.1 An introduction to model theory

An algebraist who has had some acquaintance with model theory may be surprised to see barely an ultraproduct in the pages which follow. Partly, this is because we usually work within a model of a complete theory. But also, ultraproducts provide specific realisations of structures and elements, the existence of which is consistent: yet, usually all we need is the fact that such realisations exist. This existence is embodied in the completeness theorem, in the form which states that any (finitely) consistent set of formulas has a realisation in some elementary extension (one may give a proof of this theorem using ultraproducts). So, when a realisation of a type is required, I will normally call on this existence theorem, rather than produce a specific ultraproduct. In any case, when we have realised a type in some (unspecified) elementary extension, it will often be possible to cut that extension down to a more economical one, should we wish to do so.

The descriptions given here of model-theoretic concepts will mostly be phrased in terms of modules rather than more general structures: but I do not, at this stage, use any features peculiar to modules.

Our model theory will be set within a certain context - for us here the context is that of modules over some fixed ring (for others it might be groups, or ordered fields,...). The context having been established, there is usually an obvious set of operations, relations and constants, from which one may construct the most basic terms and relations, using unknowns and particular elements of particular structures,

For example, if our context is that of modules over the ring R, then we always have the operation of addition of elements in a module and so we put the symbol "+" into our language. Within each module there is a distinguished element: the zero for the addition, so we will have a constant symbol "0" in the language. Then, for each element $r \in R$, we have the multiplication by r, which is simply an endomorphism of the underlying abelian group of the module; therefore we add to our language a function symbol for each element of the ring. Our modules will be right modules and I will use the normal notation "$(-)r$" for this function symbol.

We want to be able to use our language to define natural subsets of structures. In the context of fields, examples are the sets of those n-tuples which satisfy a given system of polynomial equations - that is, the subvarieties of affine n-space. To be able to define such subsets, we need some "unknowns" or **variables**: $u, v, w, v_0, v_1, \ldots$; often we will allow ourselves to use parameters from a given structure in addition to these. With what we now have to hand, we can build up **terms**: polynomials in the case of fields; R-linear combinations in the case of R-modules. Thus a typical term in the module case may be brought to either the form "0" or to the form "$v_1r_1 + v_2r_2 + \ldots + v_nr_n$" (or just $\Sigma_1^n v_i r_i$ for short), where the r_i are elements of the ring and the v_i are variables: if we were allowing parameters in the formulas, then some of the variables might be replaced by (symbols for) elements from a certain module.

Having defined the terms, we may define the most basic formulas. In modules there are no special relation symbols (such as "$\leqslant$") to deal with and so we simply add "=" to our language, then define the **atomic** formulas to be those expressions of the form $t_0 = t_1$ where t_0 and t_1 are terms. In the module case such an expression may, of course, be simplified to one of the form $\Sigma_1^n v_i r_i = 0$.

Given an atomic formula φ such as that above and given any module M, one may consider φ as defining the subset of M^n consisting of all n-tuples which satisfy it. This subset will in fact be a subgroup, since it is the solution set for an R-linear equation. For example, if we were working over the ring of integers then φ could be $v6 = 0$, in which case, the set defined by φ would be the set of all elements of M annihilated by 6.

We will be interested in sets such as these but also in their boolean combinations: if φ and ψ are two formulas (by adding dummy variables we can assume that they involve the same unknowns) then the formula $\varphi \wedge \psi$, read as "φ *and* ψ", defines the intersection of the corresponding definable subsets (in any module), and is termed the **conjunction** of φ and ψ (e.g., $v6 = 0 \wedge v4 = 0$ defines the set of all elements annihilated by 12). Repeated conjunction will be denoted by use of "$\bigwedge$" (in the same way that "Σ" is used for repeated addition). So now we have expanded our notion of definable subset to encompass solution sets of <u>systems</u> of R-linear equations.

If φ is a formula, then the complement of the subset it defines (in any given module) is defined by the formula $\neg\varphi$, which is read as "*not* φ" and is called the **negation** of φ (e.g., $v \neq 0$ - being $\neg(v = 0)$).

Since unions may be obtained by use of intersection and complement, it is not necessary to introduce them separately, but it is useful to have a notation for them; so we write $\varphi \vee \psi$, read as "φ *or* ψ" and called the **disjunction** of φ and ψ, for the formula $\neg(\neg\varphi \wedge \neg\psi)$ which defines the union of the sets defined by φ and ψ.

The formulas which may be obtained from the atomic formulas by applying $\wedge$ and $\neg$ (and $\vee$) are the **quantifier-free** formulas (e.g., $v6 = 0 \wedge v2 \neq 0 \wedge v3 \neq 0$). We see in that in modules these define precisely the boolean combinations of solution sets of (systems of) linear equations. These correspond to what are termed the "constructible" sets in algebraic geometry, and in that case one need go no further on account of the Chevalley/Tarski Theorem (see [vdD8?]) which says that the image of a constructible set under a morphism is again constructible.

In modules (as in most contexts) life is not so simple (though I use the word with trepidation!). The problem is that even the most straightforward morphisms between modules - namely projections from one power of a module to another - may take a subset defined in the above way to one which is not so defined.

For example, suppose that we are working with modules over the ring, $\mathbb{Z}$, of integers. Let us consider $\mathbb{Z}$ as a module and look at the subset of $\mathbb{Z}\times\mathbb{Z}$ consisting of all pairs (a,b) with $a=b2$: this is just the subset defined by the quantifier-free formula $v_1=v_2 2$. If we project to the first copy of $\mathbb{Z}$, then the image of this definable subset is the set of all even integers, and it is easily seen ($\mathbb{Z}$ is torsion-free) that this set cannot be defined by a quantifier-free formula. In fact the natural way to define this set is to use an existential quantifier - for it is the set of all elements of $\mathbb{Z}_{\mathbb{Z}}$ which satisfy the condition expressed by the formula $\exists v_2\,(v_1=v_2 2)$.

Thus we are lead to expand our notion of formula by saying that if φ is a formula and if v is a variable then $\exists v\,\varphi$ is a formula, read as "there exists v such that φ" (no matter if the variable v does not occur (unquantified) in φ; for then the prefixing of the existential quantifier simply has no effect). For example $v_1=v_2 2$ becomes $\exists v_2(v_1=v_2 2)$. As is usual, we call "$\exists v$" an **existential quantifier**. The **universal quantifier** "$\forall v$" ("for all v") need not be introduced separately since $\forall v\,\varphi$ and $\neg\exists v\,\neg\varphi$ define the same set. In terms of definable sets, the existential quantifier corresponds to projection. For example, in the language of fields, the formula $\exists v\,(vw=1)$ defines, in any field K, the projection to the first coordinate of the subset $\{\alpha\beta : \alpha\beta=1\}$ of K^2.

Finally we say that a **formula** is an expression built up from our atomic formulas using $\wedge, \neg, \exists$ (and $\vee, \forall, \rightarrow, \leftrightarrow$ as derived symbols: $\varphi\rightarrow\psi$ ("φ implies ψ") being $\neg\varphi\vee\psi$; $\varphi\leftrightarrow\psi$ being $(\varphi\rightarrow\psi)\wedge(\psi\rightarrow\varphi)$). By the **language** L_R will be meant the set of all these formulas.

Any formula φ may be brought to an equivalent formula in **normal form** which has the shape: a string of quantifiers followed by a quantifier-free formula which latter is called the **matrix** of φ - a singularly appropriate term in modules, as it turns out!

A word on reading formulas: "$\neg$" has first priority; then "$\wedge$"; then "$\vee$"; then "$\rightarrow$" and "$\leftrightarrow$"; then "$\exists$" and "$\forall$". Thus, for example $\neg\varphi\vee\psi$ means $(\neg\varphi)\vee\psi$. Brackets and spacing will be used to increase readability.

Given a formula φ, one commonly displays those variables which may be substituted for - these are called the **free** or **unquantified** variables - and one writes $\varphi(v)$, $\varphi(v_1,v_2)$, $\varphi(\overline{v})$,... as appropriate. For example, the formula $wr=v$ has both variables v,w free; the formula $\exists w\,(wr=v)$ has only v free. The formula $\forall v\,(vr=0)\wedge\exists w\,(ws=v)$ has only v free but also, that variable occurs free only in the second conjunct so, when we substitute an element for v in the formula, it is only the last occurrence which is substituted for. This is in accordance with a common-sense reading of the formula, which recognises that the first conjunct is a sentence - an expression with no variables free, which makes sense as it stands and so in any given module is either true or false. This is in contrast with the second conjunct, which requires a value to be assigned to its free variable before it becomes a statement. The replacement of "v" in its first two occurrences by some other variable would not, in any essential sense, change the formula and would be good practice: indeed such inessential changes which increase readability will often be made without comment.

Given a formula $\varphi(\overline{v})$, a module M, and a tuple $\overline{a}$ in M and which matches $\overline{v}$, we write $M\models\varphi(\overline{a})$ if $\overline{a}$ lies in the subset defined by φ. A number of points are to be made. The subset defined by φ is of course a subset of M^n, where $n=l(\overline{v})$ (by $l(\overline{v})$ is meant the length of the tuple $\overline{v}$): this subset is usefully denoted $\varphi(M)$. Assumptions such as the matching of $\overline{a}$ and $\overline{v}$ for length normally will not be made explicit and may be assumed. In saying that the tuple $\overline{a}$ is **in** M, I mean that the entries of $\overline{a}$ lie in M. One may regard $\varphi(\overline{a})$ as a formula which involves the entries of $\overline{a}$ as parameters (the entries of $\overline{a}$ replace occurrences of entries of $\overline{v}$), and so may read "$M\models\varphi(\overline{a})$" as "$M$ **satisfies** $\varphi(\overline{a})$" (also: "$\overline{a}$ satisfies φ in M") for, if $\varphi(\overline{a})$ is read in a common-sense way, then we see that $\overline{a}$ lies in $\varphi(M)$ iff the sentence $\varphi(\overline{a})$ is true in M. Similarly, if σ is a **sentence** - that is, a formula with no variables occurring free - then one writes $M\models\sigma$, saying that M satisfies σ, if the sentence σ, when **interpreted** (read) in M, is true. If Φ is a, possibly infinite, set of formulas (a type, for instance), then define $\Phi(M)=\bigcap\{\varphi(M):\varphi\in\Phi\}$.

Example 1 Let us take the ring to be the ring of integers: then the modules are just the abelian groups. Although the language for $\mathbb{Z}$-modules strictly contains that for abelian groups (which has just 0 and $+$), nevertheless any formula in the first language clearly is equivalent to one in the second language (multiplication is just repeated addition) and so the two languages are "equivalent".

Here are some properties of a $\mathbb{Z}$-module which may be expressed by a sentence or a set of sentences:

(i) M has an element of order 2: $M \models \exists v\,(v2=0 \wedge v \neq 0)$.

(ii) M is torsion-free: $M \models \{\forall v\,(vn=0 \rightarrow v=0) : n \in \mathbb{Z}, n \neq 0\}$ (if Σ is a set of sentences then we write $M \models \Sigma$ to mean $M \models \sigma$ for each $\sigma \in \Sigma$).

(iii) M is divisible: $M \models \{\forall v\,\exists w\,(v=wn) : n \in \mathbb{Z}, n \neq 0\}$.

There is a point which is sometimes lost sight of by those to whom these ideas are unfamiliar. The range of the quantifiers $\exists v$, $\forall v$ is over module elements only and, in particular, not over elements (rather, function symbols) of the ring. Of course the ring itself is a module, but the roles of its elements as members of this module and "as" function symbols of the language are quite distinct.

As illustration, consider the property of being a torsion module. The obvious way of writing this - $\forall v\,\exists n\,(n \neq 0 \wedge vn=0)$ - is not an expresssion of our language. Lest one be tempted to increase the expressiveness of the language one should be warned that in general, the more one can express in a formal language the worse-behaved is the language: it is not enough to know that some property may be expressed formally, one must be able to make effective use of that information.

It will be shown below that there is no (less obvious) way of expressing "torsion" in this language.

I have already alluded to the possibility of enriching the language by adding in constant symbols for parameters (i.e., particular elements of a particular module). For instance, one may have an element a and a set B, both in the module M, and it may be that we are interested in the relation between a and B. This relation could be described by specifying those subsets, definable with parameters in B (one says **definable over** B), to which a belongs. One treats this formally by adding to the language a set of constants (or rather constant symbols) which are to be interpreted as the corresponding elements of B. In practice, we make no notational distinction between the constant symbols and the elements they are to represent. If the original language is denoted L then this expanded language is denoted L^B.

The idea which is described next is a key one. Its centrality is one distinction between stability theory and more classical model theory (though it is of course important in the latter). It is also well-suited to applications in algebra since it, in some way, generalises the notion of isomorphism type of the substructure generated by a set of elements.

Given a tuple $\overline{a}$ in the module M and a subset $B \subseteq M$, we will define the type of $\overline{a}$ over B (in M) to be the collection of all those subsets of M which are definable over B and which contain $\overline{a}$. This gives the right picture but we must be careful here since, by subsets of M, I mean subsets of M^n, where n is the length of $\overline{a}$ (we allow n to be infinite): thus, for example, if σ is a sentence then we regard the subset of M^n defined by σ to be the whole of M^n if σ is true in M, otherwise σ defines the empty set.

Exercise 1 Show that the type of $\overline{a}$ in M over B forms an ultrafilter in the boolean algebra consisting of all subsets of M^n $(n = l(\overline{a}))$ definable over B, ordered by inclusion. Recall that an **ultrafilter** in a boolean algebra B is a non-empty subset of B which is closed under finite intersections, is upwards closed, does not contain the zero $\emptyset$ of B (that is a **filter**) and, for every $X \in$ B, contains either X or its complement (but not, of course, both).

One may also define types to be syntactic objects as follows. Rather than concentrating on the definable subsets, we consider the formulas used to define them, and say that the type of $\overline{a}$ in M over B is: $\mathrm{tp}^M(\overline{a}/B) = \{\varphi(\overline{v},\overline{b}) : M \models \varphi(\overline{a},\overline{b})$ where φ is a formula and $\overline{b}$ is in $B\}$. We will blur the distinction between these two ways of defining types by using "$\varphi(\overline{v},\overline{b})$" both for the formula and for the set $\varphi(M,\overline{b})$ which it defines (assuming that "M" is clear from context). Note that different formulas may define the same set.

I should perhaps be a little more explicit in the definition just above. Suppose that the tuple $\bar{a}$ is $(a_\alpha)_{\alpha<\beta}$: choose and fix a corresponding sequence of variables $\bar{v}=(v_\alpha)_{\alpha<\beta}$ (since I allow $\bar{a}$ to be infinite and even uncountable, the language may have to be expanded with extra variables). Then $\mathrm{tp}^M(\bar{a}/B)$ is to be regarded as a set of formulas in this fixed sequence of variables, with v_α always corresponding to a_α. So, where in the definition one has $M\models\varphi(\bar{a},\bar{b})$ and one actually means $M\models\varphi(a_{\alpha_1},\dots,a_{\alpha_k},\bar{b})$ for suitable $\alpha_1,\dots,\alpha_k$, the formula to be put into $\mathrm{tp}^M(\bar{a}/B)$ is $\varphi(v_{\alpha_1},\dots,v_{\alpha_k},\bar{b})$.

When the context makes M clear or superfluous it may be dropped as a superscript. We may drop set brackets from B if appropriate. When B is empty or $\{0\}$ (clearly in modules one has $\mathrm{tp}(-/\emptyset)=\mathrm{tp}(-/0)$) we write $\mathrm{tp}^M(\bar{a})$ and simply refer to the type of $\bar{a}$ (in M).

The first sections of Chapter 2 are largely devoted to exploring, in the context of modules, the sort of information that a type may contain.

Suppose that $M\leqslant M'$ is an inclusion of modules. We say that this embedding is an **elementary** one and write $M\prec M'$ if, for every (finite) $\bar{a}$ in M, one has $\mathrm{tp}^M(\bar{a})=\mathrm{tp}^{M'}(\bar{a})$. One also says that M' is an **elementary extension** of M or that M is an **elementary substructure** of M'. Since, by definition, the type of $\bar{a}$ in M includes all sentences true in M, one has, if $M\prec M'$ that, in particular, M and M' are **elementarily equivalent** – that is, they satisfy exactly the same sentences of the language – and we write $M\equiv M'$. Easy examples (such as $2\mathbb{Z}\leqslant\mathbb{Z}$) show that even if $M\leqslant M'$ and $M\simeq M'$, it need not be the case that M is an elementary substructure of M'. The elementary embeddings between structures give the most natural category of structures, since they preserve both the definable subsets (if $M\prec M'$ and φ is a formula, perhaps with parameters from M, then $\varphi(M)=M\cap\varphi(M')$) and those pairs of formulas which define the same subsets (that can be expressed by a sentence – in fact every sentence may be read as such a statement).

The (complete) **theory** of a module M is the set, $\mathrm{Th}(M)$, of all sentences of the language which are true in M. In general a **theory** is any consistent set, T, of sentences of the language: **consistency** means that no formal contradiction may be deduced from T (equivalently, from any finite subset of T). A **model** of the theory T is any module M which satisfies all the sentences in T: one writes $M\models T$. Any theory of the form $\mathrm{Th}(M)$ is **complete** in the sense that, for every sentence σ, either σ or $\neg\sigma$ lies in the theory.

One theory which will always be assumed to be in the background (included in any theory considered) is the set of axioms for R-modules. Given the ring R, one may set down the axioms (sentences) stating that $(?,+,0)$ is an abelian group (a finite set of sentences) and that the elements of R act to form a ring of linear maps under pointwise addition and composition (can be taken to be a finite set of sentences if R is finite, but not otherwise). (Exercise: write these down.) The theory of R-modules certainly is not complete: $\forall v\,(v=0)$ is a sentence which is true in the zero module but false in any non-zero module, so neither it nor its negation lies in the **deductive closure** (set of logical consequences of) this theory (usually I don't distinguish between a set of sentences and its deductive closure). Most of our work will be with complete theories.

Let us summarise:

$\mathrm{Th}(M)=\{\sigma : \sigma$ is a sentence and $M\models\sigma\}$;

$\mathrm{Mod}(T)=\{M : M\models T\}$ is the set of models of T;

$M\models T$ iff $M\models\sigma$ for each $\sigma\in T$.

$\Phi\vdash\Psi$ – "Φ proves Ψ" – means that every formula in Ψ is a logical consequence of the set of formulas Φ.

If $\mathcal{C}$ is a class of modules, write $\mathcal{C}\models T$ if each member of $\mathcal{C}$ satisfies each sentence in T.

Also, one says that a class is **axiomatisable** or **elementary** if there is a, possibly infinite, set of sentences T such that the class is exactly the class of models of T. A class is elementary iff it is closed under ultraproducts and elementary substructures.

An absolutely central result is the compactness/completeness theorem which links syntax (formal consistency) with semantics (existence of a model). It says that if T is a theory (a formally consistent set of sentences in some first-order language) then T has a model. Reflecting our concentration on complete theories, will be our use of this theorem in the following form (obtained as a corollary by adding in new constant symbols corresponding to the entries of $\bar{v}$).

Suppose that $\Phi(\bar{v})$ is a set of formulas consistent with the complete theory T (by this is meant that, if $\{\varphi_1(\bar{v}), \varphi_2(\bar{v}), \ldots, \varphi_n(\bar{v})\}$ is any finite subset of $\Phi(\bar{v})$, then any, equivalently every, model of T satisfies $\exists \bar{v} \bigwedge_{i=1}^{n} \varphi_i(\bar{v})$: that is, "the intersection of the corresponding subsets, $\varphi_i(M)$, is non-empty"). Then, given any model M of T, there is an elementary extension M' of M and there is $\bar{a}$ in M' such that $M' \models \Phi(\bar{a})$: we say that M' contains a **realisation** of $\Phi(\bar{v})$.

Example 2 Take $R = \mathbb{Z}$ and let T be the complete theory of the module $M = \bigoplus \{ \mathbb{Z}_n : n \geqslant 2\}$ where $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is the abelian group of integers modulo n – notice that M is a torsion module.

Consider the set of formulas $\Phi(v) = \{ "vn \neq 0" : n \geqslant 1 \}$ which says that "v" is a torsionfree element. Any finite subset of Φ is clearly equivalent to a single formula, $vn_0 \neq 0$, where n_0 is the lowest common multiple of the integers appearing (we can be looser than this: take n_1 to be the product of the integers appearing and note that consistency of $vn_1 \neq 0$ certainly implies consistency of the subset). Then certainly $M \models \exists v\, (vn_0 \neq 0)$, since a generator b (say) of $\mathbb{Z}_{n_0+1}$ satisfies $bn_0 \neq 0$. Thus $\Phi(v)$ is consistent with T. So, by the compactness/completeness theorem, there is an elementary extension $M' \succ M$ and an element $a \in M'$ with $M' \models \Phi(a)$. Since $an \neq 0$ for each $n \geqslant 1$, a is a torsionfree element of M'. Since M itself is torsion, one sees that it really was necessary to go to a proper elementary extension, in order to realise the set Φ. I have also justified my earlier claim that "torsion" is not a property which can be axiomatised in this language: for $M \equiv M'$.

Exercise 2 Show that "all elements are n-torsion" is axiomatisable for any given n.

The compactness/completeness theorem is relevant also to the notion of type introduced earlier. Let T be complete and suppose that $A \subseteq M \models T$. A (complete) **1-type** over A (T is understood) is a set, $p(v)$, of formulas in the expanded language L^A with at most the variable v free, which is consistent (i.e., finitely satisfied in M) and which is a maximal such consistent set. In terms of definable subsets, a 1-type over A is just an ultrafilter in the boolean algebra of subsets (of M) definable over A. In order to specify the type $p(v)$, it is of course enough to give a subset of $p(v)$ whose deductive closure, modulo the axioms of T, is $p(v)$ (i.e., give a filter base).

One may see that the set, $\Phi(v)$, in the example above does not generate a complete type in this way: given $n \geqslant 2$ it is consistent to add to $\Phi(v)$ a formula saying that v is divisible by n, but it is also consistent to add the negation of this formula (exercise!): thus the deductive closure of Φ is not maximal consistent (it is a filter but not an ultrafilter).

In the same way one may define n-**types** (types in n free variables) and even I-**types** for any index set I (such as an ordinal).

The notation $S_n{}^T(A)$ is used for the set of n-types over A (the "T" is dropped when convenient) and $S_I{}^T(A)$ is used for the set of I-types over A. Actually, the set $S_n{}^T(A)$ comes equipped with a natural topology which has, as a basis of clopen sets, the $\mathcal{O}_\varphi = \{ p \in S_n{}^T(A) : \varphi \in p \}$ for φ a formula. (Thus, types may be thought of as neighbourhood systems, of actual or potential points, in the topology which has the definable sets as its clopen sets.) This space may be seen (exercise) to be a totally disconnected Hausdorff space which, by the completeness/compactness theorem, is compact. At most points, I will consider $S_n{}^T(A)$ simply as notation for a set but, for some purposes, the topology is used explicitly. The notation $S^T(A)$ is used for $S_n{}^T(A)$ when n is fixed but unspecified. A notation for all the types in finitely many free variables is $D^T(A) = \bigcup \{ S_n{}^T(A) : n \in \omega,\ n \geqslant 1\}$.

If $A \subseteq B (\subseteq M \models T)$ then one may say more (define a finer topology) using elements of B as parameters than using just those of A, and there is indeed the obvious restriction map $q \mapsto q \restriction A$, which is a continuous surjection from $S(B)$ to $S(A)$ (here $q \restriction A$ is just $q \cap L^A$).

If $q \in S(B)$ is such that $q \upharpoonright A = p$, then we say that q is an extension of p to B. Every type over A has at least one extension to a type over B.

The import of the completeness theorem here is this. The two ways in which types have been introduced are equivalent: a set of formulas in L^A is a type (modulo T) iff it is the type of a tuple in some model (of T, containing A). There is another way of seeing types: as orbits, under the automorphism group, of a saturated model (see below).

A module M is **weakly saturated** if, for every $n \in \omega$, every type of $S_n{}^T(0)$ ($= S_n{}^T(\emptyset)$) is realised in M (here, T is the complete theory of M). Given an infinite cardinal κ, the module M is said to be **κ-saturated** if for every $A \subseteq M$ with $|A| < \kappa$ the model M is A-saturated: that is, every type in $S_1{}^T(A)$ is realised in M. Saturation is a kind of compactness property of models. It is a fact that, given $A \subseteq M \models T$, there is an elementary extension, M', of M which is A-saturated (hence, exercise, there is one which is κ-saturated): it may also be supposed that M' is **A-homogeneous**, in the sense that any two n-tuples have the same type over A iff they lie in the same orbit of the group, $\mathrm{Aut}_A M'$, of automorphisms of M' which fix A pointwise. Beware that there are a number of more or less strong notions of "homogeneity": that above is what I will mean by the term. More generally, say that M is **κ-homogeneous** if, whenever $\bar{a}$ and $\bar{b}$ are (matching) sequences of elements from M, of length strictly less than κ and with the same type, then there is $f \in \mathrm{Aut}\, M$ ($= \mathrm{Aut}_\emptyset M$) with $f\bar{a} = \bar{b}$. If M is any module and κ any cardinal, then there is a κ-saturated, κ-homogeneous elementary extension of M. By the unadorned term homogeneous, I mean $\aleph_0$-homogeneous ($= \emptyset$-homogeneous).

One has that $\aleph_0$-saturated implies weakly saturated but (exercise) not conversely. Also, if $\alpha < \kappa$, then a κ-saturated module realises all α-types over sets of cardinality strictly less than κ.

A model is **prime** if it elementarily embeds in every model: a model is **atomic** if every n-type realised in it is isolated (in the space $S_n(0)$).

Finally, the reader should be introduced to the monster model. This is a model so saturated that all "small" situations may be found within it. To be a little more specific, one chooses a cardinal κ which is very large (and has very large cofinality) - say an inaccessible if such exists, or simply a cardinal much larger than any we may care about. Let the **monster model** $\widetilde{M}$ be any κ-saturated model of the complete theory we are working with. All work is then assumed to take place inside $\widetilde{M}$. Thus, in saying that M is a model of T, one implies that M is embedded as an elementary submodel of $\widetilde{M}$. There may also be non-elementary embeddings of M in $\widetilde{M}$, but then the image of such an embedding would not be counted as a model - it would only be algebraically isomorphic to a model. Similarly, if A is a set (of parameters) then A is supposed to be a subset of $\widetilde{M}$ and, in particular, A comes equipped with its type in $\widetilde{M}$. The monster model perhaps takes a little getting used to, but it is very convenient: an alternative is a more category-theoretic approach (see §5.4).

Since $\widetilde{M}$ is extremely saturated, equivalence of "situations" is equivalence under the action of the group of automorphisms of $\widetilde{M}$. Thus two tuples have the same type over A iff there is an automorphism of $\widetilde{M}$, fixing A pointwise, taking one to the other. Also, if f is an automorphism of $\widetilde{M}$ and $A \subseteq \widetilde{M}$ then there is induced a homeomorphism $S(A) \longrightarrow S(fA)$ given by $p \longmapsto fp \equiv \{\varphi(\bar{v}, f\bar{a}) : \varphi(\bar{v}, \bar{a}) \in p(\bar{v})\}$. Clearly $\bar{c}$ realises p iff $f\bar{c}$ realises fp.

I finish this section by stating, for ease of reference, some of the basic theorems (look for them in the texts mentioned at the beginning of the section). They are stated for the particular case of modules. By the cardinality of T is meant its cardinality as a set of formulas: so if $|R| = \kappa$ then $|T| = \max(\kappa, \aleph_0)$.

<u>Theorem 1.1</u> (Completeness theorem) *If T is a theory and if σ is a sentence, then σ is true in every model of T iff T proves σ.* □

<u>Theorem 1.2</u> (Compactness theorem) *If Φ is a set of sentences, then Φ has a model iff every finite subset of Φ has a model. In particular, if every finite*

subset of the set of formulas $\Phi(\bar{v})$ is satisfied in the module M, then there is an elementary extension of M and there is $\bar{a}$ in that extension such that $\Phi(\bar{a})$ holds. □

Theorem 1.3 (Upwards and Downwards Löwenheim-Skolem Theorems) *(a) If $\kappa \geq |T|$ is a cardinal and if M is a module of cardinality no more than κ then M has an elementary extension of cardinality κ.*
(b) If $|T| \leq \kappa$, if $|M| \geq \kappa$ and if A is a subset of M of cardinality no more than κ, then there is an elementary submodule of M which contains A and is of cardinality κ. □

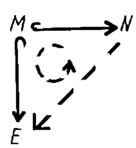
Theorem 1.4 (Joint embedding property for complete theories) *If M and M' are models of the complete theory T, then there is a model of T which contains both M and M' as elementary substuctures.* □

Theorem 1.5 (Saturated extensions) *If M is any module then M has a κ-saturated, κ-homogeneous elementary extension.* □

Theorem 1.6 (Prime models) *Suppose that T is a countable complete theory. Then the space of types, $S_n{}^T(0)$, either is countable or has cardinality $2^{\aleph_0}$. A model M of T is the prime model of T iff every type realised in M is isolated. The theory T has a prime model iff every formula extends to an isolated type (that is, iff the isolated points are dense in $S_n{}^T(0)$ for each $n \in \omega$). In particular, if T is countable with only countably many n-types for each $n \in \omega$, then T has a prime model.* □

1.2 Injective modules and decomposition theorems

On almost any page of this book, the reader will be able to find the term "pure-injective module" (=algebraically compact module). These modules play a very significant role in the model theory of modules: in some sense they are "typical" - see 2.27. They are generalisations of injective modules and their properties reflect this. Indeed, in appropriate categories they become precisely the injective objects.



A module E is **injective** if any diagram as given may be completed as shown (solid arrows show given morphisms; circling arrows denote commutative polygons). This is easily seen to be equivalent to the condition that E is a direct summand in every containing module (exercise - use pushouts or the homomorphism extension property).

A module M over the ring R is said to be **divisible** if, whenever c is a **right non-zero-divisor**, or **right regular**, element of R (i.e., for all $r \in R$, $cr=0$ implies $r=0$), the module M satisfies the sentence $\forall v \exists w \, (v = wc)$. That is (exercise), M is injective over embeddings of the sort $cR \hookrightarrow R$ with c a right regular element of the ring. Thus divisibility is an **elementary property** (one which may be axiomatised in our formal language) and also injectivity implies divisibility. For commutative domains, the converse - that divisibility is enough for injectivity - holds iff R is a Dedekind domain. Therefore, for such rings, injectivity is an elementary property. One of the results of [ES71] is that injectivity is an elementary property iff R is **right noetherian** (i.e., has the ascending chain condition - acc - on right ideals).

Example 1 Take $R=\mathbb{Z}$. The (direct-sum-) indecomposable injective modules are the module $\mathbb{Q}$ of rationals and, for each prime p, the Prüfer group $\mathbb{Z}_{p^\infty}$. The latter may be described as the limit (union) of the sequence of natural embeddings $\mathbb{Z}_p \hookrightarrow \mathbb{Z}_{p^2} \hookrightarrow \mathbb{Z}_{p^3} \hookrightarrow \ldots \hookrightarrow \mathbb{Z}_{p^n} \hookrightarrow \ldots$ ($n\in\omega$, $n\geqslant 1$). Note that $\mathbb{Q}$ is torsion-free and $\mathbb{Z}_{p^\infty}$ is a p-group with elements of unbounded finite order.

It has long been known (see [Kap54; p74]) that every injective abelian group is a direct sum of copies of these groups.

Example 2 Recall that R is **semisimple artinian** iff it is a finite product of full matrix rings (of various sizes) over (various) division rings D_i: say $R=\times_{i=1}^{n} M_{n_i}(D_i)$. Over such a ring, every module is injective and, conversely, this property of the modules characterises the semisimple artinian rings.

It is well-known (see any treatment of these rings) that every module over such a ring is a direct sum of indecomposable (injective) modules and there are, up to isomorphism, exactly n (as above) indecomposables: to each simple component ring $M_{n_i}(D_i)$ of R there corresponds the (simple) minimal right ideal (row) of this matrix ring.

An embedding $M \overset{f}{\hookrightarrow} M'$ is **essential** if, for all morphisms $M' \xrightarrow{g} M''$, if gf is monic then so is g. So, for example, if $M'=fM\oplus M_1$ for some $M_1\neq 0$ then, taking g to be the projection from M' to fM, one sees that f is not essential - in fact this is the antithesis of an essential embedding. An essential embedding may be thought of as one where the codomain sits "tightly" over the image of the morphism.

It is easy to check (exercise) that, with notation as above, f is essential iff, for every nonzero $m'\in M'$, there are $m\in M$ and $r\in R$ such that $m'r=fm\neq 0$ (this relation means that if m' is sent to zero by a morphism then so must be the element $m'r$ of M). That is, f is essential iff for all nonzero $m'\in M'$ one has $m'R\cap fM\neq 0$. For example, the (canonical, or indeed, any) embedding $\mathbb{Z}\hookrightarrow\mathbb{Q}$ is essential.

The module E is an **injective hull** (or **envelope**) of M if E is a minimal injective extension of M: that is, E is injective and, if $M\leqslant E'\leqslant E$ with E' injective, then $E'=E$. If the context requires one to be more accurate, one says that the embedding $M\hookrightarrow E$ is an injective hull.

<u>Theorem 1.7</u> [Ba40], [EcSc53] *Every module M has an injective hull, $M\hookrightarrow E(M)$. This module is unique up to M-isomorphism (that is, if E' also is an injective hull of M then there is an isomorphism between E' and $E(M)$ which fixes M pointwise). The inclusion $M\hookrightarrow E(M)$ is a maximal essential extension of M (that is, the inclusion $M\hookrightarrow E(M)$ is essential and, if $E(M)\longrightarrow M'$ is not an M-isomorphism, then the composition $M\longrightarrow M'$ is not essential).* ▫

Example 3

(i) Take $R=\mathbb{Z}$. Then: $E(\mathbb{Z}_{p^n})=\mathbb{Z}_{p^\infty}$; $E(\mathbb{Z}_6=\mathbb{Z}_2\oplus\mathbb{Z}_3)=\mathbb{Z}_{2^\infty}\oplus\mathbb{Z}_{3^\infty}$; $E(\mathbb{Z})=\mathbb{Q}=E(\mathbb{Z}_{(p)})$, where $\mathbb{Z}_{(p)}$ is the localisation of $\mathbb{Z}$ at the prime p - $\mathbb{Z}_{(p)}=\{m/n\in\mathbb{Q} : p\nmid n\}$.

(ii) Take $R=\mathbb{Z}_4$ to be the ring of integers modulo 4. Then: $E(\mathbb{Z}_4)=\mathbb{Z}_4$; $E(\mathbb{Z}_2)=\mathbb{Z}_4$.

It is in general the case that $E(M\oplus N)=E(M)\oplus E(N)$, where the equality sign is interpreted in the natural way (namely as "can be chosen to be"). The hulls introduced later will not commute with direct sums, unless these sums are understood in the right category. This is because they are defined with respect to a certain surrounding context. What will be seen later is that which remains after the special features of the injective ("quantifier-free") case have been subtracted.

The next result says that two injectives, each embeddable in the other, are isomorphic.

<u>Theorem 1.8</u> [Bu65] *Suppose that* $\varphi: M \hookrightarrow N$ *and* $\psi: N \hookrightarrow M$ *are embeddings between injective modules. Then* $M \simeq N$.

Proof We have $N = X \oplus \varphi(M) = X \oplus \varphi(\psi(N) \oplus M')$ (say) $= X \oplus (\varphi\psi)(X) \oplus (\varphi\psi)\varphi(M) \oplus \varphi(M') = X \oplus (\varphi\psi)(X) \oplus (\varphi\psi)^2(X) \oplus \ldots \oplus (\varphi\psi)^n(X) \oplus \ldots$. Set Y to be the submodule $(\varphi\psi)(X) \oplus (\varphi\psi)^2(X) \oplus \ldots \oplus (\varphi\psi)^n(X) \oplus \ldots$ of $\varphi(M)$. We may write $\varphi(M)$ as $E(Y) \oplus Z$ say. Note that $E(Y) \simeq X \oplus E(Y)$ (take the first factor out from the injective hull). Then $N = X \oplus \varphi(M) = X \oplus E(Y) \oplus Z \simeq E(Y) \oplus Z = \varphi(M) \simeq M$, as required. □

In these notes we will be concerned not just with types, but with parts of types consisting of certain kinds of formulas. In the injective case, the formulas of interest are the $\bigwedge$-atomic ones. These are the conjunctions of atomic formulas. In the context of R-modules, a formula $\theta(\bar{v})$ is $\bigwedge$-atomic iff it has (or is equivalent to one of) the form $\bigwedge_{i=1}^{n} t_i(\bar{v}) = 0$ where $t_i(\bar{v}) = 0$ - a typical atomic formula - is a linear equation (possibly involving parameters) with unknowns from $\bar{v}$. In other words, a $\bigwedge$-atomic formula is a matrix equation of the form $\bar{x}H = 0$, where H is a rectangular matrix with entries in R and $\bar{x}$ is a row of free variables and parameters (and 0 is a zero tuple).

A $\bigwedge$-atomic type is a consistent set of $\bigwedge$-atomic formulas - namely one with every finite subset having a solution (to require maximal consistent as in the definition of type would be inappropriate). If $\bar{a}$ is in M and $B \subseteq M$ then the $\bigwedge$-atomic type of $\bar{a}$ over B is $\mathrm{tp}_0(\bar{a}/B) = \{\theta(\bar{v}) : \theta(\bar{v})$ is a $\bigwedge$-atomic formula and $M \models \theta(\bar{a})\}$ (we need not mention M, since truth of quantifier-free formulas does not depend on context).

Essential embeddings were defined above, both in terms of morphisms and in terms of elements. Now we see how they are defined in terms of $\bigwedge$-atomic types.

<u>Proposition 1.9</u> *The embedding* $i: M \hookrightarrow M'$ *is essential iff, for all* $a \in M'$, *the* $\bigwedge$*-atomic type,* $\mathrm{tp}_0(a/M)$, *of* a *over* M *is maximal with respect to inclusion in the set,* $\{\mathrm{tp}_0(a_1/M) : a_1 \in M_1,\ M_1$ *any extension of* $M\}$, *of all possible* $\bigwedge$*-atomic types over* M.
In particular, every element of $E(M)$ *has maximal* $\bigwedge$*-atomic type over* M.

Proof $\Rightarrow$ It is immediate from the element-wise reformulation of the definition of "essential" that, if $a \in M'$, then the embedding $j: M \hookrightarrow M + aR$ is essential. Suppose that $a_1 \in M_1 \geqslant M$ is such that $\mathrm{tp}_0(a_1/M) \supseteq \mathrm{tp}_0(a/M)$.

Define $M + aR \xrightarrow{f} M + a_1R$ by: $f \upharpoonright M = \mathrm{id}_M$; $fa = a_1$. This is well-defined since, if $m + ar = 0$ (where $m \in M$, $r \in R$) then "$m + vr = 0$" $\in \mathrm{tp}_0(a/M)$ and so, by assumption, "$m + vr = 0$" $\in \mathrm{tp}_0(a_1/M)$ - that is $m + a_1r = 0$, as required.

Now fj is monic so, since j is essential, f must be monic.

Let "$vr + m = 0$" $\in \mathrm{tp}_0(a_1/M)$: that is, suppose $a_1r + m = 0$. Then $f(ar + m) = 0$ so, since f is monic, $ar + m = 0$. In other terms "$vr + m = 0$" $\in \mathrm{tp}_0(a/M)$. Hence $\mathrm{tp}_0(a/M) = \mathrm{tp}_0(a_1/M)$ and maximality of $\mathrm{tp}_0(a/M)$ is established.

$\Leftarrow$ Suppose that one has $i: M \hookrightarrow M'$ and $M' \xrightarrow{f} M''$ with fi monic. It must be shown that f is monic.

Let $a \in M'$. If "$vr = m$" $\in \mathrm{tp}_0(a/M)$ then, from $ar = m$ (really, "$= i.m$"), one deduces $fa.r = m$. Thus $\mathrm{tp}_0(fa/M) \supseteq \mathrm{tp}_0(a/M)$ (here I am identifying M, iM, and fiM - such identifications are convenient and commonly will be made: indeed, this one is implicit even in the statement of the result). By maximality of $\mathrm{tp}_0(a/M)$ it follows that $\mathrm{tp}_0(fa/M) = \mathrm{tp}_0(a/M)$.

In particular, if $fa = 0$ then "$v = 0$" is in this $\bigwedge$-atomic type, and so $a = 0$. Thus f is indeed monic, so i is essential, as required. □

Exercise 1 Show that the statement of **1.9** holds also for tuples in place of single elements. (It is quite typical that the properties of 1-tuples are not essentially different from those of longer tuples.)

Lemma 1.10 *The class of injective modules is closed under direct products and direct summands but is not, in general, closed under direct sums.*

Proof The first statement is an easy generality. For an example establishing the second statement, see Ex.16.2/2 for instance. □

Theorem 1.11 [Mat58; 2.5] *The following conditions on the ring R are equivalent:*
(*i*) *R is right noetherian;*
(*ii*) *every direct sum of injective R-modules is injective;*
(*iii*) *every injective module has an essentially unique decomposition as a direct sum of indecomposable injective submodules.* □

This theorem links a finiteness condition on the lattice of right ideals with a decomposition property in a certain class of modules. Theorems of this sort are a major theme in these notes.

The rather peculiar phrase "essentially unique" occurs in the theorem. The point is that if one has some decomposition $M = \bigoplus_I M_i$, where the M_i are indecomposable, then it is almost always the case that there are other ways of decomposing M as a direct sum of indecomposables: $M = \bigoplus_J N_j$ say (for example, in the 2-plane there are many choices for pairs of coordinate axes). "Essential uniqueness" means that one may infer the existence of a bijection $\pi : I \longrightarrow J$ such that $M_i \simeq N_{\pi i}$ for each $i \in I$.

The kind of decomposition appearing in **1.11** is rather strong. Since a direct sum of injectives is not in general injective, one might reasonably ask when an injective module may be expressed as the *injective hull* of a direct sum of indecomposable injectives. It is worth noting that, in such a decomposition, any finitely many factors may be separated out as direct summands, so the adding of "injective hull of" is perhaps not a great leap in complication. In [Pop73] these are called KRSG-decompositions ("Krull-Remak-Schmidt-Gabriel) and one has the following theorem generalising **1.11** (although, by working in appropriate categories, one may obtain **1.12** as a corollary of the general version of **1.11**).

Theorem 1.12 [Mat58; 2.4 Remark(2), 2.7], [War69a; Thm 3], [Gab62] *The following conditions on the ring R are equivalent:*
(*i*) *every right ideal, I, of R may be written as the intersection of two right ideals, the first of which is $\cap$-irreducible, the second of which strictly contains I;*
(*ii*) *every injective R-module has an indecomposable direct summand;*
(*iii*) *every injective R-module E has the form $E(\bigoplus E_i)$, where the E_i are indecomposable (injective) direct summands of E. This decomposition is essentially unique.*

These conditions are satisfied if every R-module has Gabriel dimension so, in particular, if R has Krull dimension. □

For the above result, see [Pop73; §§5.3, 5.5]; also, for Krull and Gabriel dimension, see [GR73], [GR74]. Krull dimension is a finiteness condition on the lattice of right ideals of the ring (see §10.5). It will also be seen (§4.3) that every injective module has the form $E(\bigoplus_i E_i) \oplus E_C$ where the E_i are indecomposable injectives and E_C has no (nonzero) indecomposable direct summands. (The zero module will be counted as indecomposable or not (usually not), depending on which convention simplifies any particular statement.)

The reader is likely aware of decomposition theorems similar to Matlis'. There are generalisations to Σ-injective modules (see Ex 3.2/2). There are similar results for: modules over rings of finite representation type (Chpt.11); finitely generated modules over right artinian rings (Chpt.11); projective modules over right perfect, left coherent rings (Chpt.14). Another example is abelian groups of bounded exponent: any abelian group of exponent n may be written as $\bigoplus\{\mathbb{Z}_{p^k}{}^{(\kappa_{p,k})} : p^k | n\}$ for suitable cardinals $\kappa_{p,k}$.

All of these, bar the finitely generated modules over right artinian rings (which require a different sort of treatment), will be derived as special cases of Garavaglia's decomposition theorem **3.14** for totally transcendental modules.

That theorem, generalisations of it, and detailed analyses of the indecomposable factors which occur, constitute one main strand which runs through these notes. It is a strand which weaves together model theory and algebra. A major motivation for such decomposition theorems is that they should be stepping stones towards a good understanding of the structures under consideration.

Before finishing this section, I will say just a little about indecomposable injectives themselves, for one should have some knowledge of the building blocks, as well as of the ways in which they may be put together.

Suppose that E is an indecomposable injective and let a be a non-zero element of E. The $\bigwedge$-atomic type of a is determined by its **annihilator** $\text{ann}(a)=\{r\in R : ar=0\}$ - a right ideal of R. Since E is indecomposable it is not difficult to see, using $E(M\oplus N)=E(M)\oplus E(N)$, that $\text{ann}(a)$ is **$\cap$-irreducible** in the lattice of right ideals: that is, if $\text{ann}(a)=J\cap K$, where J, K are right ideals, then either $\text{ann}(a)=J$ or $\text{ann}(a)=K$. Conversely, if I is a $\cap$-irreducible right ideal, then the injective hull of $1+I$ (in, say, R/I) - an element whose annihilator is exactly I - is indecomposable. Furthermore, with $a\in E$ as above, if b is any non-zero element of E then, by **1.9**, the elements b and a are very closely linked. These points will be generalised and expanded upon below.

CHAPTER 2 POSITIVE PRIMITIVE FORMULAS AND THE SETS THEY DEFINE

If K is an algebraically closed field then the sets of n-tuples which may be defined by positive quantifier-free formulas are precisely the sub-varieties of affine n-space. The Chevalley-Tarski theorem says that every definable subset of affine n-space is a boolean combination of such sub-varieties (is "constructible"). The point is that the existential quantifiers introduced by projection may be eliminated: one says that algebraically closed fields have (complete) elimination of quantifiers.

For comparison one may consider the theory of groups. Here the definition of a subset may require arbitrarily large numbers of alternations of quantifiers, and there seems to be no hope of understanding the shape of a general definable set.

Modules are definitely closer to algebraically closed fields than to groups in this regard. For modules have a relative elimination of quantifiers: it turns out that every definable subset of a module is a boolean combination of "pp-definable" cosets. A pp-definable coset is simply the projection of the solution set to a (not necessarily homogeneous) system of R-linear equations. Therefore, such a coset is definable by a formula with only existential quantifiers prefixing a conjunction of atomic formulas (a "positive primitive" formula): we say that modules have pp-elimination of quantifiers. It is this fact which brings the model-theoretic and algebraic aspects of modules close together.

This description of the definable subsets is the key to the model-theoretic analysis of modules.

The reader should know that the pace of this chapter is rather leisurely so as to accommodate a wide variation in readers' backgrounds. A number of examples are introduced and many of these are developed further in the text. There are also quite a few exercises: many are straightforward and are designed to give the reader a chance to test his or her understanding of what may be unfamiliar material.

The first section is devoted to the sets defined by positive primitive formulas. Since they are projections of solution sets of systems of linear equations, these sets are subgroups and cosets of such subgroups. It is unusual for every subgroup of a module to be (so) definable, but the set of pp-definable subgroups of any given module does form a sublattice of the lattice of all subgroups of the module. A recurring theme in these notes is the effect of various finiteness conditions which may be imposed upon the lattice of pp-definable subgroups. There are a number of examples at the end of the section.

The second section concentrates on pp-types. These are descriptions of where (actual or potential) elements lie, in the sense that they specify precisely those pp-definable cosets to which such an element belongs. Associated to each pp-type is a certain (possibly infinitely) definable subgroup which is a measure of the amount of information in the pp-type (the extent to which it ties down an element). Although pp-types are simply the pp-parts of complete types we will see later that, associated to them, are certain concepts and constructions which do not depend on any over-theory. (For instance, it turns out (§2.6) that complete theories of modules are built up from a common pool of "components". Moreover, the "building blocks" for (sufficiently saturated) models of the various theories are copies of the "hulls" of pp-types (§4.1).)

The notion of pure embedding (the pp-analogue of elementary embedding) and the polynonymous pure-injective modules provide the subject matter for the third section. In these notes the typical morphism between modules is a pure embedding and, although the pure-injective modules are not (except in one sense) typical modules, they are the ones whose structure is the most clear-cut. We see the first example of an equivalence between a finiteness condition on the lattice of pp-definable subgroups and an algebraic structural condition (2.11).

The key result - pp-elimination of quantifiers - which allows us to begin to understand the definable subsets of modules, is established in the fourth section. The section opens with Neumann's Lemma (2.12) - a result used many times to prove consistency of a type. Then it is shown that: any first-order statement about a module is equivalent to declarations about values of indices of various pp-definable subgroups, one in another; within any given module, every definable subset is a finite boolean combination of pp-definable cosets. It follows that every type is determined by its pp-part (2.20).

A crop of rather immediate consequences is reaped in the fifth section. These include a number of results which were obtained, but with considerably more labour, before the tools of the fourth section were available. In particular it is seen that: the direct sum and direct product of any family of modules are elementarily equivalent (2.24); two modules are elementarily equivalent iff each embeds purely in a module elementarily equivalent to the other (Exercise 2.5/3); any pure embedding between elementarily equivalent modules is an elementary one (2.26).

Modules ("representations") have their origin in the idea of investigating a group, ring or other structure by studying its action on various vector spaces or abelian groups. Thus it is often the category of modules, rather than individual modules in isolation, which is of interest. It is fortunate, therefore, that we may readily compare and relate modules which are not necessarily elementarily equivalent. In fact there is a natural ordering (§6) on complete theories of modules, under which there is even a "largest" such theory (2.32), with all complete theories of modules being "components" of this one.

Frequently it is a technical advantage to assume that we are dealing with a class of modules which is closed under product. The fact that this often entails no real loss in generality is another outcome of the considerations in this section and of §4.4.

Our most useful source of examples is the theory of abelian groups. The concepts and results that we need are given in a supplementary section. One purpose of that section is to allow the reader to verify, without difficulty, any unsupported statements which I make in the course of using abelian groups as examples. The section also includes: the particularly simple form that pp formulas take over a principal ideal domain (2.$\mathbb{Z}$1); the complete classification of indecomposable pure-injectives over discrete rank 1 valuation domains (2.$\mathbb{Z}$3) and, indeed, over any ring all of whose localisations at maximal primes are fields or discrete rank 1 valuation domains (2.$\mathbb{Z}$10); the relevant local/global principle (2.$\mathbb{Z}$8, 2.$\mathbb{Z}$9). There is also the result that every module over a commutative ring is an elementary substructure of the product of its localisations at maximal primes (2.$\mathbb{Z}$5).

Finally, there is a supplementary section in which I indicate what has been done on modules in languages other than the finitary first-order one which is used throughout these notes.

2.1 pp formulas

The formula φ $(=\varphi(\bar{v})=\varphi(v_1,\ldots,v_n))$ is pp (short for **positive primitive**) if it is equivalent to one of the form $\exists w_1,\ldots,w_l \bigwedge_{j=1}^{m}(\sum_{i=1}^{n} v_i r_{ij} + \sum_{k=1}^{l} w_k s_{kj} = 0)$ with $r_{ij}, s_{kj} \in R$. This formula may be re-written using matrix notation as:

$$\exists w_1,\ldots,w_l \ (v_1 \ldots v_n \ w_1 \ldots w_l)\begin{pmatrix} r_{11} & \cdots & r_{1m} \\ \vdots & & \vdots \\ r_{n1} & \cdots & r_{nm} \\ s_{11} & \cdots & s_{1m} \\ \vdots & \cdots & \vdots \\ s_{l1} & \cdots & s_{lm} \end{pmatrix} = 0$$

Here, as elsewhere, "0" denotes a zero matrix of appropriate size. In a more compact notation, this becomes $\exists \bar{w}\ (\bar{v}\,\bar{w})H=0$ where the matrix H has a natural block decomposition $\begin{pmatrix} R \\ S \end{pmatrix}$ say. So yet another way of expressing this pp formula is "$\exists \bar{w}\ (\bar{v}R=-\bar{w}S)$" where R, S are the above matrices.

If we are allowing parameters from a set A in our formulas then a pp formula is as above, except that the zeroes on the right hand side of the equation(s) may be replaced by elements of A

(actually, by linear combinations of elements of A, but there is no harm in assuming, if convenient, that our sets of parameters are modules).

Algebraically, a pp formula expresses solvability of a system of linear equations. Put in terms of definable sets, a pp formula defines an image under projection of the solution set for a system of R-linear equations (the system is homogeneous if there are no extra parameters).

In most parts of these notes I will use pp formulas rather than the corresponding matrices. Those who are not very comfortable with this approach may easily translate everything said into terms of matrices and solution sets. In fact such (occasional) translation is a useful exercise since it often reveals what "purely algebraic" situation is being generalised. For instance, the representation $\exists \bar{w}\,(\bar{v}R=-\bar{w}S)$ suggests that one may think of pp formulas as generalised divisibility statements.

I will now give some examples of pp formulas and, whilst doing so, I will take the opportunity to establish some terminology.

Example 1

(i) Annihilator conditions: if $r\in R$ then the formula $vr=0$ is satisfied by an element a of a module M iff r **annihilates** a (i.e., $ar=0$).

(ii) Divisibility conditions: if $r\in R$ then the formula $\varphi(v)\equiv \exists w\,(v=wr)$ (which in matrix notation is $\exists w\,(v\ w).\begin{pmatrix}1\\-r\end{pmatrix}=0$ is satisfied by $a\in M$ - $M\models\varphi(a)$ - iff a is divisible by r in M - that is, iff there exists $b\in M$ with $a=br$.

In this second example it matters very much what module M is being considered. For instance, the element 2 is divisible by the ring element 2 in the abelian group $\mathbb{Z}$ of integers, but this element is not divisible by 2 in the group $2\mathbb{Z}$ of even integers. It is this kind of dependence on context which makes consideration of conditions expressed by pp formulas more subtle than the "purely algebraic" (=quantifier-free) conditions exemplified by (i).

(iii) Tuples of elements tend to display behaviour no more complicated than that of single elements (for some explanation of this see §10.T). So, generalising (i), we may let $r_1,\dots,r_n$ be elements of the ring and consider the pp formula $\varphi(v_1,\dots,v_n)\equiv \sum_{i=1}^{n} v_i r_i=0$, which expresses a generalised annihilator condition. In another notation, if $a=(a_1,\dots,a_n)$ is an n-tuple from or in M (that is, whose entries lie in M) then $M\models\varphi(\bar{a})$ iff $\bar{a}.\bar{r}^{\mathsf{T}}=0$ where $\bar{r}=(r_1\dots r_n)$ and "$^{\mathsf{T}}$" denotes transpose.

The use of the word "equivalent" in the first sentence of this section is somewhat ambiguous. It could mean equivalent modulo the particular complete theory under consideration, or it could mean equivalent in all modules (i.e., modulo that theory common to all R-modules). In fact the term will normally, but not always, be intended in the former sense: whenever there is some danger of ambiguity I will be more explicit.

Our first result establishes a simple but fundamental property of pp formulas.

Lemma 2.1 (Linearity of pp formulas) *Suppose that* $\varphi(\bar{v})$ *is a pp formula. Then:*

(i) $\varphi(\bar{0})$ *holds;*

(ii) $\varphi(\bar{a})$ *and* $\varphi(\bar{b})$ *implies* $\varphi(\bar{a}-\bar{b})$;

(iii) if $r\in C(R)$ *- the centre of* R *- then* $\varphi(\bar{a})\to\varphi(\bar{a}r)$, *where* $(a_1,\dots,a_n)r=(a_1r,\dots,a_nr)$ *(it is enough that* r *commute with all the elements of* R *appearing in* φ).

Proof This is an easy computation which should be performed by the reader (in his or her choice of notation). □

If we think in terms of definable sets, then the first two parts of 2.1 are obvious, since they simply state that for any module M the set defined by φ is a group - and this is clear since $\varphi(M)$ is a projection of the solution set of a homogeneous system of linear equations (and that is a group). Having observed 2.1, we turn to the subsets defined by pp formulas.

Corollary 2.2 *Let* $\varphi=\varphi(\bar{v})$, $\psi=\psi(\bar{v})$ *be pp formulas, let* $l=l(\bar{v})$ *be the length of* $\bar{v}$ *and let* M *be any module.*

(i) $\varphi(M)=\{\bar{a}\in M^l : M\models\varphi(\bar{a})\}$ *is a subgroup of* M^l *(in fact it is an* $(\mathrm{End}M, C(R))$*-sub-bimodule of* M^l *via the diagonal action).*
$\varphi(M)$ *is termed a* pp-definable subgroup *of* M *or, more accurately, a subgroup of* M^l *pp-definable in* M.
(In [Zim77] *these are termed "endlich matriziell" and in* [GJ73] *they are called "subgroups of finite definition".)*

(ii) *Suppose that we substitute specific values,* $\bar{a}$, *for the last* $l-k$ *variables in* $\bar{v}$. *Then the set* $\varphi(M,\bar{a})=\{\bar{c}\in M^k : M\models\varphi(\bar{c},\bar{a})\}$ *which is defined by the resulting pp formula (with parameters) is either empty or is a coset of the subgroup* $\varphi(M,\bar{0})$ *of* M^k: *a* pp-definable coset.
If $M'\succ M$ *is an elementary extension, then every coset of* $\varphi(M',\bar{0})$ *which intersects* M^k *non-trivially has the form* $\bar{c}+\varphi(M',\bar{0})$ *for some* $\bar{c}\in M^k$, *and the intersection of this coset with* M^k *is just* $\bar{c}+\varphi(M,\bar{0})$: *in particular, the coset is pp-definable over (i.e., using parameters from)* M.
Therefore if a pp formula defines a subgroup then that subgroup may be defined by a pp formula without parameters.

(iii) *The poset of subgroups of* M^l *pp-definable in* M *is a sublattice of the lattice of all subgroups of* M^l: *in particular the former lattice is modular.*
The intersection and sum are given by:
$\varphi(M)\cap\psi(M)=(\varphi\wedge\psi)(M)$
$\varphi(M)+\psi(M)=(\varphi+\psi)(M)$ *where we define* $(\varphi+\psi)(\bar{v})$ *to be the pp formula* $\exists\bar{u}\,\exists\bar{w}\,(\varphi(\bar{u})\wedge\psi(\bar{w})\wedge\bar{v}=\bar{u}+\bar{w})$.

(iv) $\varphi(M,\bar{a})\cap\psi(M,\bar{b})$, *if non-empty, is a coset of* $(\varphi\wedge\psi)(M,\bar{0})$ *(where* $l(\bar{a})=l(\bar{b})=l(\bar{0})$*).*
$\varphi(M,\bar{a})+\psi(M,\bar{b})$ *is a coset of* $(\varphi+\psi)(M,\bar{0})$, *provided both* $\varphi(M,\bar{a})$ *and* $\psi(M,\bar{b})$ *are non-empty.*

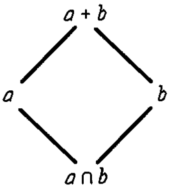
Proof (i) In view of 2.1, the only point to check is that pp-definable subgroups are stable under endomorphisms of the module.

Therefore let $f\in\mathrm{End}(M)$ and suppose that $\bar{a}\in\varphi(M)$. Take $\varphi(\bar{v})$ to have the form $\exists w_1,\dots,w_l \bigwedge_{j=1}^{m}(\sum_{i=1}^{n} v_i r_{ij} + \sum_{k=1}^{l} w_k s_{kj}=0)$. From $M\models\varphi(\bar{a})$ one deduces the existence of $\bar{b}$ in M with $\bigwedge_{j=1}^{m}(\sum_{i=1}^{n} a_i r_{ij} + \sum_{k=1}^{l} b_k s_{kj}=0)$. Applying f to this conjunction of equations yields $\bigwedge_{j=1}^{m}(\sum_{i=1}^{n} fa_i.r_{ij} + \sum_{k=1}^{l} fb_k.s_{kj}=0)$. Hence $f\bar{a}\in\varphi(M)$ as required.

(ii) Supposing $\varphi(M,\bar{a})$ to be non-empty, take two elements $\bar{c},\bar{c}'$ in it. From $\varphi(\bar{c},\bar{a})$ and $\varphi(\bar{c}',\bar{a})$, 2.1 yields $\varphi(\bar{c}-\bar{c}',\bar{0})$. Conversely if $\bar{c}_0\in\varphi(M,\bar{0})$ then from $\varphi(\bar{c},\bar{a})\wedge\varphi(\bar{c}_0,\bar{0})$, 2.1 yields $\varphi(\bar{c}+\bar{c}_0,\bar{a})$. Thus $\varphi(M,\bar{a})$ is indeed a coset of $\varphi(M,\bar{0})$.

Let $\bar{c}\in\varphi(M',\bar{a})\cap M^k$. Then $\bar{b}\in\varphi(M',\bar{a})$ iff $\bar{c}-\bar{b}\in\varphi(M',\bar{0})$ iff $\bar{b}\in\bar{c}+\varphi(M',\bar{0})$, and note that this last coset is just $\varphi(M'-\bar{c},\bar{0})$ (with an obvious abuse of notation). Also, since M is an elementary substructure of M', certainly $M\cap\varphi(M'-\bar{c},\bar{0})=\varphi(M-\bar{c},\bar{0})$.

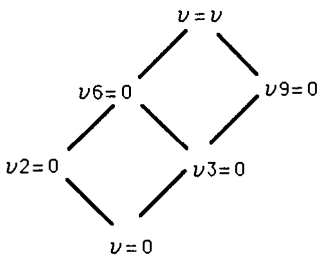
The last statement follows (for one may take $\bar{c}$ to be $\bar{0}$).



(iii) Surely it is clear that $\varphi\wedge\psi$, resp. $\varphi+\psi$, defines the intersection, respectively the sum, of $\varphi(M)$ and $\psi(M)$.
Recall that a lattice (with operations "$\wedge$" and "+") is modular if the identity $a\wedge(b+c)=(a\wedge b)+c$ holds whenever $a\geqslant c$. The point is that, for any a,b in the lattice, the intervals $[a+b,b]$ and $[a,a\wedge b]$ are isomorphic *via* the map $c(\in[a+b,a])\mapsto a\wedge c$.

(iv) The argument is, by (ii) and (iii), just elementary group theory: let me consider the second part only.

Take $\overline{m}_\varphi,\overline{m}_\psi$ such that $\varphi(M,\overline{a})=\overline{m}_\varphi+\varphi(M,\overline{0})$ and $\psi(M,\overline{b})=\overline{m}_\psi+\psi(M,\overline{0})$. Then it is trivial to verify that $\varphi(M,\overline{a})+\psi(M,\overline{b})=\overline{m}_\varphi+\overline{m}_\psi+(\varphi(M,\overline{0})+\psi(M,\overline{0}))$. ▫



Example 2 Let $R=\mathbb{Z}$ and take M to be $\mathbb{Z}_2\oplus\mathbb{Z}_9$. The lattice of pp-definable subgroups is as shown: beside each subgroup is a pp formula which defines it (the formula is not uniquely determined by the subgroup: for example the formulas $v6=0$ and $3|v$ define the same subgroup). Clearly all the groups shown are pp-definable: to see that there are no more, use the fact (**2.2**(i)) that a pp-definable subgroup is closed under endomorphisms.

The next example shows that a pp-definable subgroup need not be a submodule. The one after shows that if $\varphi(\overline{v},\overline{0})$ is pp then it need not be the case that every coset of φ is defined by $\varphi(\overline{v},\overline{a})$ for some suitable choice of parameters $\overline{a}$ (here I am using the device of identifying a formula with the set it defines).

Example 3 If the base ring is commutative then, by **2.2**(i) every pp-definable subgroup is a submodule. The simplest non-commutative rings are the simple artinian rings - full rings of matrices over division rings. For example take R to be the ring of 2×2 matrices over some field K and take M to be the ring, regarded as a right module over itself. Let e_{11} be the matrix with "1" in the (1,1)-position and zeroes elsewhere. Consider the formula $\varphi(v)\equiv\exists w\,(v=we_{11})$. Then $\varphi(R)=\begin{pmatrix}K & 0\\ K & 0\end{pmatrix}=\{\begin{pmatrix}a & 0\\ b & 0\end{pmatrix}: a,b\in K\}$ - this is a left, but not a right, ideal of R.

Example 4 To see that not every coset of $\varphi(M,0)$ need be expressible in the form $\varphi(M,b)$ for some b, take $R=\mathbb{Z}$, $T=\mathrm{Th}(\mathbb{Z}_2{}^{(\aleph_0)})$ and let $\varphi(v,y)$ be $v=y2$. Then $\varphi(M,0)=0$ for any $M\models T$ (for 0 is the only element divisible by 2).

Now, if $a\in M\models T$ is non-zero, $\{a\}=a+\varphi(M,0)$ is defined by the formula $v-a=0$ but by no formula of the form $\varphi(v,b)$ - for that would imply that $a=b2=0$.

We will be interested not just in the pp-definable subgroups, but also in arbitrary intersections of them: such a subgroup consists of the common solutions to a possibly infinite set of pp conditions and so is not necessarily definable in the sense that we have used the term. We say that a set defined by possibly infinitely many formulas is **$\bigwedge$-definable**. The $\bigwedge$-pp-definable subgroups are almost, but not quite, as well-behaved as the pp-definable ones (they are considered in [Zim77]).

Corollary 2.3 *Let $\Phi(\overline{v}),\Psi(\overline{v})$ be sets of pp-formulas; set $l=l(\overline{v})$ (we allow the possibility that l is infinite), and let M be any module.*

(i) *The set* $\Phi(M)=\{\bar{a}\in M^l : M\models\varphi(\bar{a})$ *for all* $\varphi\in\Phi\}$ *is a subgroup of* M^l *which is closed under the action of* End(M).

(ii) *If* $\Phi(\bar{v})=\Phi(\bar{v},\bar{a})$ *is a set of pp-formulas, possibly with parameters, then* $\Phi(M)$, *if non-empty, is a coset of* $\Phi(M,\bar{0})$.
If $M\prec M'$ *then every coset of* $\Phi(M',\bar{0})$ *which intersects "M" has the form* $\Phi(M'-\bar{c},\bar{0})$ *for some* $\bar{c}$ *in* M, *and the intersection of this coset with* M *is just* $\Phi(M-\bar{c},\bar{0})$.

(iii) $\Phi(M)\cap\Psi(M)=(\Phi\cup\Psi)(M)$;
$\Phi(M)+\Psi(M)\subseteq(\Phi+\Psi)(M)$, *with equality if* M *is pure-injective (see §3), where* $\Phi+\Psi=\{\varphi+\psi : \varphi\in\bigwedge\Phi,\ \psi\in\bigwedge\Psi\}$, *where* $\bigwedge\Phi$ *denotes the closure of* Φ *under (finite!) conjunction.*
Similar statements hold for cosets.

Proof (i) Since $\Phi(M)=\bigcap\{\varphi(M) : \varphi\in\Phi\}$ this part is clear from 2.2(i).

(ii) The proof is just as for 2.2(ii).

(iii) The truth of the first statement should be clear, as should be the inclusion in the second (e.g., $\Phi(M)+\bar{0}\subseteq(\Phi+\Psi)(M)$). Therefore, suppose that M is pure-injective, and let $\bar{a}\in(\Phi+\Psi)(M)$. Then the set of pp formulas $\Theta(\bar{v}_1,\bar{v}_2)=\Phi(\bar{v}_1)\cup\Psi(\bar{v}_2)\cup\{\bar{a}=\bar{v}_1+\bar{v}_2\}$ is finitely satisfied in M (see below). Since M is pure-injective we may appeal to 2.8 below, which provides us with a solution $\bar{b},\bar{c}$ in M (that is $M\models\Theta(\bar{b},\bar{c})$). Then we have $\bar{a}=\bar{b}+\bar{c}$ where $\bar{b}\in\Phi(M)$ and $\bar{c}\in\Psi(M)$, as required. ▫

Exercise 1 In the proof just above, I asserted that the set Θ of formulas is finitely satisfied in M. Similar situations will be encountered many times so, for those not familiar with this sort of argument, let me give the details (and some notational conveniences) on this occasion and leave it to the reader to provide similar arguments in future.

A typical finite subset of Θ has the form:
$\{\varphi_1,\dots,\varphi_n\}\cup\{\psi_1,\dots,\psi_m\}\cup\{\bar{a}'=\bar{v}_1+\bar{v}_2\}$ where $\varphi_1,\dots,\varphi_n\in\Phi$, $\psi_1,\dots,\psi_m\in\Psi$, and $\bar{a}',\bar{v}_1,\bar{v}_2$ are finite subsequences of $\bar{a},\bar{v}_1,\bar{v}_2$ respectively. To say that Θ is finitely satisfied in M is to say that, for every such finite subset, the sentence
$\exists\bar{w}_1,\bar{w}_2\,(\bigwedge_{i=1}^n\varphi_i(\bar{w}_1)\wedge\bigwedge_{j=1}^m\psi_j(\bar{w}_2)\wedge\bar{a}''=\bar{w}_1+\bar{w}_2)$ is satisfied in M, where our original sequences $\bar{a}',\bar{v}_1,\bar{v}_2$ have been expanded to $\bar{a}'',\bar{w}_1,\bar{w}_2$ so as to include all the variables appearing free in the various φ_i,ψ_j and where enough "coordinates" have been included so that writing $\bar{a}''=\bar{w}_1+\bar{w}_2$ makes sense. Moreover our usual conventions are in force, so writing $\varphi_i(\bar{w}_1)$ implies that every free variable of φ_i appears in $\bar{w}_1$ but not conversely. One may note that if we show satisfaction of this sentence then, certainly, we will have shown simultaneous satisfaction of the original finite set of formulas.

Set $\varphi=\bigwedge_{i=1}^n\varphi_i$ and $\psi=\bigwedge_{j=1}^m\psi_j$. Then what has to be shown is that the sentence $\exists\bar{w}_1,\bar{w}_2\,(\varphi(\bar{w}_1)\wedge\psi(\bar{w}_2)\wedge\bar{a}''=\bar{w}_1+\bar{w}_2)$ is satisfied in M. But this (for all φ,ψ and finite subsequences of $\bar{a}$) is exactly what it means for $\bar{a}$ to be in $(\Phi+\Psi)(M)$, and so are done.

Two points to note from all this are: our useful abuses of notation when dealing with infinite sequences of variables and parameters; in proving that a set is consistent (or finitely satisfied in a given module) one often replaces it and/or a typical finite subset with a (larger) set whose consistency implies that of the first.

Example 5 This next example shows that the inclusion in 2.3(iii) may be proper. Take the ring to be the ring $\mathbb{Z}$ of integers and let $T=\mathrm{Th}(\mathbb{Z})$. Choose integers p,q with greatest common divisor 1 ($p,q\neq\pm1$). Let $\Phi(v)=\{p^n|v : n\in\omega\}$ and $\Psi(v)=\{q^n|v : n\in\omega\}$.

Since $\mathbb{Z}p^n+\mathbb{Z}q^m=\mathbb{Z}$ for all $m,n\in\omega$, one has that 1 lies in $(\Phi+\Psi)(\mathbb{Z})$ (by definition of this sum) since 1 satisfies each formula $\exists v_1,v_2\,(p^n|v_1\wedge q^m|v_2\wedge 1=v_1+v_2)$. Yet $\Phi(\mathbb{Z})+\Psi(\mathbb{Z})=0$, since no non-zero integer is infinitely divisible by p or by q.

(Here the notation "|" (divides) has its usual meaning: $a|b$ iff there is c with $ac=b$.)

We must go to a (pure-injective) elementary extension of $\mathbb{Z}$ before we find an element in "Φ" and one in "Ψ" which add together to give 1. Let me describe such elements (making some use of later results). Suppose that $p=2$ and $q=3$. The pure-injective hull, $\overline{\mathbb{Z}}$, of $\mathbb{Z}$ is an elementary extension of $\mathbb{Z}$ and is the pure-injective hull of $\oplus\,\overline{\mathbb{Z}_{(p)}}$ where the sum ranges over all primes p. Split off the first two summands, to write $\overline{\mathbb{Z}}$ as $\overline{\mathbb{Z}_{(2)}}\oplus\overline{\mathbb{Z}_{(3)}}\oplus X$, where X is the hull of the sum over all primes but 2 and 3. Then the element $1_{\mathbb{Z}}$ may be written, with respect to this decomposition, as $(1,1,x)$ for some $x\in X$. Take a to be the element $(0,1,x)$ and let b be $(1,0,0)$. Then a is divisible by every power of 2 and b is divisible by every power of 3, and $a+b=1$, as required.

Exercise 2 Show the following.

(i) Every pp-definable subgroup of R_R is a left ideal.

(ii) Every finitely generated left ideal of R is a pp-definable subgroup of R_R.
In particular, if R is left noetherian then the pp-definable subgroups of R_R are precisely the left ideals of R (= the ideals, if R is commutative).

(iii) If R is weakly saturated and if every left ideal of R is right pp-definable deduce that R is left noetherian. (Actually weakly saturated is far from necessary: for example the condition that R be left coherent can replace it - see **14.16**.)

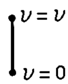
(iv) Generalise the above, replacing R_R by an arbitrary module.

Exercise 3 Characterise the rings over which every pp-definable subgroup in every module is a submodule.

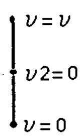
There now follows a long list of examples which are frequently used to illustrate various points (and consideration of which has often suggested general results). Background concerning the abelian groups is in §2.$\mathbb{Z}$.

Example 6

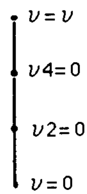
(i) $R=\mathbb{Z}$, $M=\mathbb{Z}_n$ (= $\mathbb{Z}/\mathbb{Z}n$, $n\neq 0$).



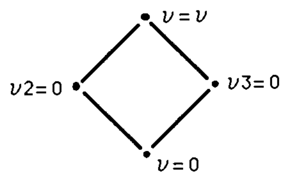
$\mathbb{Z}_2$



$\mathbb{Z}_4$

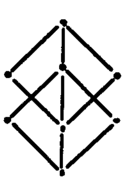


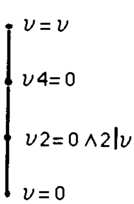
$\mathbb{Z}_8$



$\mathbb{Z}_6$

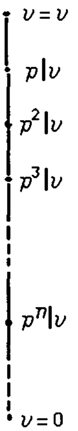
The pp-definable subgroups of M are just its subgroups (for example, by Exercise 2(ii) above). So one has, for instance, the above lattices of pp-definable subgroups.
Beside each pp-definable subgroup, I have given a pp formula which serves to define it. In general, such a formula is far from unique. For example, in the case of $\mathbb{Z}_4$ the formulas $v2=0$ and $2|v$ (i.e. $\exists w\,(v=w2)$) are equivalent, defining the same subgroup.





On the other hand, for $M=\mathbb{Z}_2\oplus\mathbb{Z}_4$ the subgroup lattice has the first shape opposite, whereas the lattice of pp-definable subgroups is just the chain shown, as the reader may easily verify (using **2.$\mathbb{Z}$1**, some of our more general results proved later or just **2.2**(i)).

In particular, rather different groups (or theories) may have isomorphic lattices of pp-definable subgroups.



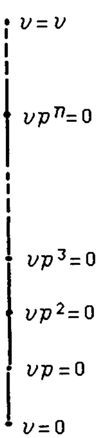
Exercise 4 Find the lattice of pp-definable subgroups of $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ (it is not simply a chain - note that a new factor emerges - cf. Ex 7(i) below).

(ii) $R=\mathbb{Z}$; $M=\mathbb{Z}_{(p)}$ - the localisation $\{m/n : m,n\in\mathbb{Z},\ p\nmid n\}$ of the module $\mathbb{Z}$ at the prime p.

Once again (by Exercise 2(ii) above) the pp-definable subgroups of M are just the $\mathbb{Z}_{(p)}$-ideals (there are far more $\mathbb{Z}_{(p)}$-ideals in the elementary extension $\overline{\mathbb{Z}(p)}$ of $\mathbb{Z}_{(p)}$, though there are "no more" pp-definable subgroups). Thus, for each $n\in\omega$, we have the pp-definable subgroup consisting of all those elements which are divisible by p^n; together with the zero subgroup these give all the pp-definable subgroups.

Thus there is a single infinite descending chain (opposite) of order type $\omega+1$.

(iii) $R=\mathbb{Z}$; $M=\mathbb{Q}$ (the rationals regarded as an abelian group).
Then $\mathbb{Q}$ has only the trivial pp-definable subgroups: itself and 0 (to prove this consider its endomorphism ring).

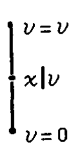


(iv) $R=\mathbb{Z}$; $M=\mathbb{Z}_{p^\infty}$ (the Prüfer group of type p, p being a prime: one may think of $\mathbb{Z}_{p^\infty}$ as the limit (union) of the system of canonical embeddings $\mathbb{Z}_p \hookrightarrow \mathbb{Z}_{p^2} \hookrightarrow \mathbb{Z}_{p^3} \hookrightarrow \ldots$ or as the multiplicative group of all p^n-th roots of unity for $n\in\omega$).

Yet again, the subgroups of M all are pp-definable. They are: the whole group M and, given any $n\in\omega$, the set of all elements of order dividing p^n (of course in any proper elementary extension of M there will be subgroups - for example M itself! - which are not pp-definable).

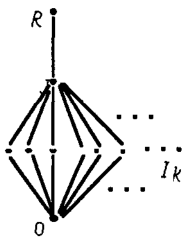
Thus there is a single infinite ascending chain of order type $\omega+1$.

(v) $R=K[X]/\langle X\rangle^2=K[x : x^2=0]$ where K is a field and $\langle X\rangle$ is the ideal generated by X. I will use J for the Jacobson radical $\langle x\rangle$ of R, where x is the image in R of X. Take $M=R$.



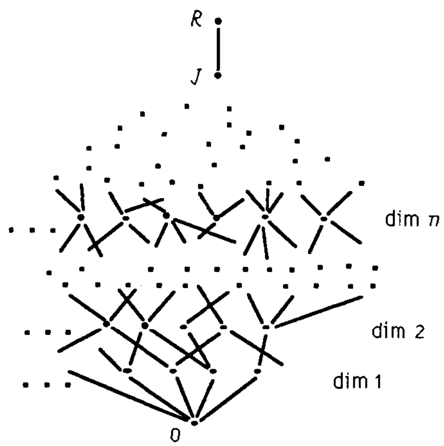
Once again, invoking Exercise 2(ii) (note that R is commutative noetherian) one has that the pp-definable subgroups are just the ideals. There are only three of these: R itself; the Jacobson radical $J=\langle x\rangle$, which may be defined for example by "$x|v$" or by "$vx=0$"; and 0.

(vi) $R=K[X,Y]/\langle X,Y\rangle^2=K[x,y : x^2=y^2=xy=yx=0]$ where K is a field. Take $M=R_R$.



Yet again, the pp-definable subgroups are just the ideals. If K is infinite then there will be infinitely many ideals of K-dimension 1. The only ideal of K-dimension 2 is the Jacobson radical and this may be defined by $\exists v_1, v_2\ (v = v_1 x + v_2 y)$ or more simply by $vx = 0$. The ideals of K-dimension 1 are parametrised by the projective line $P(K) = K \cup \{\infty\}$ over K where, for $k \in P(K)$, we set $I_k = \langle x + yk \rangle$ ($I_\infty = \langle y \rangle$), this ideal being definable by $(x + yk) | v$. (It should be noted that commutativity of the ring has been used heavily.)

(vii) $R = K[X_n : n \in \omega]/\langle X_n : n \in \omega \rangle^2 = K[x_n\ (n \in \omega) : x_i x_j = 0\ (i,j \in \omega)]$ where K is a field. Take $M = R_R$.

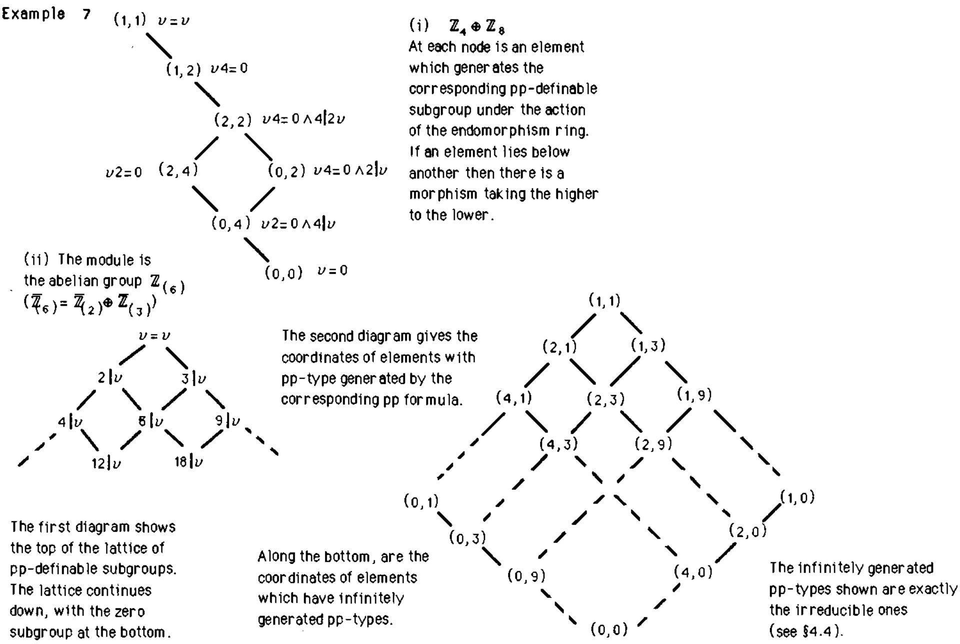


Now, in this case not every ideal is pp-definable: for example the reader may try in vain to define $\langle x_n : n$ is even$\rangle$. It is left as an exercise (perhaps better left until some general results are available, since a direct verification is rather tedious; alternatively, see [Z-HZ78; Thm 5]) to show that, apart from R, 0 and the Jacobson radical J (infinitely generated, so there is no choice but to define it by an annihilator condition), the pp-definable subgroups are the finite-dimensional K-subspaces of R (that all of them are pp-definable is easy to see, but the converse requires some work).

(viii) All the preceding examples have been over commutative rings. The simplest non-commutative examples are: R a division ring; M an R-vectorspace. Then M is simply a direct sum of copies of R_R. By **2.1** it is clear that if $\varphi(v)$ is a pp formula with one free variable, then $\varphi(R)$ is either R or 0. It then follows by **2.10** (a direct proof is not difficult) that $\varphi(M)$ is either M or 0. The situation for pp formulas in more than one free variable is not much more complicated.
Somewhat more interesting behaviour is displayed by simple artinian rings - that is, rings of the form $M_n(D)$ where D is a division ring and $M_n(D)$ denotes the ring of $n \times n$ matrices over D. Take $M = R_R$.
Then the pp-definable subgroups are easily seen to be just the left ideals - the sums of columns of $M_n(D)$.

Exercise 5 Compute the lattice of pp-definable subgroups of $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8$ and that of $\mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(3)} \oplus \mathbb{Z}_{(5)}$ (cf. the example below).

Example 7



2.2 pp-types

A partial pp-type is the specification of a filter of pp-definable cosets (a filter in the poset of all pp-definable cosets, ordered by inclusion). In terms of formulas, we say that $p(\bar{v})$ is a partial pp-type if it is a set of pp-formulas, possibly with parameters, which is consistent modulo whichever theory is being considered. Here "$\bar{v}$" may be a tuple of arbitrary length. As in the first definition, I will usually make the tacit assumption that partial pp-types are deductively closed: closed under finite conjunction (intersection) and implication of pp formulas ("upwards closed"). In contrast with complete types, we do not require that a (partial) pp-type be an ultrafilter in the poset of pp-definable cosets. A partial pp-type which has no extra parameters is a filter of subgroups, so certainly is realised (by the zero tuple). To be inconsistent, a deductively closed set of pp-formulas must contain two formulas which define different cosets of the same subgroup.

In the definition of partial pp-type a context (complete theory) is implicit but we will see that pp-types and their associated structures have an existence outside such a specified context (cf. §4.1) and we should think of them as being associated with the category of modules itself (cf. Chapter 8).

If $\bar{c}$ is in M and if $A\subseteq M$ then the pp-type of $\bar{c}$ in M over A is the filter consisting of those cosets, pp-definable over A, to which $\bar{c}$ belongs: $\mathrm{pp}^M(\bar{c}/A)=\{\varphi(\bar{v},\bar{a}) : \varphi(\bar{v},\bar{a})$ is pp, $\bar{a}$ is in A, and $M\models\varphi(\bar{c},\bar{a})\}$. One may also consider $\mathrm{pp}^M(\bar{c}/A)$ to be the corresponding set of formulas. If A is empty or $\{0\}$, we write simply $\mathrm{pp}^M(\bar{c})$. If M can safely be omitted we may do so.

If p is a type then the set of pp formulas in it - $p^+=\{\varphi : \varphi$ is pp, $\varphi\in p\}$ - is the pp-part of p, and is a typical pp-type (so $\mathrm{pp}^M(\bar{c}/A)$ is a pp-type).

It is pp-types, rather than partial pp-types, that we will be interested in. We have to make the distinction, because a partial pp-type need not be the pp-part of a complete type (see Ex16.1/2).

The pp-type of an element (or tuple) "contains" (information which gives) the annihilator of that element, but in general it contains much more: in particular it carries information about how the element sits within the containing module. It will be seen that, nevertheless, pp-types may usefully be thought of as generalised right ideals (at least in their role as annihilators). This point of view is a useful one, both for seeing what should be true and for suggesting how to prove it. It is very much from this perspective that these notes have been written.

Exercise 1 Let $c \in M$ and set $p = \mathrm{pp}^M(c)$. Show that $\{r \in R : "vr=0" \in p(v)\}$ is the annihilator of c. Suppose further that $N \leqslant M$. Interpret the right ideal $(N:c) = \{r \in R : cr \in N\}$ in terms of $\mathrm{pp}^M(c/N)$.

Example 1 Let R be the ring of integers and take $M = \mathbb{Z}_{\mathbb{Z}}$. Let n be a non-zero integer. Since every element of $\mathbb{Z}$ is torsionfree, there are no non-trivial annihilator conditions (= quantifier-free pp formulas) in $\mathrm{pp}^{\mathbb{Z}}(n)$. What $\mathrm{pp}^{\mathbb{Z}}(n)$ does contain is a precise description of the extent to which n is divisible. So for each integer m dividing n, $\mathrm{pp}^{\mathbb{Z}}(n)$ contains the formula "$m|v$" (that is, $\exists w\,(v = wm)$): all this information is of course summed up in the single formula "$n|v$" $\in \mathrm{pp}^{\mathbb{Z}}(n)$. Since $\mathbb{Z}$ is torsionfree it follows easily from **2.ℤ1** that there is essentially no more information in the pp-type of n. Thus the formula "$n|v$" "generates" this pp-type.

It is immediate that if n' also is an integer, then $\mathrm{pp}^{\mathbb{Z}}(n) \subseteq \mathrm{pp}^{\mathbb{Z}}(n')$ iff $n|n'$; moreover $\mathrm{pp}^{\mathbb{Z}}(n) = \mathrm{pp}^{\mathbb{Z}}(n')$ iff $n = \pm n'$.

Observe that pp-types contain only "positive" information. In particular, from the fact that $\mathrm{pp}^{\mathbb{Z}}(n)$ is generated by (implied by) a single formula it does not follow (and it is in fact false) that the full type $\mathrm{tp}^{\mathbb{Z}}(n)$ is isolated (implied by a single formula).

Example 2 Let R be the ring of integers and take the module M to be the localisation, $\mathbb{Z}_{(p)}$, of $\mathbb{Z}$ at the prime p. Let $q(v)$ be the pp-type generated by $\{p^n|v : n \in \omega\}$. It is easy to check (see Example 1.6(ii)) that q is not "generated" by any single formula (that is, the filter q is not **principal** = generated as a filter by a single element). Now, although the element 0 satisfies this pp-type (and is in fact the only element of $\mathbb{Z}_{(p)}$ to do so) it is consistent that there is a non-zero element whose pp-type contains (so is) q - for the set $q(v) \cup \{v \neq 0\}$ is finitely satisfied so is consistent. Therefore there is an elementary extension of $\mathbb{Z}_{(p)}$ which contains a non-zero element divisible by every power of p (and hence by every prime power, since divisibility by the others is part of the theory of $\mathbb{Z}_{(p)}$). It follows that a copy of the rationals, $\mathbb{Q}$, splits off in such an elementary extension. As an easy consequence of some later results it then follows that $\mathbb{Z}_{(p)} \oplus \mathbb{Q}$ is an elementary extension of $\mathbb{Z}_{(p)}$.

A pp-type, p, in finitely many free variables is **finitely generated** if there is a single pp-formula $\varphi \in p$ (equivalently a finite number of formulas in p - replace them by their conjunction) such that φ is equivalent to p, in the sense that every formula in p is a consequence of φ. There is of course a context implicit here: φ proves p modulo some (usually complete) theory. This is a point to watch: it may make sense for p (regarded as a set of formulas) to be a pp-type in more than one theory, but it may well happen that p is finitely generated modulo one theory but not modulo another. Thus, being finitely generated is not in general an intrinsic property of a pp-type. If one prefers to think in terms of a filter of pp-definable cosets, then changing the context in which a pp-type is regarded may result in some identifications and, conversely, some scissions, of cosets.

Example 3 Take $R=\mathbb{Z}$; $T=\mathrm{Th}(\mathbb{Z}_2\oplus\mathbb{Z}_4)$; $c=(1,0), d=(0,2)$ both in $\mathbb{Z}_2\oplus\mathbb{Z}_4$. Now, c,d both have order 2 but, whereas $\mathrm{pp}(c)$ is generated by the formula "$v2=0$", the element d is also divisible by 2 and, in fact, $\mathrm{pp}(d)$ is generated by $\exists w\,(v=w2)$.

Thus elements with the same annihilator (quantifier-free type) may well be differentiated by their pp-types.

In analogy with the notation $S^T(A)$ for complete types over A we use the notation $S^+(A)$ for the set of all pp-types (pp-parts of complete types) over A. If we wish to be specific about the over-theory and /or the number of free variables then we may display these. In §4 the exact connection between $S(A)$ and $S^+(A)$ will be established: for the moment one notes that there is a surjection $S(A)\twoheadrightarrow S^+(A)$ given by $p\mapsto p^+$ - it will be shown (2.20) that this map is a bijection.

In our picture of a pp-type as a filter of definable cosets we are making an implicit identification of formulas with the sets they define. This convention is extremely useful and will be employed frequently. The reader unaccustomed to this convention may feel rather uncomfortable with it on account of the question: "definable in which model?". Essentially, it does not matter which model is chosen, so long as it contains all parameters mentioned and, when infinite sets of formulas (such as types) are discussed, the model should be supposed to be sufficiently saturated that it contains witnesses for the (partial) types which are being considered. For definiteness one may assume that, unless otherwise indicated, "definable subset" means definable subset of the monster model.

Associated with any filter of pp-definable cosets is the filter of corresponding pp-definable subgroups. The intersection of these subgroups - a $\bigwedge$-pp-definable subgroup - is a good algebraic measure of that filter. Let us introduce some notation. If p is a pp-type we set $\mathcal{G}(p)=\{\varphi(\bar{v},\bar{0}) : \varphi(\bar{v},\bar{y})$ is a formula without parameters which is represented in $p\}$, where we say that the formula $\varphi(\bar{v},\bar{y})$ is **represented** in p if $\varphi(\bar{v},\bar{a})\in p$ for some $\bar{a}$. We allow ourselves to consider $\mathcal{G}(p)$ either as a set of formulas or as a filter of subgroups, as convenient. Set $G(p)=\bigcap\mathcal{G}(p)$ to be the intersection of all these subgroups. Clearly $G(p)$ will be pp-definable (in the monster model), rather than just $\bigwedge$-pp-definable, iff p is finitely generated. On those few occasions when we are taking account of the model in which our formulas are being interpreted, we use an appropriate superscript: thus $\mathcal{G}^M$ and G^M. These notations are extended to complete types, p, by setting $\mathcal{G}(p)=\mathcal{G}(p^+)$, etc.

Exercise 2

(i) Show that if $p\in S^+(A)$, if M is $|A|^+$-saturated, and if $G^M(p)$ is pp-definable then p is finitely generated. Give an example to show that one needs some kind of saturation assumption on M.

(ii) Improve on (i) by showing that $|T|^+$-saturation is enough.

Exercise 3 (For those who know what a definable type is: see [Pi83; §1] or [Poi85; §11.b], for instance.) Show that if $p\in S^+(A)$ then there is $B\subseteq A$ with $|B|\leqslant|T|$ and with p the definable extension (with the obvious meaning for incomplete types) of $p\restriction B$ to A.

<u>Lemma 2.4</u> *Suppose that M is weakly saturated and let $p\in S_n(0)$. Suppose that H is a subgroup of M^n which is pp-definable in M. If $H\geqslant G^M(p)$ then $H\in\mathcal{G}(p)$.*

Proof Let φ be a pp formula defining H (that is, $\varphi(M)=H$). By 2.2(ii) it may be supposed that φ has no parameters. Consider $\mathcal{G}(p)$ as a set of pp formulas. If this set did not entail φ then $\mathcal{G}(p)\cup\{\neg\varphi\}$ would be a consistent set of formulas - so would extend to at least one type $q\in S_n(0)$. By hypothesis there would then be $\bar{a}$ in M realising q. But that would be a tuple lying in $\mathcal{G}(p)$ but not in H - contrary to hypothesis. Thus (some finite subset of) $\mathcal{G}(p)$ entails φ and so $\varphi\in\mathcal{G}(p)$ ($\mathcal{G}(p)$ being a filter). □

It follows that if p and q are (pp-)types then $G(p)>G(q)$ iff $\mathcal{G}(p)\subset\mathcal{G}(q)$.

That some saturation hypothesis on M is required in 2.4 (as in other situations where we need witnesses to infinite sets of formulas) is illustrated by the example which follows.

Example 4 Take $\mathbb{Z}$ for the ring and $\mathbb{Z}_{(p)}$ for the module. For p take the type over $\mathbb{Z}_{(p)}$ of any element of $\overline{\mathbb{Z}(p)}\setminus\mathbb{Z}_{(p)}$ (see §2.Z). Since $\mathbb{Z}_{(p)}$ is algebraic over the element 1, what is essentially the same type may even be found over $\{1\}$.

For every $n\in\omega$ there is some formula of the form $p^n|v-a$ in $p(v)$, so "$p^n|v$"$\in\mathcal{G}(p)$. Since $\bigcap_n \mathbb{Z}_{(p)}p^n=0$, it follows that $G^M(p)=0$.

Yet the formula $v=0$ does not lie in $\mathcal{G}(p)$. For otherwise, p would have to contain a pp formula $\varphi(v,\bar{a})$ with $\varphi(M,\bar{0})=0$: so $\varphi(M,\bar{a})$, being a coset of $\varphi(M,\bar{0})$, would be a singleton $\{c\}$ say. Thus p would contain the formula $v=c$ and c would be the sole realisation of p in any model. On the other hand the model $\mathbb{Z}_{(p)}$ satisfies $\exists v\,\varphi(v,\bar{a})$, since its elementary extension $\overline{\mathbb{Z}(p)}$ satisfies this. So the only possibility is that $c\in\mathbb{Z}_{(p)}$ - in contradiction to our choice of p.

The last few lines of argument in the example above are standard. A type is said to be **algebraic** if one of the definable sets constituting it is finite: if (as in the argument above) one of the sets constituting a type is a singleton, then the type is said to be **1-algebraic**. By an argument similar to that above, one shows that if $p\in S(A)$ is algebraic and if $M\geq A$ is any model then M contains each one of the finitely many realisations of p. An element is said to be **algebraic over** the set A if its type over A is algebraic and is said to be **definable over** A if its type over A is 1-algebraic. If M is a sufficiently saturated model containing A, then an element is algebraic iff its orbit under $\mathrm{Aut}_A M$ is finite.

For example, if we are working in the (complete) theory of algebraically closed fields of some specific characteristic then "algebraic over" has its usual meaning: "definable over" is less interesting since, in this context, it reduces to "being in the subfield generated by".

Recall that if p is a type over A and if $B\geq A$, then a type q over B is an extension of p if $q\geq p$ (equivalently, if p is the set of all formulas in q with parameters from A).

<u>Lemma 2.5</u> *Suppose that $p\in S(A)$: let $q\in S(B)$ be any extension of p to $B\geq A$. Then $G(p)\geqslant G(q)$. Furthermore $G(p)>G(q)$ iff there is some pp formula $\varphi(\bar{v},\bar{y})$ represented in q such that the group $\varphi(\bar{v},\bar{0})$ does not correspond to any pp formula represented in p.*

Proof Since q extends p, any formula represented in p certainly is represented in q, so the first inclusion is clear. The second statement follows since $G(p)>G(q)$ iff $\mathcal{G}(p)<\mathcal{G}(q)$ (by 2.4). □

Exercise 4

(i) If p is a type with $G(p)=0$ then p defines a single element. The converse is false (exercise) - a defining formula may require negations.
On the other hand, if p is algebraic then one may show (say, using 2.16 and 2.12 below) that $G(p)$ is finite. Note also that "being algebraic over" is more general than "being a term in". For take $M=\mathbb{Q}_{\mathbb{Z}}$ and $A=\{1\}$. Since one has unique division in $\mathbb{Q}$, every element of $\mathbb{Q}$ is algebraic over 1.

(ii) One may well have $q\geq p$ with $G(p)=G(q)$, yet with some pp formula represented in q but not in p (for a given coset may be defined by many different pp formulas using a variety of parameters).

The group $G(p)$ itself turns out not to be the best invariant of the pp-type p, for it may be that one cannot avoid expanding $\mathcal{G}(p)$ when extending p to a type over a model. Rather we look at the **connected component** $G_0(p)=\bigcap\mathcal{G}_0(p)$ where, regarded as a filter of subgroups, $\mathcal{G}_0(p)$ is the set of pp-definable subgroups H such that there exists $G\in\mathcal{G}(p)$ with the index

$[G:G\cap H]$ finite: $G_0(p)=\bigcap\{H$: there is $G\in\mathcal{G}(p)$ with $G/G\cap H$ finite$\}=\bigcap\{H$: there is $G\in\mathcal{G}(p)$ with $H\leqslant G$ and $[G:H]$ finite$\}$.

In general we will say that a group H **has finite index** in the group G if the index $[G:G\cap H]$ is finite. We say that a $\bigwedge$-pp-definable subgroup is **connected** if there is no pp-definable subgroup of finite index in it (in the sense just defined).

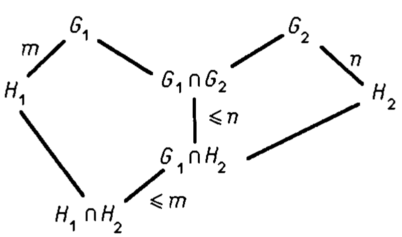
To see that the connected component is well-defined we need first to note the following. If, in some model M, the index of the pp formula ψ in the pp formula φ is finite - equal to n say - then in every $M'\equiv M$ one has $[\varphi(M'):\varphi(M')\cap\psi(M')]=n$. This follows since "$[\varphi:\psi]=n$" - that is $[\varphi(\tilde{M}):\psi(\tilde{M})]=n$ - is expressed by the sentence:

$\exists\bar{v}_1,\dots,\bar{v}_n\,(\bigwedge_{i=1}^{n}\varphi(\bar{v}_i)\wedge\bigwedge_{i\neq j=1}^{n}\neg\psi(\bar{v}_i-\bar{v}_j)\wedge\forall\bar{v}\,(\varphi(\bar{v})\to\bigvee_{i=1}^{n}\psi(\bar{v}-\bar{v}_i)))$.

Thus we may write "$[\varphi:\psi]=n$" without ambiguity: note that if ψ is not contained in φ then that expression will be taken to mean that the index of $\psi\wedge\varphi$ in φ is n.

Lemma 2.6 *Let* p *be a type or pp-type. Then:*
(a) $\mathcal{G}_0(p)$ *is a filter of pp-definable subgroups;*
(b) $G_0(p)$ *is a connected* $\bigwedge$*-pp-definable subgroup;*
(c) *if* p *is a type over a model* M *then* $G_0(p)=G(p)$.

Proof (a) To see that $\mathcal{G}_0(p)$ is upwards closed, let $G\in\mathcal{G}(p)$, let H be a pp-definable subgroup of finite index in G and let K be a pp-definable subgroup containing H. Since $\mathcal{G}(p)$ is upwards closed, the group $G+K$ lies in it. By modularity one has $[G+K:K]=[G:G\cap K]\leqslant[G:H]<\omega$ - so K is in $\mathcal{G}(p)$.



To show that $\mathcal{G}_0(p)$ is closed under intersection, let G_1,G_2 be in it and take pp-definable subgroups H_1,H_2 with, say, $[G_1:H_1]=m$ and $[G_2:H_2]=n$ for some integers m,n. Then $H_1\cap H_2$ has finite index in $G_1\cap G_2\in\mathcal{G}(p)$ since, by modularity (see the diagram), one has $[G_1\cap G_2:G_1\cap H_2]\leqslant[G_2:H_2]$ and $[G_1\cap H_2:H_1\cap H_2]\leqslant[G_1:H_1]$.

(b) Let $\psi(\bar{v})$ be any pp formula not in $\mathcal{G}_0(p)$. Consider the following set $\Psi(\bar{w})$ of pp formulas with $\bar{w}=(\bar{v}_n)_{n\in\omega}$ where $l(\bar{v}_n)=l(\bar{v})$ for all n:

$\{\varphi(\bar{v}_n):\varphi\in\mathcal{G}_0(p)$ and $n\in\omega\}\cup\{\neg\psi(\bar{v}_n-\bar{v}_m): m,n\in\omega,\ m\neq n\}$.

Informally, Ψ says that there are infinitely many elements in $G_0(p)$ lying in pairwise distinct cosets of ψ. So to show that ψ has infinite index in $G_0(p)$ it will be enough to show that Ψ is consistent.

If this were not so, then some finite subset of Ψ would be inconsistent. But it is easy to see that inconsistency of such a finite subset can amount to no more than the assertion that there is φ in the filter $\mathcal{G}_0(p)$ such that the index $[\varphi:\psi]$ is finite. But that would be enough to place ψ in $\mathcal{G}_0(p)$ - contrary to choice of ψ.

Thus $G_0(p)$ is indeed connected.

(c) It must be shown that if the set of parameters over which p is defined is a model then $G(p)$ is connected. It will be sufficient to show that if $G\in\mathcal{G}(p)$ and if H is pp-definable of finite index in G then $H\in\mathcal{G}(p)$. Since $G\in\mathcal{G}(p)$, there is a pp formula, $\varphi'(\bar{v},\bar{a})$, in p with $\varphi'(\tilde{M},\bar{0})=G$.

Set $\varphi(\bar{v})$ to be $\varphi'(\bar{v},\bar{0})$ and let the pp formula $\psi(\bar{v})$ define H: also let n be the index of H in G. Then the sentence: $\exists\bar{v}_1,\dots,\bar{v}_n\,(\bigwedge_1^n\varphi(\bar{v}_i)\wedge\forall\bar{v}\,(\varphi(\bar{v})\to\bigvee_1^n\psi(\bar{v}-\bar{v}_i))$ holds in the monster model, so also holds in M. Thus there are $\bar{a}_1,\dots,\bar{a}_n$ in M such that

$\varphi(\tilde{M}) = \bigcup_1^n \bar{a}_i + \psi(\tilde{M})$. Also, since $\varphi'(\bar{v},\bar{a})$ is in p, M satisfies $\exists \bar{v}\, \varphi'(\bar{v},\bar{a})$: say $\bar{b}$ is in $M \cap \varphi'(\bar{v},\bar{a})$.

Let $\bar{c}$ be any realisation of p. Then $\varphi'(\bar{c}-\bar{b},\bar{0})$, that is $\varphi(\bar{c}-\bar{b})$, holds. So, for some $i \in \{1,...,n\}$, one has $\psi(\bar{c}-\bar{b}-\bar{a}_i)$. Thus $\psi(\bar{v}) \in \mathcal{G}(p)$, as required. ▫

Example 5

(i) If p is a type over 0 then $G(p)$ is just the subgroup defined by p^+.

(ii) Take $R=\mathbb{Z}$ and $T=\mathrm{Th}(M_0=\mathbb{Z}_4{}^{(\aleph_0)})$. Let $p(v) \in S_1(M_0)$ be the type which says $v2=0$ and $v \neq m$ for each $m \in M_0$ - the type of any element of order two which is not in M_0. Let $q(v) \in S_1(M_0)$ be given by $v \neq m$ for each $m \in M_0$ and $(v-a)2=0$ for some element $a \in M_0$ of order 4. Thus q is the type of an element of order 4, which lies outside M_0 but which differs from the element $a \in M_0$ only by an element of order 2 (so a realisation of q is not "completely independent" of M_0). Note that these are indeed descriptions of complete types p,q over M_0 (consider automorphisms: if two elements satisfy say, q, then there is an automorphism (of the monster model) taking one to the other).

Then $G(p)=G(q)=M2$ even though $p \neq q$.

It is not difficult to see (use **2.23**) that, in this example, if $G > H$ are pp-definable subgroups then $[G:H]$ is infinite. So for any type p' one has $G_0(p')=G(p')$.

Exercise 5 Let R be any ring and let $a,b \in R$. Show that $\mathrm{pp}^R(a) \leqslant \mathrm{pp}^R(b)$ iff $b \in Ra$. Hence $\mathrm{pp}^R(a)=\mathrm{pp}^R(b)$ iff $Rb=Ra$.

2.3 Pure embeddings and pure-injective modules

Most of the concepts and results introduced in this section belong in a much more general context: but these notes are about modules, so I confine the discussion to that case.

When one takes a "purely algebraic" viewpoint, an embedding is simply a morphism which preserves annihilators. But here we are interested in pp-types rather than in annihilators (= $\bigwedge$-atomic types) and so a morphism which merely preserves annihilators and does not keep track of pp-types is often too weak for our purposes. For example, the canonical injection $\mathbb{Z}_2 \longrightarrow \mathbb{Z}_4$ strictly increases the pp-type of the non-zero element of $\mathbb{Z}_2$ (since its image is divisible by 2). We will single out those morphisms which preserve pp-types.

Let us note first that all morphisms are non-decreasing on pp-types.

Lemma 2.7 *Let $M \xrightarrow{f} M'$ be any morphism and let $\bar{a}$ be in M. Then $\mathrm{pp}^M(\bar{a}) \subseteq \mathrm{pp}^{M'}(f\bar{a})$.*

Proof This is clear since (by the proof of **2.2**(i)) if $M \models \varphi(\bar{a})$, with φ pp, then $M' \models \varphi(f\bar{a})$ ("morphisms preserve pp formulas"). ▫

In particular, if $M \subseteq M'$ is an inclusion of modules and if $\bar{a}$ is in M, then $\mathrm{pp}^M(\bar{a}) \subseteq \mathrm{pp}^{M'}(\bar{a})$ (here I have identified elements of M with their images in M' - I seldom (need to) distinguish between embeddings and inclusions). This embedding is said to be **pure** if for every finite tuple $\bar{a}$ in M one actually has $\mathrm{pp}^M(\bar{a})=\mathrm{pp}^{M'}(\bar{a})$, and in that case one writes $M \prec^+ M'$. The notation "$-\prec^+\rightarrow$" will occasionally be used.

The definition of purity may be re-phrased as follows: $M \prec^+ M'$ iff for every pp formula φ and $\bar{a}$ in M one has $M \models \varphi(\bar{a}) \Leftrightarrow M' \models \varphi(\bar{a})$. Otherwise said, M is pure in M' iff every finite system of linear equations with coefficients in R and parameters from M and with a solution in M' already has a solution in M. We also say that M is **pure in** M', and the terms **pure submodule** and **pure extension** are used in the obvious ways.

Example 1

(i) If M is a direct summand of M' then M is pure in M' (given a solution in M', project it to M).

(ii) $\mathbb{Z}_2$ is not a pure submodule of $\mathbb{Z}_4$ (this was noted above). In particular, notice that there are essentially two different embeddings of $\mathbb{Z}_2$ into $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ (exercise: how many are there of $\mathbb{Z}_4$ into $\mathbb{Z}_4 \oplus \mathbb{Z}_8$? - take "different" to mean "not conjugate by an automorphism").

(iii) Often we will encounter pure but non-split embeddings - the split embeddings are simply the extreme case of pure embeddings. I point out just two examples now. The first is the canonical embedding of the localisation $\mathbb{Z}_{(p)}$ into its completion (see §2.$\mathbb{Z}$) $\overline{\mathbb{Z}_{(p)}}$. As for the second: a module is said to be **absolutely pure** if every embedding of it into any other module is a pure embedding. Such modules are not in short supply: every module over a (von Neumann) regular ring (see Chapter 16) is absolutely pure and, unless the ring is actually semisimple artinian, there will be some which are not injective.

Exercise 1 If M is pure in M' and if M'/M is finitely presented, then M is a direct summand of M'. Indeed, an embedding $M \leqslant M''$ is pure iff M is a direct summand of every M' with $M \leqslant M' \leqslant M''$ and M'/M finitely presented (see [Laz69; 2.4], also see [Ri79; §1.F] for other equivalents if R is a finite-dimensional algebra). Deduce that every absolutely pure module is injective iff the ring is right noetherian. [Hint: for "$\Leftarrow$" use a pushout; for "$\Rightarrow$" use 1.11.]

On occassion it will be useful to us to have a notation to express that an embedding is split. There is no particularly standard notation for this: I will use "$M|N$" to express that M is a direct summand of N (*via* a particular embedding) - one may read this as "M is a factor of N" or "M divides N".

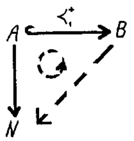
Given a class of embeddings one may define a notion of injectivity with respect to the class. From the class of all embeddings, one derives the usual notion of injectivity. From the class of all pure embeddings, we obtain the pure-injective modules: the next theorem characterises these modules in various ways. Notice that they may be defined as being those modules which are saturated for pp formulas. Most parts of the result are particular instances of a rather general theorem which may be found in [Myc64], [Weg66]; for abelian groups, see [Łos57], for modules, see [War69; Thm 2], [St67] and [Zim77; 2.3] (see the comments after Thm 2 in Warfield's paper for further details and references). (One may consult [Hal79] for another algebraic example).

<u>Theorem 2.8</u> *The following conditions on a module N are equivalent, and any module N which satisfies these conditions is said to be* **pure-injective** (*or* **algebraically compact**).

(*i*) *Every system of equations over (i.e., with parameters in) N which is finitely satisfied in N actually has a solution in N. Here the system may be in any number - finite or infinite - of unknowns.*

(*ii*) *Every partial pp-type (in one variable) over N which is finitely satisfied in N is actually realised in N.*

(*iii*) *If N is purely embedded in the module M then this embedding is split: that is, $M = N \oplus M'$ for some M'.*



(*iv*) N *is injective over pure embeddings. That is, any diagram as given may be completed to a commutative diagram as shown.*

(*v*) *If* $\bar{a}$ *is in* M, *if* $\bar{b}$ *is in* N *and if* $\mathrm{pp}^M(\bar{a}) \leq \mathrm{pp}^N(\bar{b})$ *then there is a morphism* $f: M \longrightarrow N$ *with* $f\bar{a} = \bar{b}$.

Proof (i)⇒(ii) (Actually, this will follow from the other implications but the direct proof is worth seeing.) Suppose that the system, $\Phi(\bar{v})$, of pp formulas over N is finitely satisfied in N. Strip off the existential quantifiers from the formulas in Φ by replacing $\varphi(\bar{v}) \equiv \exists \bar{w}\, \theta(\bar{v}, \bar{w})$ (where θ is $\bigwedge$-atomic) by $\theta(\bar{v}, \bar{w}_\varphi)$: we can arrange that if $\varphi \neq \psi$ then $\bar{w}_\varphi$ and $\bar{w}_\psi$ have no variables in common.

Let $\Theta(\bar{v}, \bar{w})$ be the resulting system of equations over N. Clearly $\Phi(\bar{v})$ is equivalent to "$\exists \bar{w}\, \Theta(\bar{v}, \bar{w})$", where the latter is $\{\exists \bar{w}\, \theta(\bar{v}, \bar{w}) : \theta \in \Theta\}$. By (i), Θ is realised in N since it is finitely satisfied in N (for Φ is). Therefore $\Phi(\bar{v})$ is realised in N.

(ii)⇒(i) Let $\Theta(\bar{v})$ be a system of equations over N which is finitely satisfied in N. Suppose that $\bar{v} = (v_\beta)_{\beta < \alpha}$. In the obvious notation, let $\Phi(v_0)$ be the partial pp-type $\exists(\bar{v} \setminus \{v_0\})\, \Theta(\bar{v})$ - thus we existentially quantify out all the free variables but v_0. Since $\Theta(\bar{v})$ is finitely satisfied in N, so is $\Phi(v_0)$. Since the latter is a partial pp-type it has, by (ii), a solution $a_0 \in N$.

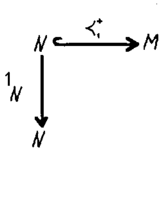
Thus we replace the first free variable by a solution for it: and of course a solution for $\Theta(a_0 {}^\frown (\bar{v} \setminus \{v_0\}))$ will yield a solution for Θ. Continue, by (transfinite) induction, replacing free variables by solutions for them, until one reaches a complete solution for the original system Θ.

(i)⇒(v) Consider the elements of M as strung out in a (likely infinite) tuple $\bar{m}$, of which $\bar{a}$ is a sub-tuple, and set $p_0(\bar{v}, \bar{a}) = \mathrm{tp}^0(\bar{m}/\bar{a})$ to be the $\bigwedge$-atomic type of M over $\bar{a}$. Without difficulty one sees that $\mathrm{pp}^M(\bar{a})$ is equivalent to $\exists \bar{v}\, p_0(\bar{v}, \bar{a})$ (notation as before). So, by the hypothesis of (v), the partial pp-type $\exists \bar{v}\, p_0(\bar{v}, \bar{b})$ is satisfied in N. Thus $p_0(\bar{v}, \bar{b})$ is a system of equations over N which is finitely satisfied in N. By assumption (i), there is $\bar{c}$ in N such that $p_0(\bar{c}, \bar{b})$ holds.

Define a map $M \longrightarrow N$ by $\bar{m} \mapsto \bar{c}$ (that is, the β-th entry of the tuple $\bar{m}$ is sent to the β-th entry of $\bar{c}$). This is a morphism, since any linear relation involving the entries of $\bar{m}$ is included in $p_0(\bar{v}, -)$ and hence is satisfied by the corresponding members of $\bar{c}$. Moreover, since $\bar{a}$ occurs as a part of $\bar{m}$ and since $p_0(\bar{v}, \bar{a})$ includes the relevant equations expressing this fact, it follows that this morphism takes $\bar{a}$ to $\bar{b}$, as required.

(v)⇒(iv) Given a pure embedding $A \prec^+_1 B$ and a morphism $A \xrightarrow{f} N$ as in the statement of (iv), one notes first that, on writing A as the parameter string $\bar{a}$, purity of the embedding yields $\mathrm{pp}^B(\bar{a}) = \mathrm{pp}^A(\bar{a}) \leq \mathrm{pp}^N(f\bar{a})$, where the inclusion is by 2.7. So, by (v), there is a morphism $B \xrightarrow{g} N$ with $g\bar{a} = f\bar{a}$ - as required.

(iv)⇒(iii) This is immediate on applying (iv) to the diagram below.



(iii)⇒(ii) Let $p(\bar{v})$ be any partial pp-type over N which is finitely satisfied in N. Then $p(\bar{v})$ is realised, by $\bar{c}$ say, in some elementary extension, M, of N. Since $N \prec M$ one has in particular $N \prec^+_1 M$. By (iii), this embedding is split; say $M = N \oplus M'$ for some M'. Write $\bar{c} = (\bar{n}, \bar{m}) \in N \oplus M'$ accordingly. Since pp formulas are preserved by morphisms (2.7) one obtains, on projecting $p(\bar{c})$ to the "N-component", $N \models p(\bar{n})$. Thus there is, indeed, already a solution for p in N. □

Corollary 2.9 *If a module is $|T|^+$-saturated, then it is pure-injective. In particular, every module is elementarily equivalent to a pure-injective module.*

Proof This is immediate from 2.8(ii). □

Examples and Exercises 2

(i) Injective modules are pure-injective (2.8(iv)).

(ii) Finite modules are pure-injective (2.8(i) or (ii)) but are not necessarily injective ($\mathbb{Z}_2$ is an example).

(iii) The class of pure-injective modules is closed under direct summands and direct products, but not in general under infinite direct sums: for example it will follow from 2.11 that $\overline{\mathbb{Z}(p)}^{(\aleph_0)}$ is not pure-injective.

(iv) If N has the descending chain condition on pp-definable subgroups then (since it therefore has dcc on pp-definable cosets) it is pure-injective.

(v) The module $\overline{\mathbb{Z}(p)}$ is pure-injective but does not have dcc on pp-definable subgroups (Ex1.6/6(ii)).

(vi) Countable pure-injective modules necessarily have dcc on pp-definable subgroups (3.11 below).

(vii) If $M_{C(R)}$ (i.e., M regarded as a module over the centre of R) has dcc then, by (v) above and 2.1, M must be pure-injective. Important examples are the finitely generated modules over Artin algebras.

Exercise 3 (cf. Exercise 11.3/3) Let N be pure-injective and let $S = \mathrm{End}(N)$.

(i) Let $a \in N$ and set $p = \mathrm{pp}^N(a)$. Then $p(N) = Sa$.

(ii) Every finitely generated S-submodule of N is $\bigwedge$-pp-definable in N_R.

(iii) If $_SN$ is noetherian then the $\bigwedge$-pp-definable subgroups of N_R are precisely the S-submodules of N. (Also see [Zim77; Prop3 and following].)

The next lemma says that pp formulas "commute with" direct sums and direct products. This is far from being true of general formulas. The lemma is very useful, both in general considerations and in the analyses of particular examples.

Lemma 2.10 *Suppose that $\varphi(\bar{v})$ is a pp formula and let $\{M_i : i \in I\}$ be any set of modules. Then:*

(a) $\varphi(\bigoplus_I M_i) = \bigoplus_I \varphi(M_i)$;

(b) $\varphi(\prod_I M_i) = \prod_I \varphi(M_i)$;

(c) *If $\bar{a}_i$ is in M_i $(i \in I)$ and if $\bar{a} \in \prod_i M_i$ has i-th component $\bar{a}_i$, then* $\mathrm{pp}(\bar{a}) = \bigcap_i \mathrm{pp}(\bar{a}_i)$.

Proof Let $\bar{a} = (\bar{a}_i)_i \in (\prod_I M_i)^{l(\bar{v})} = \prod_I M_i^{l(\bar{v})}$. On projecting $\varphi(\bar{a})$, one obtains (by 2.7) $\varphi(\bar{a}_i)$ for each $i \in I$. Thus $\varphi(\prod M_i) \leq \prod \varphi(M_i)$.

Conversely, if $\varphi(\bar{a}_i)$ holds (in M_i, equally in $\prod_j M_j$ since, being a direct summand, M_i is pure in the product), and this for each i, then it is an easy exercise to see that $\varphi(\bar{a})$ holds (patch together witnesses for the existential quantifiers on each coordinate).

Thus $\varphi(\bar{a})$ holds iff $\varphi(\bar{a}_i)$ holds for each $i \in I$. So (a), (b) and (c) follow immediately. □

Exercise 4 (if you know what a reduced power is) [Gar80a] Let M be a module, let I be an index set and let $\mathcal{F}$ be any filter on I. Suppose that φ is pp and let $\bar{m} = (\bar{m}_i)_{i \in \mathcal{F}}$ be an element of the reduced power $M^I/\mathcal{F}$. Show that $M^I/\mathcal{F} \models \varphi(\bar{m})$ iff $\{i \in I : M \models \varphi(\bar{m}_i)\} \in \mathcal{F}$. Hence any reduced power of a module has the "same" lattice of pp-definable subgroups as M.

It turns out to be a significant question whether or not a direct sum of pure-injectives is pure-injective. In the injective case, closure under direct sums corresponds to the right noetherian condition (more precisely, to dcc on annihilators). There is an analogous correspondence for pure-injectives. For now, we make the following definition and prove a useful characterisation.

Say that the module N is **Σ-pure-injective** (or **Σ-algebraically compact**) if $N^{(\aleph_0)}$ is pure-injective (such a module N is, in particular, pure-injective).

Theorem 2.11 [Zim77; 3.4], [GJ76; Thm] *The module N is Σ-pure-injective iff N has the dcc on pp-definable subgroups.*

Proof ⇒ (The basic argument is due to Bass and appears in [Ch60; Prop 4.1].) Suppose that $\varphi_0(N) > \varphi_1(N) > \ldots > \varphi_n(N) > \ldots$ is a strictly descending chain of pp-definable subgroups of N. For each $i \in \omega$ choose some element, a_i, in the gap $\varphi_i(N) \setminus \varphi_{i+1}(N)$. Set $b_n = (a_0, a_1, \ldots, a_{n-1}, 0, 0, \ldots) \in N^{(\aleph_0)}$.

Consider the partial pp-type $\Phi(\bar{v}) = \{\varphi_n(v - b_n) : n \in \omega, \ n \geqslant 1\}$ over $N^{(\aleph_0)}$. This set is finitely satisfied in $N^{(\aleph_0)}$ since, for $k \leqslant n$, one has $\varphi_k(b_{n+1} - b_k)$.

If $N^{(\aleph_0)}$ were pure-injective there would, by 2.8, be a solution $c = (c_n)_{n \in \omega}$ for Φ in $N^{(\aleph_0)}$. A contradiction is produced from the fact that c lies in the direct sum, $N^{(\aleph_0)}$, rather than in the direct product, $N^{\aleph_0}$. For there is some $m \in \omega$ such that $c_n = 0$ for $n \geqslant m$. But then, from $\varphi_{m+1}(c - b_{m+1})$ one has, on projecting to the m-coordinate, that $\varphi_{m+1}(c_m - a_m)$ holds. But that gives $\varphi_{m+1}(-a_m)$ (equivalently $\varphi_{m+1}(a_m)$) - contrary to choice of a_m.

⇐ If there is a strict inclusion of pp-definable subgroups $\varphi(N^{(\aleph_0)}) > \psi(N^{(\aleph_0)})$ then, by 2.10, the inclusion $\varphi(N) \geqslant \psi(N)$ also is strict. So if N has dcc on pp-definable subgroups then the same condition holds for $N^{(\aleph_0)}$. Therefore, as noted in Exercise 2(iv), we have that $N^{(\aleph_0)}$ is pure-injective, as required. □

Exercise 5 It is easy to find an example of an infinite direct sum of Σ-pure-injective modules which is not Σ-pure-injective [consider any direct sum of finite abelian groups which is not of bounded exponent: i.e., for which there is no $n \in \mathbb{Z}$ $n \neq 0$ which annihilates every element]. For a description of exactly when a direct sum of Σ-pure-injective modules is Σ-pure-injective, see [Zim77; 3.3]. Analogously, [Zim77; 3.7] describes the pure-injective direct sums of pure-injective modules.

Exercise 6 A module N has dcc on pp-definable subgroups iff for each/any $l \geqslant 1$ its power N^l has dcc on pp-definable subgroups.

Exercise 7 [GJ76; Thm], [Zim77; 3.4] The following conditions on N are equivalent: N is Σ-pure-injective; $N^{(I)}$ is pure-injective for all/some infinite index set I; $N^{(\aleph_0)}$ is a direct summand of $N^{\aleph_0}$; $N^{(I)}$ is a direct summand of N^I for each/any infinite index set I.

Example 3

(i) Some Σ-pure-injectives are: $\mathbb{Z}_n$ for any integer $n \geqslant 2$, in fact any abelian group of bounded exponent; any injective module over a right noetherian ring or indeed any **Σ-injective** module E (that is, a module E such that $E^{(\aleph_0)}$ is injective: see below); any finitely generated module over an Artin algebra. Exercise: show that a module E is Σ-injective iff it is injective and Σ-pure-injective.

(ii) Some pure-injectives which are not Σ-pure-injective are: $\overline{\mathbb{Z}_{(p)}}$ for any prime p; any injective but not Σ-injective module - for example, any injective over a regular ring which is not the injective hull of a semisimple module.

(iii) Products of different Σ-pure-injectives need not be Σ-pure-injective. Consider $\prod\{\mathbb{Z}_n : n \geqslant 2\}$: observe that this module splits off a copy of $\overline{\mathbb{Z}_{(p)}}$ (and obviously a direct summand of a Σ-pure-injective is Σ-pure-injective) or else use 2.11.

Corollary 2.11a (Bass, see [Ch60; Prop 4.1]) *An injective R-module E is Σ-injective iff R has the acc on annihilators of elements of E. In particular, every injective R-module is Σ-injective iff R is right noetherian.* □

The proof is left as an exercise.

Exercise 8 Rososhek [Ros78] calls a module M "purely correct" if every module N such that each of M and N is purely embedded in the other is actually isomorphic to M.
(i) Show that if M is not Σ-pure-injective then $M^{(\aleph_0)}$ is not purely correct.
(ii) Deduce that every injective R-module is purely correct iff R is right noetherian [Ros78; Thm 1].
(iii) Deduce that every R-module is purely correct iff R is right pure-semisimple (cf. §11.1) [Ros78; Thm 3].
(This is how Rososhek proves the direction "⇒" in [Ros?; Thm 2, Thm 3]. The proof of the direction "⇐" is much simplified if one uses the uniqueness of the decomposition of a totally transcendental module as a direct sum of indecomposables (3.14, 4.A14).)

It will be seen in Chapter 3 that if M is a totally transcendental module then M has the dcc on pp-definable subgroups (one may try this as an exercise now - compare Exercise 2(vi) above). In fact, the converse also will be established. In order to do that, we need to know the connection between types and pp-types - given a pp-type p how many complete types are there with p as their pp-part?. The answer is: exactly one. It will be shown that if $p,q \in S(A)$ are such that $p^+ = q^+$ then $p = q$. In fact, given a complete type p, p^+ together with the set, $\neg p^-$, of negations of all those pp-formulas not in p^+ serves to prove p.

Showing this involves looking at the interactions of the various pp-definable subgroups and cosets: that is what we turn to next.

2.4 pp-elimination of quantifiers

In this section it is shown that, in any theory of modules, every formula is equivalent to a boolean combination of pp formulas together with sentences having a closely prescribed form. In particular, in any complete theory of modules, every definable set is a boolean combination of pp-definable cosets.

The first lemma is a crucial component in the proof of pp-elimination of quantifiers. It will also be used at a number of other points. Actually, I state it in a form which is stronger than is needed for most, but not all, its applications.

I use the following notation. If $H \geqslant K$ are subgroups of some group, then by $[aH:K]$, I will mean just $[H:K]$ - the number of cosets of K required to cover the coset aH. One may generalise this: if X is a union of cosets of subgroups, all of which contain K, then by $[X:K]$, I mean the minimum number of cosets of K required to cover X.

Theorem 2.12 (Neumann's Lemma) [Neu54; 4.4] *Suppose that we have a coset contained in a finite union of cosets: $aH \subseteq \bigcup_1^n a_iH_i$ where H and the H_i are subgroups (of some group to which a and the a_i belong). Then one may omit from this cover of aH all cosets of the form a_iH_i where $[H:H\cap H_i] > n!$. In particular, all cosets "of infinite index in aH" may be omitted.*

Proof Let us make some immediate simplifications. It may be supposed that the coset aH is the subgroup H - replace the original inclusion by $H \subseteq \bigcup a^{-1}a_iH_i$. Also, replace each H_i by $H \cap H_i$ so that it may be assumed that each subgroup H_i is contained in H. So we have $H = \bigcup_1^n a_iH_i$.

Assume that the cover is minimal in the sense that for each $j\in\{1,...,n\}$ the group H is not contained in the union $\bigcup\{a_iH_i : i\in\{1,...,n\},\ i\neq j\}$. Suppose that there are r distinct subgroups among the H_i.

The proof proceeds by establishing a series of claims.

Claim 1 Each index $[H:H_i]$ is finite.

This is proved by induction on r, the case $r=1$ being obvious.

Choose $j\in\{1,...,n\}$. By minimality of the cover and since $r\geqslant 2$, there is $g\in H\setminus\bigcup\{a_iH_i : H_i=H_j\}$. Then the complete coset gH_j lies in this difference and, in particular, $gH_j\subseteq\bigcup\{a_iH_i : H_i\neq H_j\}$. Thus $H_j\subseteq\bigcup\{g^{-1}a_iH_i : H_i\neq H_j\}$.

It follows that H may be covered by finitely many cosets of those H_i which are unequal to H_j,. For if k is such that $H_k=H_j$ then, by the above paragraph, $a_kH_k\subseteq\bigcup\{a_kg^{-1}a_iH_i : H_i\neq H_j\}$. There are only $r-1$ such subgroups; so by the induction hypothesis (over all such situations) one has that each index $[H:H_i]$ is finite. But j was arbitrary and $r\geqslant 2$, so the claim follows.

Thus the weaker (last) statement of the theorem is proved.

Claim 2 For some i, $[H:H_i]\leqslant n$.

Set $K=\bigcap H_i$ and note that, by Claim 1, the index $[H:K]$ is finite - equal to m say. Assume for a contradiction that for all $i\in\{1,...,n\}$ one has $[H:H_i]=[H:K]/[H_i:K]>n$. This gives $[H_i:K]<m/n$ and hence $[a_iH_i:K]<m/n$.

Therefore one has: $[\bigcup_1^n a_iH_i:K]\leqslant\sum_1^n[H_i:K]<n(m/n)=m=[H:K]$. This contradicts the fact that $H=\bigcup_1^n a_iH_i$, and so the claim follows.

Claim 3 For each i, $[H:H_i]\leqslant n!$.

Again this is by induction on r, with the case $r=1$ being covered by Claim 2. So, by that claim, one may suppose that $r\geqslant 2$ and that (say) $[H:H_1]\leqslant n$. Fix $i\geqslant 2$: if $H_i=H_1$ then we are finished, so we may assume that $H_i\neq H_1$.

Let $g\in H\setminus\bigcup\{a_jH_j : j\neq i\}$. As in the proof of the first claim, one has $gH_1\subseteq\bigcup\{a_kH_k : H_k\neq H_1\}$. Now, it may well be that this cover of gH_1 is not minimal; nevertheless, in cutting it down to a minimal cover indexed by, say, $X\subseteq\{1,...,n\}$, the coset a_iH_i must be retained (since g lies in only that coset). Thus we obtain a minimal cover of the coset gH_1 involving at most $r-1$ subgroups, one of which is H_i. To apply the induction hypothesis in its precise form, we translate to what is necessarily a minimal cover of H_1: $H_1\subseteq\bigcup\{g^{-1}a_kH_k : k\in X\}$; and then simplify to the minimal cover $H_1=\bigcup\{(g^{-1}a_kH_k)\cap H_1 : k\in X\}$.

There are no more than $r-1$ subgroups involved in this union of no more than $n-1$ cosets so, by induction, the index in H_1 of each subgroup, $H_1\cap H_k$, involved is at most $(n-1)!$. In particular $[H_1:H_1\cap H_i]\leqslant(n-1)!$. Hence $[H:H_i]\leqslant[H:H_1\cap H_i]=[H:H_1].[H_1:H_1\cap H_i]\leqslant n.(n-1)!=n!$ as required, and the theorem is proved. □

The above proof, which is due to W. Hodges, actually shows that $[H:\bigcap_1^n H_i]\leqslant n!$. This, and related results, have also been considered by Tomkinson [Tom87] where it is shown that this bound is, in general, exact. Also see [Bd84a; 3.5] for an infinitary version.

The main result of this section is that, in any module, every definable subset is a boolean combination of pp-definable subsets. Indeed, every formula in the language of R-modules is equivalent, in every module, to an "invariants sentence" (this says something about indices of pp-definable subgroups in each other) together with a boolean combination of pp formulas. I will present the proof of this in terms of the boolean algebra of definable subsets. Every definable subset may be built up from those defined by atomic formulas, using the boolean operators "$\cap$", "$\cup$" and "$\neg$" (complement) together with the quantifier "$\exists$". Therefore, in order to show that every formula is a boolean combination of formulas of a certain kind, it is enough to establish the induction step for "$\exists$". That is, we assume inductively that $\psi(v,\overline{w})$ is a boolean combination of the required sort, and show that $\exists v\, \psi(v,\overline{w})$ is also of the required sort. In other words, it is enough to show how to eliminate an existential quantifier. I will use the convention of identifying a formula with the set which it defines in an (unspecified) module.

Before embarking on the proof, let me introduce some notation and terminology.

If κ, λ are cardinals write $\kappa \doteq \lambda$ if both are infinite or if $\kappa=\lambda$: also set $\kappa \leqslant \lambda$ if both are infinite or if $\kappa \leqslant \lambda$.

Let M be any module and suppose that φ, ψ are pp formulas (in the same number of free variables). Set $\mathrm{Inv}(M,\varphi,\psi) \doteq |\varphi(M)/\varphi(M)\cap\psi(M)|$. Of course $\mathrm{Inv}(M,\varphi,\psi)=\mathrm{Inv}(M,\varphi,\varphi\wedge\psi)$ and, in fact, I will normally use the notation with the implicit assumption that ψ implies φ.

The following observation is very important: the statement $\mathrm{Inv}(-,\varphi,\psi)>k$ $(k\in\omega)$ is an elementary one (it is in fact $\exists\forall$). For one has $\mathrm{Inv}(M,\varphi,\psi)>k$ iff $M\models \forall \overline{v}_1,\dots,\overline{v}_k\ \exists\overline{v}\,(\varphi(\overline{v})\wedge \bigwedge_{i=1}^{n}\neg\psi(\overline{v}-\overline{v}_i))$. Consequently, for any finite k, the property $\mathrm{Inv}(-,\varphi,\psi)=k$ is expressed by the formula $\mathrm{Inv}(-,\varphi,\psi)>k \wedge \neg\mathrm{Inv}(-,\varphi,\psi)>k+1$ and, also, the property of $\mathrm{Inv}(-,\varphi,\psi)$ being infinite is an (infinitely) axiomatisable one. Hence, for T a complete theory of modules, one may define $\mathrm{Inv}(T,\varphi,\psi)$, without ambiguity, to be $\mathrm{Inv}(M,\varphi,\psi)$, where M is any model of T. I will use the term **invariants** for these. If a given invariant is infinite, its actual value usually is not of interest and I simply say that its value is "∞" and, especially after this chapter, correspondingly use "$=$" instead of "$\doteq$".

These invariants - a given finite value or infinity - are therefore invariants for any complete theory of modules. They may be discerned in [Bau76], are nearer the surface in [Mon75] and [Mrt75] and were explicitly noted and used by Garavaglia ([Gar7?] and subsequent papers). (In their more particular form for abelian groups, they are already in [Sz55].)

Exercise 1 Define the following relation on intervals in the lattice of pp-definable subgroups ([Zg84; §1]): if $\varphi\geqslant\varphi'\geqslant\psi'\geqslant\psi$ then $[\varphi,\psi]\geqslant[\varphi',\psi']$; if $\varphi\geqslant\psi$ and θ is arbitrary, then $[\varphi,\psi]\geqslant[\varphi+\theta,\psi+\theta]$ and $[\varphi,\psi]\geqslant[\varphi\wedge\theta,\psi\wedge\theta]$; then extend to a transitive relation. The main consequence is that if $[\varphi,\psi]\geqslant[\varphi',\psi']$ then the second interval is no more complex (lattice-theoretically) than the first.

Show that if $[\varphi,\psi]\geqslant[\varphi',\psi']$ and if M is any module, then $\mathrm{Inv}(M,\varphi,\psi)\geqslant\mathrm{Inv}(M,\varphi',\psi')$.

Example 1 Take $R=\mathbb{Z}$ and $M=\mathbb{Z}_{p^\infty}$. Take $\varphi(v)$ to be "$v=v$" and $\psi(v)$ to be "$vp=0$". Also let $\theta(v)$ be "$v=0$".

Then $\mathrm{Inv}(M,\varphi,\psi)=\aleph_0$ (or just "∞" if we care only that the invariant is infinite); $\mathrm{Inv}(M,\psi,\theta)=p$; $\mathrm{Inv}(M^2,\psi,\theta)=p^2$; ... $\mathrm{Inv}(M^{(\aleph_0)},\psi,\theta)=\infty$.

Exercise 2

(i) Show that if $M\equiv M^{(\aleph_0)}$ (or simply if $M\equiv M^2$) then each invariant of M is either 1 or infinite.

(ii) Show that if R is an algebra over an infinite central subfield and if M is any R-module then each invariant of M is either 1 or infinite.

Exercise 3 [Kue73] Let $R=\mathbb{Z}$ and let p be a prime. If the sentence $\mathrm{Inv}(-,vp=0,v=0)>1 \wedge \mathrm{Inv}(-,v=v,p|v)=1$ is satisfied in the abelian group M then that group is infinite. Find a "dual" sentence.

It will follow from the argument below that every sentence in the language of modules is equivalent to a boolean combination of **invariants conditions** (i.e., assertions that certain invariants are less than/equal to/or greater than certain integers). I will use the term **invariants statement** to mean a boolean combination of invariants conditions.

Let us proceed with the proof, assuming inductively that $\psi(v,\overline{w})$ is a boolean combination of pp formulas. A little elementary set theory allows one to see that the set $\psi(v,\overline{w})$ may therefore be expressed as $\bigcup_j (\varphi_j(v,\overline{w}) \cap \bigcap_i \neg\psi_{ij}(v,\overline{w}))$ where the φ_j and ψ_{ij} are pp (I am using the fact that an intersection of pp formulas is pp). Therefore $\exists v\, \psi(v,\overline{w})$ is equivalent to the formula $\exists v \bigvee_j (\varphi_j(v,\overline{w}) \wedge \bigwedge_i \neg\psi_{ij}(v,\overline{w}))$. Now, there exists a point in a union of sets iff there exists a point in one of the sets, so the latter formula is equivalent to $\bigvee_j \exists v (\varphi_j(v,\overline{w}) \wedge \bigwedge_i \neg\psi_{ij}(v,\overline{w}))$. Therefore, it will be enough to show how to eliminate the quantifier from a formula of the form $\exists v (\varphi(v,\overline{w}) \wedge \bigwedge_{i=1}^{n} \neg\psi_i(v,\overline{w}))$ where φ and the ψ_i are pp.

Let us imagine first that we are working inside some definite module. Then we will see just what information about the module we used for the elimination procedure.

It may be assumed that $\psi_i(v,\overline{w}) \to \varphi(v,\overline{w})$ for each i. For any $\overline{b}$ in the module M, the formula $\varphi(v,\overline{b}) \wedge \bigwedge_i \neg\psi_i(v,\overline{b})$ simply defines the "set" $\varphi(v,\overline{b}) \setminus \bigcup_i \psi_i(v,\overline{b})$ (i.e., $\varphi(M,\overline{b}) \setminus \bigcup_i \psi_i(M,\overline{b})$). So what we have to do is express the condition "$\varphi(v,\overline{b}) \setminus \bigcup_i \psi_i(v,\overline{b})$ is non-empty" as a boolean combination of pp formulas; equivalently, we must express the condition "$\varphi(v,\overline{b}) \setminus \bigcup_i \psi_i(v,\overline{b})$ is empty" as such a boolean combination.

By Neumann's Lemma (**2.12**) one may drop from the union, $\bigcup_i \psi_i(v,\overline{b})$, those $\psi_i(v,\overline{b})$ such that $\psi_i(v,\overline{0})$ is of infinite index in $\varphi(v,\overline{0})$, without changing the truth of the statement "$\varphi(v,\overline{b}) \setminus \bigcup_i \psi_i(v,\overline{b})$ is empty". Note that whether $\psi_i(v,\overline{b})$ is dropped depends only on $\mathrm{Inv}(M,\varphi(v,\overline{0}),\psi_i(v,\overline{0}))$ (this is independent of $\overline{b}$) and on whether $\psi_i(v,\overline{b})$ is empty (that is, on whether $\overline{b}$ satisfies the pp condition $\exists v\, \psi_i(v,\overline{b})$). I now claim that the truth of "$\varphi(v,\overline{b}) \setminus \bigcup_i \psi_i(v,\overline{b})$ is empty" depends only on the following data:

(i) the invariants $\mathrm{Inv}(M,\theta(v,\overline{0}),\chi(v,\overline{0}))$ where θ is either φ or a conjunction of certain of the ψ_i and where χ is a conjunction of certain of the ψ_i (this is independent of $\overline{b}$);

(ii) the "pattern of intersections" of the cosets $\psi_i(v,\overline{b})$ - by this I mean the formulas of the form $\pm\exists v \bigwedge_K \psi_k(v,\overline{b})$ where K is a subset of the full index set I (this depends on which of the pp conditions, $\exists v \bigwedge_K \psi_k(v,\overline{w})$, are satisfied by $\overline{b}$).

For, one is asking whether a certain coset is covered by a finite collection of "sub-cosets" all of which have finite index in it. So I am claiming that all one needs to know are the indices, in one another, of intersections of the various subgroups involved, together with information about which of the cosets of these are disjoint. With the right picture in mind this is "obvious" so, rather than make a long digression here to prove this rigourously, I have placed the detailed proof below (**2.14**) and encourage the reader to accept it for the moment.

Therefore, in order to determine whether the set $\varphi(v,\overline{b}) \setminus \bigcup_i \psi_i(v,\overline{b})$ is empty we need to know:

(i) about certain of the invariants of M, and

(ii) some of the pp-type of $\overline{b}$

(the exact requirements may be extracted from the discussion above).

If the set is to be empty then, by Neumann's Lemma, there are only finitely many possibilities for the values of the invariants; for any subgroup with index in $\varphi(v,\overline{0})$ strictly greater than $n!$ may be discounted. Also, for each such possibility, there are only finitely many partial pp-types for $\overline{b}$ in the formulas φ, ψ_i which make the set empty.

Therefore, there is a formula $\sigma(\overline{w})$ (a disjunction of formulas of the sort: boolean combinations of invariants conditions and boolean combinations of pp formulas) such that in any module M one has $\forall\overline{w}\,(\exists v\,(\varphi(v,\overline{w}) \wedge \bigwedge_i \neg\psi_i(v,\overline{w})) \leftrightarrow \sigma(\overline{w}))$.

Let me now state formally what has been established. The result (indeed, a more specific one) was established by Szmielew for abelian groups [Sz55; 4.22]. This was extended by Eklof and Fisher [EF72; §5] to Dedekind domains. Fisher attempted, using "structural" arguments, to prove it for modules over any ring but was successful only in some very special cases [Fis75]. Baur succeeded in proving the general result [Bau76; Thm], using the combinatorics plus a back-and forth argument. Independently, Monk [Mon75] gave a proof for abelian groups which works just as well over any ring. Also independently, Mart'yanov [Mrt75; Thm 1] proved the corollary 2.15 below for abelian groups with a finite number of linear operators and specified subgroups (which implies the result for modules over any ring). The proof given here is like that in [Mon75] and [Zg84].

<u>Corollary 2.13</u> *Let R be any ring. If $\chi(\bar{v})$ is any formula in the language of R-modules then there is an invariants statement σ and a boolean combination $\theta(\bar{v})$ of pp formulas such that $\mathcal{M}_R \models \forall \bar{v}\, (\chi(\bar{v}) \leftrightarrow \sigma \wedge \theta(\bar{v}))$.* □

A rigorous proof of the claim which I made above is now given ([Gar78; Lemma6] is essentially this; a related but somewhat different proof is in [Zg84]).

The following notation will be useful. Let $Y = \bigcup_\lambda Y_\lambda$ be a representation of a set Y as a union of cosets Y_λ (of various subgroups of a given group) and let X be a coset such that the corresponding subgroup, X^0, is contained in each subgroup $Y_\lambda{}^0$ corresponding to a coset, Y_λ, involved in the union. Then define $[Y:X]$ to be the number of translates of X (i.e., cosets of X^0) required to cover Y. Also set $[\emptyset : X] = 0$.

The following properties are obvious.

(c1) If Y is itself a coset then $[Y:X] = [Y^0 : X^0]$.

(c2) If Z is a coset with $Z^0 \leq X^0$ then $[Y:Z] = [Y:X].[X:Z]$.

(c3) If Y and Y' are unions of cosets with $[Y:X]$ and $[Y':X]$ both defined, then $[Y \cup Y' : X] = [Y:X] + [Y':X] - [Y \cap Y' : X]$ provided either side is finite.

<u>Lemma 2.14</u> *Let $H_1, \ldots, H_n, K_1, \ldots, K_n$ be groups with the H_i subgroups of G (say), and the K_i subgroups of G' (say). For each $i = 1, \ldots, n$ let C_i, respectively D_i, be a coset of H_i, respectively K_i, if non-empty. Suppose that the following two conditions are satisfied:*

(a) for all $I \subseteq J \subseteq \{1, \ldots, n\}$ the indices $[\bigcap_I H_i : \bigcap_J H_j]$ and $[\bigcap_I K_i : \bigcap_J K_j]$, if finite, are equal;

(b) for every $I \subseteq \{1, \ldots, n\}$ one has $\bigcap_I C_i = \emptyset$ iff $\bigcap_I D_i = \emptyset$.

Then $[\bigcup_1^n C_i : \bigcap_1^n H_i] = [\bigcup_1^n D_i : \bigcap_1^n K_i]$. In particular, $G = \bigcup_i C_i$ iff $G' = \bigcup_i D_i$.

Proof The proof is by induction on n, the case $n = 1$ being clear by condition (b).

So assume that the result holds for all such situations where the number (n) of cosets is k. Then:

$[\bigcup_1^{k+1} C_i : \bigcap_1^{k+1} H_i] = [\bigcup_1^k C_i : \bigcap_1^{k+1} H_i] + [C_{k+1} : \bigcap_1^{k+1} H_i] - [(\bigcup_1^k C_i) \cap C_{k+1} : \bigcap_1^{k+1} H_i]$ (by (c3))

$= [\bigcup_1^k C_i : \bigcap_1^k H_i].[\bigcap_1^k H_i : \bigcap_1^{k+1} H_i] + [C_{k+1} : \bigcap_1^{k+1} H_i] - [\bigcup_1^k (C_i \cap C_{k+1}) : \bigcap_1^k (H_i \cap H_{k+1})]$
(by (c2))

$= [\bigcup_1^k D_i : \bigcap_1^k K_i].[\bigcap_1^k K_i : \bigcap_1^{k+1} K_i] + [D_{k+1} : \bigcap_1^{k+1} K_i] - [\bigcup_1^k (D_i \cap D_{k+1}) : \bigcap_1^k (K_i \cap K_{k+1})]$

$= \ldots$

$= [\bigcup_1^{k+1} D_i : \bigcap_1^{k+1} K_i]$ (reversing the steps), as required.

The main step (the third equality) has equality between its corresponding terms justified by the induction hypothesis in conjunction with assumptions (a) and (b). □

<u>Corollary 2.15</u> *Every sentence in the language of R-modules is equivalent, modulo the theory of R-modules, to a boolean combination of invariants conditions.* □

Corollary 2.16 *Given a complete theory* T *of* R*-modules, every formula is equivalent modulo* T *to a boolean combination of pp formulas.* □

Thus, working modulo a complete theory of modules, a formula $\varphi(\overline{v})$ may be supposed to have the form $\bigvee_k(\bigwedge_j(\varphi_{jk}(\overline{v}) \wedge \bigwedge_i \neg\psi_{ijk}(\overline{v})))$, where the φ_{jk} and ψ_{ijk} are pp.

Many consequences of 2.15 will be given in §5. The consequences of 2.16 will occupy us for the remainder of these notes. Another way of stating 2.16 is the following.

Corollary 2.17 [Bau76; Lemma2] *Let* M *be a module and let* $\overline{a}$ *and* $\overline{b}$ *be in* M. *Then* $\mathrm{tp}^M(\overline{a}) = \mathrm{tp}^M(\overline{b})$ *iff* $\mathrm{pp}^M(\overline{a}) = \mathrm{tp}^M(\overline{b})$. *That is, elements (or tuples) have the same type iff they have the same pp-type.* □

We now have the following extremely useful criterion for elementary equivalence of modules. It is stated explicitly in [Gar79; Thm 2].

Corollary 2.18 $M \equiv M'$ *iff* $\mathrm{Inv}(M,\varphi,\psi) \doteq \mathrm{Inv}(M',\varphi,\psi)$ *for every pair* φ,ψ *of pp formulas in one free variable (with* $\psi \rightarrow \varphi$).

Proof This is immediate from 2.15 (for the bracketed statement, just replace ψ by $\psi \wedge \varphi$). □

The criterion of 2.18 is one which may often be checked (see, for instance, the examples at the end of the section). Of course, to check it, one needs some understanding of general pp formulas over the base ring (as is the case with $\mathbb{Z}$) or, since it is true that the set of all $\mathrm{Inv}(-,\varphi,\psi)$ is highly redundant, it may be that, for a particular class of modules, one can reduce those invariants which have to be checked to a more managable collection, perhaps one with some direct algebraic significance (e.g., as in the case of Σ-injective modules (§6.1 and [Pr82]), modules over valuation domains (§10.V), or modules over regular rings (§16.2)).

As a further corollary one has the following.

Corollary 2.19 *Every complete theory of modules may be axiomatised by sentences of the form* $\mathrm{Inv}(-,\varphi,\psi)=k$ *and* $\mathrm{Inv}(-,\varphi,\psi)>k$ *(with* $k\in\omega$) *where* φ,ψ *are pp formulas in one free variable.*

Proof $M \equiv M'$ iff $\mathrm{Inv}(M,\varphi,\psi) \doteq \mathrm{Inv}(M',\varphi,\psi)$ for all such φ,ψ - and this happens iff (i) $\mathrm{Inv}(M,\varphi,\psi)=k$ whenever $\mathrm{Inv}(M',\varphi,\psi)=k$, and (ii) $\mathrm{Inv}(M,\varphi,\psi)\geqslant k$ whenever $\mathrm{Inv}(M',\varphi,\psi)\geqslant k$, for all such φ,ψ. (Since the theory is complete, disjunctions need not be considered.) □

Sabbagh [Sab70a; Cor 3 to Thm 4] had already shown that every complete theory of modules is axiomatisable by sentences involving no more than one change of quantifier (from "∃" to "∀" or *vice versa*) and that any theory of modules may be axiomatised by sentences involving no more than two changes of quantifier. These results follow from 2.18 and 2.15 respectively, and the results here are more informative regarding the content of these sentences.

The expression of 2.16 which will be used most frequently is the following one. Recall that for $p\in S(A)$ $p^+=\{\varphi : \varphi$ is pp and $\varphi\in p\}$: also set $p^-=\{\varphi : \varphi$ is pp with parameters in A and $\varphi\notin p\}$ and $\neg p^-=\{\neg\varphi : \varphi\in p^-\}$. We also set $\tilde{p}=\{\chi : \chi$ is a boolean combination of pp formulas and $\chi\in p\}$.

Theorem 2.20 *Let* T *be a complete theory of modules and let* $A\subseteq M\vDash T$. *Suppose that* p *is a type over* A.
Then $p^+\cup\neg p^-$ *proves* p.
In particular, the map $S(A)\longrightarrow S^+(A)$ *defined by* $p\longmapsto p^+$ *is a bijection.* □

Exercise 4 Determine all complete theories of abelian groups satisfying $T=T^{\aleph_0}$ (see §2.$\mathbb{Z}$).

Monk [Mon75] noted that his proof of pp-elimination of quantifiers (for abelian groups) gives a primitive recursive procedure for replacing an arbitrary formula by an equivalent boolean combination of invariants conditions and pp formulas. Weispfenning [Wei83a] works through the pp-elimination of quantifiers for abelian structures in general, noting that for countable languauges the procedure is primitive recursive.

The next results, which arose in discussion with Macintyre and Point, show that the elimination of quantifiers is even to some extent independent of the ring.

Corollary 2.21 *Let* $\chi(\bar{v})$ *be a formula in the language of* R*-modules. Then there is an invariants statement* σ *and a boolean combination* $\theta(\bar{v})$ *of pp formulas, which involve only the "ring elements" (i.e., functions, allowing formal addition and composition) appearing in* χ, *such that* $\mathcal{M}_R \models \forall\bar{v}\,(\chi(\bar{v}) \leftrightarrow \sigma \wedge \theta(\bar{v}))$.

Proof Let R_1 be the subring of R generated by the elements appearing in χ (we may assume that 1 appears). Any R-module may be considered as an R_1-module *via* the inclusion $R_1 \hookrightarrow R$.

Apply 2.13 to $\mathcal{M}_{R_1}$ to obtain σ, θ of the form required. Then, since any R-module is *a priori* an R_1-module, one has from $\mathcal{M}_{R_1} \models \forall\bar{v}\,(\chi(\bar{v}) \leftrightarrow \sigma \wedge \theta(\bar{v}))$ that, in particular, $\mathcal{M}_R \models \forall\bar{v}\,(\chi(\bar{v}) \leftrightarrow \sigma \wedge \theta(\bar{v}))$. □

Corollary 2.22 *Let* $\chi^{\bar{r}}(\bar{v})$ *be a formula in the language of modules over rings, where the "ring element" (function) symbols ocurring in* χ *are* $\bar{r}=(r_0,\dots,r_n)$ *with* r_0 *the identity function.*
Let R^* *be the free ring on* n *generators* $e_0=1,e_1,\dots,e_n$ *and let* $\chi^{\bar{e}}(\bar{v})$ *be the formula obtained by replacing* $(r_0,\dots,r_n)$ *by* $\bar{e}=(e_0,\dots,e_n)$. *Choose, by* 2.13, *a sentence* $\sigma^{\bar{e}}$ *and a boolean combination* $\theta^{\bar{e}}(\bar{v})$ *of pp formulas such that* $\mathcal{M}_{R^*} \models \forall\bar{v}\,(\chi^{\bar{e}}(\bar{v}) \leftrightarrow \sigma^{\bar{e}} \wedge \theta^{\bar{e}}(\bar{v}))$.
Then for any ring R *and any* $1+n$*-tuple* $\bar{s}=(s_0=1,s_1,\dots,s_n)$ *of elements of* R, $\mathcal{M}_R \models \forall\bar{v}\,(\chi^{\bar{s}}(\bar{v}) \leftrightarrow \sigma^{\bar{s}} \wedge \theta^{\bar{s}}(\bar{v}))$.

Proof Let R_1 be the subring of R generated by $\{s_0,\dots,s_n\}$. Let $R^* \twoheadrightarrow R_1$ be the ring morphism taking e_i to s_i. Consider R_1-modules as R^*-modules *via* this morphism. Then $\mathcal{M}_{R^*} \models \forall\bar{v}\,(\chi^{\bar{e}}(\bar{v}) \leftrightarrow s^{\bar{e}} \wedge \theta^{\bar{e}}(\bar{v}))$ implies $\mathcal{M}_{R_1} \models \forall\bar{v}\,(\chi^{\bar{s}}(\bar{v}) \leftrightarrow \sigma^{\bar{s}} \wedge \theta^{\bar{s}}(\bar{v}))$ (the action of $\bar{e}$ on R_1-modules being given by $\bar{s}$).

Now, consider R-modules as R_1-modules *via* the inclusion $R_1 \hookrightarrow R$. In the same way, we deduce $\mathcal{M}_R \models \forall\bar{v}\,(\chi^{\bar{s}}(\bar{v}) \leftrightarrow \sigma^{\bar{s}} \wedge \theta^{\bar{s}}(\bar{v}))$, as required. □

That is, if the elimination of quantifiers is done over the free ring, then the "same" sentence σ and formula θ may be used over any ring (of course, it need not be the case that an elimination of quantifiers performed over R may be pulled back to the free ring).

Suppose that M is some structure for a certain language. Let $B(M)$ denote the poset of all subsets of M which are definable with parameters from M. Then $B(M)$ is an atomic boolean algebra. Marcja and Toffalori [MT84] consider the connection between a theory T and the various $B(M)$ for M a model of T. In particular, they say that a countable theory T is p-$\aleph_0$-categorical if, for any two countable models M and M' of T, one has $B(M) \simeq B(M')$.

In [Tof86], Toffalori shows that any countable theory of modules is p-$\aleph_0$-categorical. His proof is quite long and uses the structure theory developed by Ketonen [Ket78] for countable atomic boolean algebras (as well as pp-elimination of quantifiers and Neumann's Lemma).

Subsequently, Piron [Pir87] gave a simpler proof, using a criterion of Vaught for partial isomorphism of boolean algebras. He showed that, over any ring, if the modules M and M' are elementarily equivalent, then $B(M)$ and $B(M')$ are partially isomorphic (so, if countable, are isomorphic). (Exercise: [Pir87] show that $B(M) \simeq B(M')$ does not imply $M \equiv M'$.)

Examples 2

(i) $R=\mathbb{Z}$, $M=\mathbb{Z}_{(p)}$, $M'=\mathbb{Z}_{(p)}\oplus\mathbb{Q}$. Since (§2.$\mathbb{Z}$) every pp formula reduces to annihilator and divisibility statements, it is easy to check that M and M' have the "same" invariants and hence $\mathbb{Z}_{(p)}\equiv\mathbb{Z}_{(p)}\oplus\mathbb{Q}$. For both modules are torsionfree, so only simple divisibility need be considered. Moreover, in each module every element is divisible by every prime not equal to p. So it need only be checked that
$\mathrm{Inv}(\mathbb{Z}_{(p)}, "p^n|v", "p^{n+1}|v")=\mathrm{Inv}(\mathbb{Z}_{(p)}\oplus\mathbb{Q}, "p^n|v", "p^{n+1}|v")$ for each $n\in\omega$. That this is so follows by 2.10, since $\mathbb{Q}p^n=\mathbb{Q}$ for each n.
Also, if $a=p\in\mathbb{Z}_{(p)}$ and $\bar{a}=(p,1)\in\mathbb{Z}_{(p)}\oplus\mathbb{Q}$ then it is easily seen that $\mathrm{pp}^M(a)=\mathrm{pp}^{M'}(a')$ and hence $\mathrm{tp}^M(a)=\mathrm{tp}^{M'}(a')$ (by 2.17).

(ii) $R=\mathbb{Z}$, $T=\mathrm{Th}(\mathbb{Z}_{p^\infty})$. Any model of T must be divisible and without q-torsion for all primes $q\neq p$, so must have the form $\mathbb{Z}_{p^\infty}{}^{(\kappa)}\oplus\mathbb{Q}^{(\lambda)}$ for suitable κ,λ.
Since $\mathrm{Inv}(\mathbb{Z}_{p^\infty}, vp=0, v=0)=p$, it must be that $\kappa=1$. Since $\mathbb{Z}_{p^\infty}$ is infinite there must be some λ with $\mathbb{Z}_{p^\infty}\oplus\mathbb{Q}^{(\lambda)}\equiv\mathbb{Z}_{p^\infty}$ (for T has models in all infinite cardinalities!). Now, the endomorphism ring of $\mathbb{Q}$ is just $\mathbb{Q}$ so, by 2.1, $\mathbb{Q}$ has no non-trivial proper pp-definable subgroups: in particular each invariant of $\mathbb{Q}$ is either 1 or ∞. Hence $\mathbb{Q}^{(\mu)}\equiv\mathbb{Q}^{(\lambda)}$ for all non-zero μ,λ. Therefore (2.18) $\mathbb{Z}_{p^\infty}\oplus\mathbb{Q}^{(\lambda)}$ is a model of T for every value of $\lambda\geqslant 0$; so these are precisely the models of T.

(iii) $R=\mathbb{Z}$, $T=\mathrm{Th}(M=\mathbb{Z}_2{}^3\oplus\mathbb{Z}_4{}^{(\aleph_0)})$. Since T contains the sentence $\forall v\,(v4=0)$, every model of T is an abelian group of exponent 4 and hence is a sum of copies of $\mathbb{Z}_2$ and $\mathbb{Z}_4$. Note that $\mathrm{Inv}(M, v2=0, v2=0\wedge 2|v)=8$ since $|(\mathbb{Z}_2{}^3\oplus\mathbb{Z}_4{}^{(\aleph_0)})2/\mathbb{Z}_4{}^{(\aleph_0)}2)|=|\mathbb{Z}_2{}^3|=2^3$. Therefore one cannot add copies of $\mathbb{Z}_2$ to M and retain a model of T. On the other hand, every invariant of $\mathbb{Z}_4{}^{(\aleph_0)}$ is 1 or infinite (since we have infinitely many copies of $\mathbb{Z}_4$) - here we are using the result (2.10) that pp formulas commute with direct sum. Thus a sum of copies of $\mathbb{Z}_4$ is elementarily equivalent to $\mathbb{Z}_4{}^{(\aleph_0)}$ iff it is a sum of infinitely many copies.
Hence the models of T are precisely the $\mathbb{Z}_2{}^3\oplus\mathbb{Z}_4{}^{(\kappa)}$ with $\kappa\geqslant\aleph_0$.

Exercise 5 Let $R=K[x,y : x^2=y^2=xy=yx=0]$ with K an infinite field (see Ex 2.1/6(vi)). Show that the models of $\mathrm{Th}(R)$ are just the sums $R^{(\kappa)}$ ($\kappa\geqslant 1$) of copies of R. (This exercise does become easier with some of the tools developed later.)

2.5 Immediate corollaries of pp-elimination of quantifiers

A number of the corollaries presented in this section were obtained before pp-elimination of quantifiers was available. Proofs which appeal to pp-elimination of quantifiers are generally much shorter than the original ones.

First we see how the invariants behave with respect to pure embeddings, direct sums and products. Like many results of this section these "computational aids" will often be used without explicit reference.

Lemma 2.23 *Let M,N,M_i $(i\in I)$ be any modules.*

(*a*) *If* $M\prec^{+}N$ *then* $\mathrm{Inv}(N,\varphi,\psi)=\mathrm{Inv}(M,\varphi,\psi).\mathrm{Inv}(N/M,\varphi,\psi)$ *for all pp* φ,ψ *and so* $N\equiv M\oplus(N/M)$. *In particular,* $\mathrm{Inv}(M,\varphi,\psi)\leqslant\mathrm{Inv}(N,\varphi,\psi)$.

(*a*)′ *If* $M\prec N$ *then* M *is elementarily equivalent to* $M\oplus(N/M)$: *if, moreover,* $N\prec N'$ *then* $N/M\prec^{+}N'/M$.

(*b*) $\mathrm{Inv}(M\oplus N,\varphi,\psi)=\mathrm{Inv}(M,\varphi,\psi).\mathrm{Inv}(N,\varphi,\psi)$.

(*c*) $\mathrm{Inv}(\bigoplus_i M_i,\varphi,\psi)\doteq\prod_i\mathrm{Inv}(M_i,\varphi,\psi)=\mathrm{Inv}(\prod_i M_i,\varphi,\psi)$.

Proof (c) This follows from 2.10. That result gives $\varphi(\bigoplus M_i)/\psi(\bigoplus M_i)=\bigoplus_i(\varphi(M_i)/\psi(M_i))$, and similarly for products. Then just treat the various cases: (i) for some i, $\varphi(M_i)/\psi(M_i)$ is infinite; (ii) for infinitely many i, $|\varphi(M_i)/\psi(M_i)|>1$; (iii) not (i) nor (ii).

(b) This is a special case of (a) (and (c)).

(a) One may use the fact (see **2.27**) that $M \prec \overline{M}$ then, as in [Sab70a; Cor 3 to Thm 1], use ultraproducts to split the pure-exact sequence $M \hookrightarrow N \twoheadrightarrow N/M$. Alternatively, one may argue directly as follows (compare [Zg84; proof of 2.2(1)]).

It is claimed that if M is pure in N then $\varphi(N)/\varphi(M) = \varphi(N/M)$ for every pp formula φ: clearly this will be sufficient. Certainly if $N \models \varphi(\overline{a})$ then $N/M \models \varphi(\pi\overline{a})$, where $\pi: N \twoheadrightarrow N/M$ is the canonical projection. Noting that, since M is pure in N one has $\varphi(M) = M \cap \varphi(N)$, it follows that $\varphi(N)/\varphi(M) \leqslant \varphi(N/M)$.

Now suppose that $N/M \models \varphi(\pi\overline{a})$ - say this formula is $\exists \overline{w}\, \theta(\overline{w}, \pi\overline{a})$ where θ is $\bigwedge$-atomic. Let $\overline{b}$ in N be such that $N/M \models \theta(\pi\overline{b}, \pi\overline{a})$. Regard the zero element of each equation of θ as a separate parameter: so replace $\theta(\overline{w}, \overline{v})$ by $\theta_0(\overline{w}, \overline{v}, \overline{0})$, say. Then one has $N/M \models \theta(\pi\overline{b}, \pi\overline{a}, \overline{0})$.

It follows (since θ_0 is a conjunction of equations and by definition of the quotient N/M) that there is $\overline{c}$ in $M = \ker\pi$ with $N \models \theta_0(\overline{b}, \overline{a}, \overline{c})$. Thus $N \models \exists \overline{w}, \overline{v}\ \theta_0(\overline{w}, \overline{v}, \overline{c})$. Since M is pure in N and $\overline{c}$ is in M one has $M \models \exists \overline{w}, \overline{v}\ \theta_0(\overline{w}, \overline{v}, \overline{c})$ - say $\overline{b}', \overline{a}'$ are in M with $\theta_0(\overline{a}', \overline{b}', \overline{c})$ true (in M, equally in N). Apply **2.1** to $\theta_0(\overline{b}, \overline{a}, \overline{c})$ and $\theta_0(\overline{b}', \overline{a}', \overline{c})$ to obtain $\theta_0(\overline{b}-\overline{b}', \overline{a}-\overline{a}', \overline{0})$. In particular $N \models \exists \overline{w}\, \theta_0(\overline{w}, \overline{a}-\overline{a}', \overline{0})$ and hence $N \models \exists \overline{w}\, \theta(\overline{w}, \overline{a}-\overline{a}')$ (since $\theta_0(\overline{w}, \overline{v}, \overline{0})$ is just $\theta(\overline{w}, \overline{v})$). That is, $N \models \varphi(\overline{a}-\overline{a}')$. But $\pi(\overline{a}-\overline{a}') = \pi\overline{a}$ since $\overline{a}'$ is in $M = \ker\pi$.

Thus we have shown $\pi\overline{a} \in \varphi(N)/\varphi(M)$, as required. The second assertion follows by **2.18**.

(a)′ The first part is immediate from (a), (b) and **2.18**. The second is just a special case of the (well-known) fact that if $A \leqslant B \prec^{*} C$ then $B/A \prec^{*} C/A$ (the method of the proof of (a) shows it). □

Exercise 1 Show that $\overline{\mathbb{Z}(p)}/\mathbb{Z}_{(p)}$ is a direct sum of copies of $\mathbb{Q}$.
[Hint: note that $\mathbb{Z}_{(p)}$ is pure in $\overline{\mathbb{Z}(p)}$.]

Exercise 2 Show that if A, B are abelian groups and if $A \oplus A \equiv B \oplus B$ then $A \equiv B$.

<u>Corollary 2.24</u> [SE71; §1.1], [Sab70a; Cor 2 to Thm 4] *Let* M, N, M_i $(i \in I)$ *be any modules.*

(a) $\bigoplus_i M_i \equiv \prod_i M_i$.

(b) $M^{(\kappa)} \equiv M^{(\aleph_0)} \equiv M^{\aleph_0} \equiv M^{\kappa}$ *for any infinite* κ.

Proof This is immediate from **2.23** and **2.18**. □

<u>Proposition 2.25</u> [Sab71] *If* $M \prec^{*} N \prec^{*} M'$ *and* $M \equiv M'$ *then* $M \prec N$ *(and* $N \prec M'$*).*

Proof By **2.23**(a), one has $\mathrm{Inv}(M,\varphi,\psi) \leqslant \mathrm{Inv}(N,\varphi,\psi) \leqslant \mathrm{Inv}(M',\varphi,\psi) \doteq \mathrm{Inv}(M,\varphi,\psi)$ for any pp φ, ψ. So by **2.18** one concludes $M \equiv N$.

Let $\overline{a}$ be in M. Since M is pure in N one has, by definition, that $\mathrm{pp}^M(\overline{a}) = \mathrm{pp}^N(\overline{a})$. So by **2.17** one concludes that $\mathrm{tp}^M(\overline{a}) = \mathrm{tp}^N(\overline{a})$. Thus $M \prec N$. □

Exercise 2 [Sab70a; Thm 4] If $M \prec^{*} N' \equiv N \prec^{*} M' \equiv M$ then $M \equiv N$.

<u>Corollary 2.26</u> Fisher: see [Ek72; p341]; also [EF72; 2.5] for abelian groups, and [Sab71; Thm 2] *If* M *is elementarily equivalent to* N *then any pure embedding of* M *into* N *is an elementary embedding.* □

<u>Corollary 2.27</u> [Sab70a; Cor 4 to Thm 4] *Every module is elementarily equivalent to, indeed, is an elementary substructure of, its pure-injective hull.*

Proof This follows since any sufficiently saturated module is pure-injective, by **2.25** and then by definition of pure-injective hull (see §4.2). □

Corollary 2.28 *For any set* $\{M_i\}_i$ *of modules,* $\oplus_i M_i \prec \prod_i M_i$.

Proof Taking $M=\oplus_i M_i$ and $N=\prod_i M_i$ in 2.26, this follows from 2.24. □

Proposition 2.29 *For any module* M *one has* $M \equiv M^{(\aleph_0)}$ *iff* $M \equiv M^2$.

Proof This is immediate from 2.18 and 2.23. □

If $T=\mathrm{Th}(M)$ and $M \equiv M^{(\aleph_0)}$ then write $T=T^{\aleph_0}$ and say that T is **closed under products**. If $T=\mathrm{Th}(M)$ is any complete theory of modules then set $T^{\aleph_0}$ to be $\mathrm{Th}(M^{\aleph_0})$: this is well-defined.

Exercise 4 If T is a complete theory of modules then $T=T^{\aleph_0}$ iff the class of models of T is closed under products (in $\mathcal{M}_R$).

Exercise 5

(i) Show that the pure-injective hull of $\oplus_p \overline{\mathbb{Z}(p)}$ is $\prod_p \overline{\mathbb{Z}(p)}$, where p ranges over the positive primes.
[Hint: use the invariants and show that $\mathbb{Q}$ cannot be a direct summand of the second module.]

(ii) Identify: (a) $(\prod_p \overline{\mathbb{Z}(p)})/(\oplus_p \mathbb{Z}_{(p)})$; (b) $(\prod_p \overline{\mathbb{Z}(p)})/(\oplus_p \overline{\mathbb{Z}(p)})$, where p ranges over all the positive primes.

(iii) Describe the pure-injective hull of $\mathbb{Z}_{(p)}{}^{(\aleph_0)}$ (p fixed).

(iv) Give an example of pure-injective modules N_i ($i \in \omega$) such that the pure-injective hull of $\oplus_i N_i$ is not the product $\prod_i N_i$. There are easy examples with the N_i all Σ-pure-injective, so find an example with none of the N_i Σ-pure-injective.
[Hint: you're looking for an irreducible type which is not neg-isolated; see Ex 7.2/2.]

(Also cf. [Fra81].)

2.6 Comparison of complete theories of modules

One of the main reasons for studying modules is that they represent various other mathematical objects. For example, a representation of a group is nothing more than a module over the corresponding group algebra. It follows that it is often not in individual modules that our interest lies but, rather, in the whole category of modules (or at least in certain subcategories). Therefore, if the study of one complete theory of R-modules were entirely unrelated to the study of any other this would indicate some weakness in the subject of the model theory of modules, especially since elementary classes tend not to be algebrically "natural" ones. Fortunately, this is far from being the case, and this section is devoted to showing how one may compare the various complete theories of modules over a given ring. In §4.7 this comparison will become more subtle and deep when we introduce Ziegler's topology.

We look first at the natural ordering of complete theories of modules ([Pr80e]). It is an ordering which is most useful when restricted to those theories which satisfy $T=T^{\aleph_0}$, but it does make sense for arbitrary complete theories. Let me use an example to illustrate the idea.

Take the ring to be the ring of integers and let T_1, T_2 be the theories of $\mathbb{Z}_2{}^{(\aleph_0)}$ and $\mathbb{Z}_2{}^{(\aleph_0)} \oplus \mathbb{Z}_4{}^{(\aleph_0)}$ respectively. By considering the invariants (cf. 2.18), one sees easily that the models of these theories are respectively the $\mathbb{Z}_2{}^{(\kappa)}$ ($\kappa \geq \aleph_0$) and the $\mathbb{Z}_2{}^{(\kappa)} \oplus \mathbb{Z}_4{}^{(\lambda)}$ ($\kappa, \lambda \geq \aleph_0$) respectively. Thus every model of T_1 is a direct summand of a model of T_2 and so in some sense we have $T_1 \leq T_2$. On the other hand, our requirements for one theory to be smaller than another will not be met simply if every model of the one is contained in a model of the other - for it is only the pure embeddings which preserve pp-types. Thus, for example, if we take T_3 to be the theory of $\mathbb{Z}_4{}^{(\aleph_0)}$ - so the models are the infinite direct sums of copies of $\mathbb{Z}_4$ - then, although it is true that every model of T_1 embeds in a model of T_3, these embeddings will not be pure, so we will not consider T_1 and T_3 to be comparable in our ordering.

Suppose then that T, T' are complete theories of modules. Write $T \leqslant T'$ if the equivalent conditions of the next result are satisfied, and in that case say that T is a component of T'. That this defines a partial order on the set of complete theories of R-modules is an immediate consequence of 2.25.

<u>Proposition 2.30</u> *Let T, T' be complete theories of modules. Then the following conditions are equivalent.*
(*i*) *Some model of T is purely embedded in some model of T'.*
(*ii*) *Each model of T is purely embedded in some model of T'.*
(*iii*) $\mathrm{Inv}(T,\varphi,\psi) \leqslant \mathrm{Inv}(T',\varphi,\psi)$ *for all pp formulas φ, ψ in one free variable.*

Proof (i)⇒(ii) Suppose $M \prec^{+} M'$ with $M \models T$ and $M' \models T'$. Let M_1 be any model of T. Then there is a common elementary extension M_2 of M and M_1. It will be enough to show that M_2 purely embeds in a model of T'.

By 2.23 we have $M_2 \equiv M \oplus (M_2/M)$; by hypothesis this purely embeds in $M' \oplus (M_2/M)$. So it will be enough to show that $M' \oplus (M_2/M)$ is a model of T'; by 2.18 it must be shown that $\mathrm{Inv}(M_2/M,\varphi,\psi) > 1$ implies $\mathrm{Inv}(M',\varphi,\psi) = \infty$. Since $M \equiv M \oplus (M_2/M)$ one has by 2.18 that $\mathrm{Inv}(M,\varphi,\psi) = \infty$; since $M \prec^{+} M'$ the desired conclusion follows.

(ii)⇒(i) This is immediate.

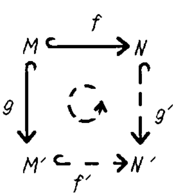
(i)⇒(iii) This is immediate by 2.23(a).

(iii)⇒(i) If $T' = (T')^{\aleph_0}$ then this is immediate. For let $M \models T$, $M' \models T'$. Then by 2.23 $\mathrm{Inv}(M \oplus M',\varphi,\psi) = \mathrm{Inv}(M,\varphi,\psi).\mathrm{Inv}(M',\varphi,\psi)$ and this is ∞ or 1 according as $\mathrm{Inv}(T',\varphi,\psi) > 1$ or $\mathrm{Inv}(T',\varphi,\psi) = 1$ (by assumption (iii)). So $M \oplus M' \models T'$ by 2.18. Therefore M is a direct summand of a model of T' as required.

For the general case, I appeal to some results of later chapters (which do not depend on 2.30 or its corollaries). Since it is the case where T and T' are closed under products which is most interesting, this early divergence from a strictly orderly presentation should not try the reader's patience unduly.

By 4.36 I may take $M \models T$, $M' \models T'$ to be discrete pure-injective models with M' being the discrete part of a $|M|^+$-saturated model. Then, by the case already dealt with, the only obstruction to M being a direct summand of M' is the possibility that there is some indecomposable direct summand, N, which occurs κ (say) times in the decomposition of M but only $\lambda < \kappa$ times in the decomposition of M'. Since M' is assumed to be very saturated (for discrete parts) this can happen only if λ is finite. If that is the case then, by 4.44, there is some T'-minimal pair φ/ψ with $\mathrm{Inv}(N,\varphi,\psi) > 1$ and $\mathrm{Inv}(T',\varphi,\psi)$ finite. But then since $\mathrm{Inv}(M,\varphi,\psi) < \mathrm{Inv}(M',\varphi,\psi)$ it must be, by 2.23, that $\kappa \leqslant \lambda$ - contradiction, as required. ◻

<u>Corollary 2.31</u> *Let $T = T^{\aleph_0}$ be a (not necessarily complete) theory of modules. Then the theory of pure submodules of models of T is axiomatised by* $\{\mathrm{Inv}(-,\varphi,\psi) = 1 : \mathrm{Inv}(M,\varphi,\psi) = 1$ *for all* $M \models T\}$. ◻



Exercise 1 Show the following (which may be used to give another proof of 2.30(i)⇒(ii). Given a diagram of embeddings of modules as shown, with g elementary and f pure, there is a completion to a commutative diagram as indicated, with g' elementary and f' pure.

Exercise 2 Determine the ordering on the set of those complete theories T of abelian groups which satisfy $T = T^{\aleph_0}$ (see §2.$\mathbb{Z}$).

Clearly $T_0 = \mathrm{Th}(0)$ is the smallest theory in the order just introduced. Is there a largest complete theory of R-modules?: indeed there is. Define $T^* = \mathrm{Th}(\oplus\{M_T : T$ is a complete theory of modules and M_T is a chosen model of $T\})$. Note that $T^* = (T^*)^{\aleph_0}$.

Proposition 2.32

(a) *T^* is the largest complete theory of modules: if T is any complete theory of modules then $T \leqslant T^*$.*

(b) *$M \models T^*$ iff $\mathrm{Inv}(M, \varphi, \varphi\wedge\psi) = \infty$ for all pp formulas φ, ψ in one free variable such that $\mathrm{Th}(\mathcal{M}_R)$ does not prove $\forall v\,(\varphi(v) \to \psi(v))$.*

Proof (a) This is immediate by construction and 2.30.

(b) The direction "$\Rightarrow$" is by construction of T^*. For if there is some module M with M not satisfying $\forall v\,(\varphi(v) \to \psi(v))$, that is with $\mathrm{Inv}(M, \varphi, \varphi\wedge\psi) > 1$, then, by the construction, $\mathrm{Th}(M^{(\aleph_0)}) \leqslant T^*$ and one has $\mathrm{Inv}(T^*, \varphi, \psi) = \infty$.

The reverse direction follows since, if M is as described and if N is any module, then necessarily $\mathrm{Inv}(M \oplus N, \varphi, \psi) \doteq \mathrm{Inv}(M, \varphi, \psi)$ for all pp φ, ψ. Thus $M \oplus N \equiv M$, whence N is a direct summand of a model of $\mathrm{Th}(M)$. Hence $\mathrm{Th}(M) = T^*$. □

This ordering on complete theories of modules is seen in [Gar79; §1]. There, Garavaglia takes a (not necessarily complete) theory T (not necessarily of modules) whose class of models is closed under direct product. Let $S(T)$ denote the set of all (complete) theories of models of T. Garavaglia shows [Gar79; Lemma 1] that $S(T)$ is a topological semigroup, where the product of T' and T'' is just $\mathrm{Th}(M' \times M'')$, where $M' \models T'$ and $M'' \models T''$, and the basic open sets have the form $U_\sigma = \{T' \in S(T) : T' \vdash \sigma\}$ where σ is a sentence. He sets $T' | T''$ if there is a model of T'' of the form $M' \oplus M$, where $M' \models T'$ (so this specialises to the ordering of this section).

Garavaglia then defines the *theory* $T' \in S(T)$ to be "T-indecomposable" if, whenever $M' \models T'$ and $M' \equiv \prod_i M_i$ ($M_i \models T$), then $M_i \models T'$ for some i. Note that a theory, rather than a structure, is indecomposable. Then he shows [Gar79; Thm 1] that every complete extension T' of T is a (possibly infinite) product of T-indecomposable members of $S(T)$.

This seems to have the flavour of Ziegler's **4.36** and perhaps neg-isolation; nevertheless, I do not see any particularly exact connections (although - see the beginning of §10.4 - in the context of m-dimension $< \infty$, two indecomposables are elementarily equivalent iff they are isomorphic).

These ideas have been extended by Nelson [Ne18?], who also proves the following: if M is any module, then its periodic power is isomorphic to $N^{(\aleph_0)}$, where the "periodic power" of M is the submodule of $M^{\aleph_0}$ consisting of all periodic functions (i.e., all $(a_i)_{i\in\omega}$ such that there exists $k \geqslant 1$ with $a_j = a_{j+k}$ for all $j \in \omega$).

The next result allows one to compare pp-types which arise from different theories.

Proposition 2.33 [Pr81; 3.1] *Let $T \leqslant T'$ be complete theories of modules. For each ordinal α, there is a natural embedding of topological spaces $j_\alpha : S_\alpha{}^T(0) \hookrightarrow S_\alpha{}^{T'}(0)$. This embedding preserves the ordering on types given by $p \leqslant q$ iff $p^+ \subseteq q^+$; also $\mathrm{im} j_\alpha$ is a closed subspace of $S_\alpha{}^{T'}(0)$ Explicitly, j_α is given on $p \in S_\alpha{}^T(0)$ by $j_\alpha p = \langle p^+ \cup \neg p^- \rangle_{T'}$, where $\langle - \rangle_{T'}$ denotes deductive closure modulo T'.*

More generally, if $A \subseteq M \models T$ then there is an entirely analogous embedding $j_\alpha{}^A : S_\alpha{}^T(A) \hookrightarrow S_\alpha{}^{T'}(A)$ (since M embeds purely in a model of T', it does make sense to think of A as a set of parameters for T').

Proof I consider the general case. Note first that if $A \subseteq M \models T$ then, choosing any pure embedding of M into a model M' of T', the pp-type of A is preserved. So $j_\alpha{}^A$ will be well-defined provided $j_\alpha{}^A p$ - the deductive closure modulo T' of $p^+ \cup \neg p^-$ in the language with parameters for A - is consistent. Since (clearly) this set decides every boolean combination of pp formulas it is, by **2.16**, already complete. Now, $p^+(M) \setminus \bigcup\{\psi(M) : \psi \in p^-\}$ is non-empty, provided M is sufficiently saturated, so the same is true with M' in place of M (for M is pure in M').

To see that $\mathrm{im} j_\alpha{}^A$ is a closed subspace of $S_\alpha{}^{T'}(A)$ (clearly $j_\alpha{}^A$ is 1-1), let $p' \in S_\alpha{}^{T'}(A)$ be in the closure of $\mathrm{im} j_\alpha{}^A$ (in the topology described in §1.1). Let $\chi(\bar{v}, \bar{a}) \in p'(\bar{v})$ be a boolean combination of pp formulas. By assumption there is $p \in S_\alpha{}^T(A)$ with $\chi(\bar{v}, \bar{a}) \in j_\alpha{}^A p$ - hence with $\chi(\bar{v}, \bar{a}) \in p$ (note that $\breve{p} = (j_\alpha{}^A p)\breve{}$). In particular $T \cup \{\exists \bar{v}\, \chi(\bar{v}, \bar{a})\}$ is consistent, and this is so for every $\chi \in (p')\breve{}$. Hence $T \cup (p')\breve{}(\bar{v})$ is consistent. Thus $p'(\bar{v}) = j_\alpha{}^A(p(\bar{v}))$ for some $p \in S_\alpha{}^T(0)$ - as required. □

Exercise 3 Describe explicitly the maps j for some examples (e.g. for $T_3 \leqslant T_2$, with notation as at the beginning of the section).

The theory T^* makes an appearance in the following context.

In [EkMz84] Eklof and Mez study the ideal structure of existentially complete (e.c.) and, more generally, algebraically complete (a.c.) D-algebras, where D is a commutative ring. One of their key lemmas is that, if R is an a.c. D-algebra and if $a, b \in R$, then $b \in RaR$ iff the pp-type of b in R_D contains the pp-type of a. This connection between the ideal structure of an a.c. D-algebra and its underlying D-module structure gives rise to the question of which D-modules can be realised as the underlying D-module of an a.c. or e.c. D-algebra. In [EkMz87] they go on to consider this question (in [EkMz85] the special case $D = \mathbb{Z}_{(p)}$ is dealt with).

They show ([EkMz87; 1.4]) that if R is an e.c. D-algebra (D commutative) then, as a D-module, R is a model of the largest theory, T^*, of D-modules: in particular, all e.c. D-algebras are elementarily equivalent as D-modules. On the other hand, they show ([EkMz87; 1.6]) that if D is a principal ideal domain but not a field, then not every model of T^* is the underlying D-module of an e.c. D-algebra. So one problem they leave open here is the characterisation of those commutative rings D over which every model of T^* may be endowed with the structure of an e.c. D-algebra.

Then it is shown [EkMz87; 2.2] that if $\kappa = |D| + \aleph_0$ and if $\lambda > \kappa$ is such that there exists a λ-saturated model of T^* of cardinality λ (cf. §3.1), then there is an e.c. D-algebra whose underlying D-module is this (necessarily unique) λ-saturated model of cardinality λ (in fact, their result is stronger than this). It is noted that there exists a λ-saturated model of cardinality λ for every $\lambda > \kappa$ iff D (commutative) is of finite representation type (see Exercise 3.1/4). So they conclude that there is, for every $\lambda > \kappa$, an e.c. D-algebra of cardinality λ, whose underlying module is λ-saturated iff D is a principal ideal ring (= commutative ring of finite representation type).

There are a number of related results and open questions. Their paper ends with an application to a problem of Fuchs.

2.$\mathbb{Z}$ pp formulas and types in abelian groups

Our most fruitful source of examples in these notes is the category of $\mathbb{Z}$-modules. By the way, I use the terms "abelian group" and "$\mathbb{Z}$-module" entirely interchangably. Algebraically, there is no distinction; model-theoretically, there is none either, since the language for $\mathbb{Z}$-modules is definitionally equivalent to that for abelian groups (the function "multiplication by n" may be defined as an iterated sum). Admittedly, theories of $\mathbb{Z}$-modules are not typical: for a start, $\mathbb{Z}$ is a commutative ring and so all pp-definable subgroups are submodules. Nevertheless, many points are well-illustrated by abelian groups and, in fact, consideration of this special case has often pointed the way to results which are true in general.

Perhaps the most atypical feature of theories of abelian groups is that they are very well understood. Szmielew's results [Sz55] on quantifier elimination and decidability provided, early on, a decision procedure for such theories and a set of invariants, of the sort defined in §2.4, which allow one to test for elementary equivalence. Other results, especially those on algebraically compact modules over $\mathbb{Z}$ and related rings, make the algebraic implications of Szmielew's work clearer and, in fact, provide a more direct route to her results. This route is taken by Eklof and Fisher in [EF72], where they extend the results to Dedekind domains.

It has been seen (2.9) that any complete theory of modules is determined by its pure-injective models. It is known, see [Kap69; Notes to §17] or below, that every pure-injective abelian group is the pure-injective hull of a direct sum of indecomposable pure-injectives. The indecomposable pure-injective abelian groups are all quite familiar. With the results of this chapter (and some later ones) most of the broad questions (and many detailed ones) about theories of abelian groups have been (or can easily be) answered.

The most significant exception is that the gap between an abelian group and its pure-injective hull is not very well understood. One consequence of this is that there is no good structure theorem for countable models of theories of abelian groups (although the truth of Vaught's Conjecture here follows nevertheless, by a crude counting argument).

One purpose of this section is to present enough material on abelian groups so as to enable the reader to appreciate (and fill in details of) examples which use these groups. In particular, a description of all the indecomposable pure-injective abelian groups is included. The first result proved is that the pp-elimination of quantifiers for modules may be further refined for abelian groups: every pp formula may be broken into a conjunction of annihilator conditions and divisibility statements. Since much of what is so for $\mathbb{Z}$-modules also holds for modules over Dedekind domains, I present the results in this generality. There is no harm, however, in reading this section as if the base ring under consideration were always $\mathbb{Z}$.

In defining Dedekind domains, I take the opportunity to define various related classes of rings which will be referred to, here and later. For more background the reader may consult [Kap70].

Suppose that R is a commutative **domain** (= has no zero-divisors). Then R has a classical **field of fractions** $Q=Q(R)$, the elements of which are the equivalence classes, $[r/s]$ with $r,s\in R$, $s\neq 0$, under the usual ring operations - see [St75; Chpt2] (r/s is equivalent to t/w iff $rw=st$). An ideal I of R is said to be **invertible** if $I^{-1}=\{q\in Q : Iq\subseteq R\}$ is such that $I.I^{-1}=R$ (note that necessarily $I.I^{-1}\subseteq R$ and that I^{-1} is an R-submodule of Q).

For example, if $R=\mathbb{Z}$ then $Q=\mathbb{Q}$ regarded as a ring, and a typical non-zero ideal, I, of $\mathbb{Z}$ has the form $n\mathbb{Z}$ for some $n\in\mathbb{Z}$. Then I^{-1} is $\{q\in\mathbb{Q} : nq\in\mathbb{Z}\}$ which is just the set of all fractions which, when written in reduced form, have denominator a divisor of n. So $II^{-1}=\mathbb{Z}$ since $1=n.(1/n)\in II^{-1}$.

A commutative domain, such as $\mathbb{Z}$ (and $K[X]$), in which every non-zero ideal is invertible, is termed a **Dedekind domain**. Other examples are the rings of integers in finite extension fields of the rational field $\mathbb{Q}$. The next result gives various standard characterisations of these rings.

<u>Theorem 2.ℤA</u> (see [Kap70; §2.3]) *Suppose that R is a commutative domain which is not a field. Then the following conditions on R are equivalent:*

(*i*) *R is a Dedekind domain;*

(*ii*) *R is noetherian, every non-zero prime ideal P of R is maximal, and each localisation $R_{(P)}$ at a non-zero prime is a discrete rank 1 valuation domain;*

(*iii*) *every ideal is a projective module (i.e., R is a* **hereditary** *commutative domain);*

(*iv*) *every ideal is uniquely a product of prime ideals;*
(*v*) *every divisible module is injective (so injective = divisible).* ◻

Recall that if R is a domain and if P is a prime ideal of R then the **localisation**, $R_{(P)}$, **of** R **at** P is the ring of equivalence classes of "fractions" r/s where $r,s \in R$, $s \notin P$.

For example, if $R = \mathbb{Z}$ then $R_{(0)} = \mathbb{Q}$ and if p is a prime then $\mathbb{Z}_{(p)}$ (that is $\mathbb{Z}_{(p\mathbb{Z})}$) is $\{m/n : m/n \in \mathbb{Q} \text{ and } p \nmid n\}$.

A **valuation domain** is a commutative domain whose ideals are linearly ordered by inclusion. A valuation domain R is **discrete rank 1** if the ideals are just $R > P > P^2 > \ldots > P^n > \ldots > 0$. An example of such a discrete valuation domain is $\mathbb{Z}_{(p)}$. For more on valuation domains see §10.V.

A commutative domain R is a **Prüfer domain** iff, for every (maximal) prime ideal P, the localisation $R_{(P)}$ is a valuation domain (note that a localisation of a valuation domain is again a valuation domain, if not a field). Recall that a commutative ring is a **principal ideal domain (PID)** if every ideal is cyclic.

Returning now to one of the main objects of this section - the description of the indecomposable pure-injective abelian groups - the first result is that which allows us to reduce general pp formulas to rather simple ones (it is also the basis of the usual proof of the decomposition theorem for finitely generated abelian groups).

<u>Theorem 2.ℤB</u> (e.g. [HH70; 7.10], [Jac74; 3.8]) *Suppose that H is a matrix over the principal ideal domain R. Then there are invertible matrices X, Y over R such that XHY is a diagonal matrix.* ◻

The next result says that, over a principal ideal domain, we have elimination of quantifiers down to pp formulas of a particularly nice form. For abelian groups, this is due to Szmielew [Sz55]. It was extended to principal ideal domains (indeed, to Dedekind domains) by Eklof and Fisher [EF72], who made heavy use of the structure theory for pure-injectives over such rings (see [Kap76; Notes to §17]).

<u>Theorem 2.ℤ1</u> *Suppose that R is a principal ideal domain (for example $R = \mathbb{Z}$).*
(*a*) *Every pp formula $\varphi(\bar{v})$ is equivalent (in every R-module) to a conjunction of formulas of the form $p^k | t(\bar{v})$ and of the form $t(\bar{v}) = 0$, where p^k is a power of the prime p and $t(\bar{v})$ is a term in (R-linear combination from) $\bar{v}$. (Recall that in this context, an element p is* prime *if $p | ab$ implies $p | a$ or $p | b$, whenever $a, b \in R$).*
Moreover, every formula of the sort $p^k | t(\bar{v})$ may be taken to have the form $p^k | p^l t'(\bar{v})$ where, if $t'(\bar{v}) = \Sigma_1^n v_i r_i$, then the greatest common divisor of $\{r_1, \ldots, r_n\}$ is 1 (and $l < k$).
(*b*) *In particular, if $\varphi(v)$ is a pp formula in one free variable then $\varphi(v)$ is equivalent, in every R-module, to a conjunction of formulas of the sort $p^k | p^l v$ (where p^k, p^l are powers of the prime p and $k > l$) and of the sort $vr = 0$ (where $r \in R$).*

Proof I follow the proof given in [Ho8?]. Since part (b) follows immediately from part (a), I prove (a).

Write the pp formula $\varphi(\bar{v})$ in the form $\exists \bar{w} (\bar{v}T + \bar{w}S = 0) \ldots (*)$ where T, S are matrices over R. By **2.ℤB**, there are invertible matrices X, Y over R such that $XSY = D = \text{diag}(r_1, \ldots, r_n)$ is a diagonal matrix (in so far as a rectangular matrix can be diagonal) with (i,i)-entry $r_i \in R$).

Since X, Y are invertible, one successively derives the following formulas equivalent to $(*)$:

$\exists\overline{w}\,(\overline{v}TY + \overline{w}X^{-1}.XSY = 0)$;

$\exists\overline{w}'\,(\overline{v}U + \overline{w}'D = 0)$....(**) where $U = TY$.

Now think of (**) as a conjunction of linear equations. If $\overline{w}' = (w_1, ..., w_m)$, then each w_i occurs (with non-zero coefficient) in at most one equation. Therefore, equivalent to (**), is the conjunction over $j \in \{1,...,m\}$ of the equations $t_j(\overline{v}) + w_j r_j = 0$ (where $t_j(\overline{v})$ is the j-entry of $\overline{v}U$).

Thus (*) is equivalent to a conjunction of equations of the form $r | t(\overline{v})$ where $r \in R$ and t is a term. As the special case, $0 | t(\overline{v})$, one has $t(\overline{v}) = 0$.

Now (see [HH70] or [Jac74]), every non-zero element has an essentially unique expression as a product of prime powers. Moreover, the g.c.d. (r,s) of elements $r, s \in R$ is 1 iff $1 \in rR + sR$. So it is easy to see that, if $(r,s) = 1$ then $rs | t(\overline{v})$ iff $r | t(\overline{v})$ and $s | t(\overline{v})$. Therefore every formula of the form $r | t(\overline{v})$ $(r \neq 0)$ may be replaced by a conjunction of formulas of the sort $p^k | t(\overline{v})$ where p is a prime.

Finally, suppose that $t(\overline{v})$ is $\Sigma_1^n v_i r_i = 0$. Write this as $(\Sigma_1^n v_i s_i)s = 0$ where s is the g.c.d. of $\{r_1, ..., r_n\}$ and the s_i are such that $s_i s = r_i$. By the argument above, if p^l is the highest power of p dividing s, then the formula $p^k | t(\overline{v})$ is equivalent to $p^k | p^l(\Sigma_1^n v_i s_i)$ - which has the form desired. Of course, if $l \geq k$ then the formula has no non-trivial content, so may be dropped.

Note that the above reduction is effective. □

The simple pp formulas which appear above are sometimes termed "basic".

Corollary 2.ℤ2 *Suppose that R is a principal ideal domain (actually* [War69; Cor 5] *it suffices that R be a Prufer domain). Let $M \leq N$ be modules. Then M is pure in N iff for every $r \in R$ one has $Nr \cap M = Mr$.*

Proof Certainly the second condition is necessary for purity. For $Mr \subseteq Nr \cap M$ always is true and, if $m = nr \in Nr \cap M$, then $N \models \exists w\,(m = wr)$ so, since M is pure in N, $M \models \exists w\,(m = wr)$ and so $m \in Mr$.

Now suppose that R is a principal ideal domain. Let $\varphi(\overline{v})$ be pp and let $\overline{a}$ be in M such that $N \models \varphi(\overline{a})$. By 2.ℤ1, $\varphi(\overline{v})$ is equivalent, in every module, to a conjunction of formulas of the form $r | t(\overline{v})$ for suitable elements r of R and terms t. Let "$r | t(\overline{v})$" be any one of these. Since $N \models \varphi(\overline{a})$, certainly $N \models r | t(\overline{a})$. So by assumption, $t(\overline{a}) \in Mr$. Hence M satisfies the conjunction of all those basic formulas and hence $M \models \varphi(\overline{a})$. Thus M is pure in N - as required. □

Theorem 2.ℤ1 will be used to describe the indecomposable pure-injectives over a discrete rank 1 valuation domain. The list one obtains may also be obtained algebraically (see [Kap69; Notes to §17], [Har59], [Hul62]). Then we use the fact that if R is a commutative ring then any indecomposable pure-injective R-module is actually a module over a localisation of R at a maximal ideal: this was noted and used by Ziegler [Zg84; 5.4] and generalises [Kap69]. Therefore, if R is a commutative ring whose localisations at maximal ideals are well enough understood that over each one has a list of indecomposable pure-injectives, then one may pull back to a list of indecomposable pure-injectives over the ring R itself. Examples of such rings are Dedekind domains and commutative regular rings (the latter are treated in Chapter 16).

Proposition 2.ℤ3 *Let R be a discrete rank 1 valuation domain with maximal ideal P and quotient field Q (regarded as a module over R). Then the indecomposable pure-injective R-modules are:*

the artinian modules R/P^n $(n \geq 1)$;

the generalised Prufer module Q/R;

the injective hull Q of R;

the completion $\overline{R}$ *of* R *in the* P*-adic topology (= the pure-injective hull of* R *- see §4.2 for the definition).*

In particular, if $R=\mathbb{Z}_{(p)}$ *then the list becomes:*

$\mathbb{Z}_{p^n}$ $(n\geqslant 1)$; $\mathbb{Z}_{p^\infty}$; $\mathbb{Q}$; $\overline{\mathbb{Z}_{(p)}}$.

Proof I follow the proof in [Zg84; 5.1]. Both Q/R and Q are easily seen to be divisible = injective (2.ℤA) and so they certainly are pure-injective. Moreover, both are **uniform** (every pair of non-zero submodules has a non-zero intersection), and hence they are both indecomposable (pure-)injectives.

Each R/P^n has dcc on submodules, hence on pp-definable subgroups (we are using that R is commutative), and so is pure-injective (Σ-pure-injective even, by **2.11**). Moreover, R/P^n has a unique minimal submodule - P^{n-1}/P^n - hence it is indecomposable (for it has dcc on submodules).

Last on the list is $\overline{R}$ (completeness of the list will be established below). Let me describe this module (ring even). The powers of the prime ideal P may be decreed to form a neighbourhood system of 0 for a linear topology on R. R has a completion in this topology: a Cauchy sequence $x_0, x_1, \ldots$ is one which is such that for every n there is n_0 with $x_i - x_j \in P^n$ whenever $i,j\geqslant n_0$. The completion, $\overline{R}$, of R in this topology is the collection of all Cauchy sequences, modulo the usual equivalence relation; it has a natural structure as a ring extension of R - in particular it is an R-module (and, as such, is easily seen to be torsionfree).

The elements of $\overline{R}$ may usefully be thought of as in 1-1 correspondence with infinite branches through the tree of cosets of the form $a+P^n$ with $a\in R$ and $n\in\omega$. That is, given $c\in\overline{R}$ and $n\in\omega$, there is a unique (mod P^n) $a_n\in R$ such that $c\equiv a_n$ mod P^n (i.e., $c-a_n\in P^n$).

That $\overline{R}$ is pure-injective is a direct consequence of the construction (note that every branch of pp-definable cosets in the tree has a point "at the bottom" (trees, of course, branch downwards)). That $\overline{R}$ is indecomposable as an R-module follows easily (consider its endomorphism ring). The pp-definable subgroups of $\overline{R}$ are just the ideals of the ring $\overline{R}$, which are $\overline{R}>P\overline{R}>P^2\overline{R}>\ldots>P^n\overline{R}\ldots>0$ (this can be proved in various ways: for example it is an immediate consequence of the fact that R is pure in $\overline{R}$).

It remains to be shown that there are no more indecomposable pure-injectives. Choose and fix a generator p of P. Let N be an indecomposable pure-injective and choose a non-zero element a of N. Consider the annihilator, $\mathrm{ann}_R(a)=\{r\in R : ar=0\}$, and the height (in the coarse sense) $\mathrm{h}(a)=\max\{n : a\in NP^n\}$ - either a member of ω or "∞". If $\mathrm{ann}_R(a)=P^{n+1}$ then $\mathrm{ann}_R(ap^n)=P$. So, replacing a if necessary, it may be supposed that $\mathrm{ann}_R(a)$ is 0 or P. Consider the possibilities.

If $\mathrm{ann}_R(a)=P$ and $\mathrm{h}(a)=n\in\omega$, then choose $b\in N$ with $bp^n=a$. Then $bR\simeq R/P^{n+1}$ since $aP=0$. Moreover, b is not divisible by p in N since $\mathrm{h}(a)=n$ so, by (a short argument using) **2.ℤ2**, bR is pure in N. But the module bR is itself pure-injective, so bR is a direct summand of N. Hence $N=bR\simeq R/P^{n+1}$.

If $\mathrm{ann}_R(a)=P$ and $\mathrm{h}(a)=\infty$ then the set, $\Phi((v_n)_{n\in\omega})=\{v_0=a\}\cup\{v_{i+1}p=v_i : i\in\omega\}$, of pp formulas is finitely satisfied in N so, by **2.8**, is realised - say by $(a_n)_{n\in\omega}$ in N. Clearly, the module generated by $\{a_n : n\in\omega\}$ is a copy of Q/P (the unique divisible torsion module). Since Q/P is injective it is a direct summand of N and so $N\simeq Q/P$.

If $\mathrm{ann}_R(a)=0$ and $\mathrm{h}(a)=\infty$, then a is uniquely (since $\mathrm{ann}_R(a)=0$) divisible in N by every power of p. Hence N contains a copy of the divisible hull of a - so N is a copy of Q.

If $\mathrm{ann}_R(a)=0$ and $\mathrm{h}(a)=n\in\omega$, then choose $b\in N$ with $bp^n=a$. Then $bR\simeq R$ (since $\mathrm{ann}_R(b)=0$) is, by **2.ℤ2**, pure in N. So $N\simeq\overline{bR}\simeq\overline{R}$ (for pure-injective hulls see §§4.1, 4.2). □

The proof actually has established the following points for modules over a discrete rank 1 valuation domain: every pure-injective module has an indecomposable direct summand - it therefore follows (see **4.A14**) that every pure-injective module is the pure-injective hull of a direct sum of copies of the modules described in **2.ℤ3**; among the indecomposable pure-injectives, the R/P^n, Q/R, and Q all are Σ-pure-injective, whereas $\overline{R}$ is not (it fails to satisfy the criterion of **2.11**).

The next task is to see that the indecomposable pure-injectives over a commutative ring may be found by looking at those over its localisations at maximal ideals. Let us look at commutative localisations more generally.

Suppose that R is commutative and let S be a **multiplicative subset** of R (so $1 \in S$, $0 \notin S$ and $r, s \in S$ implies $rs \in S$). For M an R-module, the **localisation** M_S of M at S is obtained by first factoring out the S-torsion submodule $\tau_S M = \{m \in M : ms = 0$ for some $s \in S\}$, and then forming "fractions": formal expressions m/s (or ms^{-1}) with $m \in M/\tau_S M$ and $s \in S$, under the usual equivalence and operations. In particular, R_S has a natural ring structure extending that of $R/\tau_S R$ ($\tau_S R$ is an ideal), and M_S has a natural R_S-module structure extending its structure as an R-module (see [St75; Chpt2] for details). If P is a prime ideal then $R \backslash P$ is a multiplicative subset and $M_{R\backslash P}$ is just what I earlier denoted by $M_{(P)}$ - the **localisation of** M **at** P. Let $Q_S: M \longrightarrow M_S$ be the natural morphism which takes $m \in M$ to $(m + \tau_S M)/1$ *via* $M/\tau_S M$.

Given a pp formula φ over R, we may ask what is the relation between $\varphi(M)$ and $\varphi(M_S)$: it turns out to be what one probably would expect. (There is no ambiguity in writing $\varphi(M_S)$: one may equally well regard M_S as an R-module or φ as a formula in the language of R_S (elements of R which become identified in R_S have the same action on R_S-modules).)

<u>Lemma 2.ℤ4</u> [Gar79; Lemma4], [Zg84; Lemma5.5] *Let R be a commutative ring and let S be a multiplicative subset of R. Let M be any module. Suppose that $\varphi = \varphi(\overline{v})$ is a pp formula in the language of R-modules.*
Then $\varphi(M_S) = (\varphi(M))_S$, where this latter is the R_S-submodule of $M_S{}^{l(\overline{v})}$ generated by the image of $\varphi(M)$ under Q_S.
In particular, if $m \in M$ then $Q_S(m) \in \varphi(M_S)$ iff $ms \in \varphi(M)$ for some $s \in S$.

Proof Suppose first that $\varphi(\overline{v})$ is a single equation, say $\sum_1^n v_i r_i = 0$. If $a_1, \dots, a_n \in M$ with $\sum_1^n a_i r_i = 0$ then, on applying the R-morphism Q_S, one obtains $\sum Q_S a_i . r_i = 0$. That is $Q_S(\varphi(M)) \leq \varphi(M_S)$.

Conversely, if $(a_1/s_1, \dots, a_n/s_n) \in \varphi(M_S)$, where $a_1, \dots, a_n \in M$ and $s_1, \dots, s_n \in S$, then let $s = \prod_1^n s_i$ and write $a_i/s_i = c_i/s$ for $c_i = a_i \prod_{j \neq i} s_j \in M$. Then one has $\sum (a_i/s_i) r_i = 0$, so $(\sum c_i r_i)/s = 0$, and hence $\sum c_i r_i \in \tau_S M$. Therefore there is $t \in S$ with $(\sum c_i r_i)t = 0$ - that is - $\sum c_i t . r_i = 0$. Thus $(c_1 t/1, \dots, c_n t/1) st = (c_1/s, \dots, c_n/s) = (a_1/s_1, \dots, a_n/s_1)$ is in $(\varphi(M))_S$ also, as required.

Thus the result is true for equations so, clearly, for conjunctions of equations. But pp formulas are just projections of such conjunctions of equations and $(M^n)_S = (M_S)^n$, so the result follows. □

Exercise 1 Detail the last part of the proof of **2.ℤ4**. Alternatively, re-prove the result using matrices from the beginning.

This is used to prove the following striking result of Garavaglia.

<u>Theorem 2.ℤ5</u> [Gar79; Thm3] *Suppose that R is commutative. Let M be any module.*
Then: $M \equiv \bigoplus \{M_{(P)} : P$ is a maximal ideal of $R\}$

$\equiv \oplus \{M_{(P)} : P \in \mathcal{P}$, *where* $\mathcal{P}$ *is any set of primes containing all the maximal ideals of* $R\}$.
Indeed, the natural morphism $M \longrightarrow \prod \{M_{(P)} : P$ *a maximal prime*$\}$ *is an elementary embedding.*
Here the localised modules should be understood as R*-modules via the natural morphism* $R \longrightarrow R_{(P)}$.

Proof First suppose that M is a finite R-module. Then $R/\mathrm{ann}_R M$ is a finite ring (exercise!), so there are only finitely many (maximal) primes $P_1,\ldots,P_n$ (say) containing $\mathrm{ann}_R M$. If Q is any prime not in this set then, since $(R\setminus Q) \cap \mathrm{ann}_R M \neq \varnothing$ one has $\tau_{(Q)}M = M$, and so $M_{(Q)} = 0$.

Since $R/\mathrm{ann}_R M$ is artinian (so idempotents lift modulo the Jacobson radical) and commutative, $R/\mathrm{ann}_R M$ is a finite direct product $\prod_1^n R_i$ (say) of local rings, and $P_i = \oplus_{j\neq i} R_j \oplus H_i$, H_i being the maximal ideal of R_i. Then it is not difficult to check that if e_i is the central idempotent, $(0,\ldots,1,0,\ldots,0)$, of R corresponding to R_i (so with "1" in the i-th position) then $M_{(P_i)} = Me_i$. Since the e_i are orthogonal and sum to 1, one obtains $M = M(\Sigma_1^n e_i) = \oplus_1^n Me_i = \oplus_1^n M_{(P_i)}$.

Thus if M is a finite R-module, and if $\mathcal{P}$ is any set of primes containing all the maximal primes, then $|M| = |\oplus \{M_{(P)} : P \in \mathcal{P}\}|$.

Next suppose that M is infinite and that $\mathcal{P}$ is as stated. The map $M \longrightarrow \prod \{M_{(P)} : P \in \mathcal{P}\}$ given by $a \mapsto (Q_{(P)}a)_{P\in\mathcal{P}}$ is monic. For if a is a non-zero element of M then $\mathrm{ann}_R(a) \neq R$, so there is $P \in \mathcal{P}$ with $\mathrm{ann}_R(a) \leqslant P$: then $a \notin \tau_{(P)}M$ and hence $Q_{(P)}a \neq 0$. Therefore this product is infinite: so, therefore, must be the corresponding sum $\oplus \{M_{(P)} : P \in P\}$.

In summary; we have that for any module N, $|N| \doteq |\oplus \{N_{(P)} : P \in \mathcal{P}\}|$.

Therefore, if M is any module and if $\varphi(M) \geqslant \psi(M)$ with φ, ψ being pp formulas (so, R being commutative, $\varphi(M)$, $\psi(M)$ are submodules of M), then $|\varphi(M)/\psi(M)| \doteq |\oplus \{(\varphi(M)/\psi(M))_{(P)} : P \in \mathcal{P}\}| \doteq \prod \{|(\varphi(M)/\psi(M))_{(P)}| : P \in \mathcal{P}\}$. But ($R$ is commutative so $Q_{(P)}$ is exact [St75; XI.3.4]) $(\varphi(M)/\psi(M))_{(P)} \simeq \varphi(M)_{(P)}/\psi(M)_{(P)}$, and this in turn is isomorphic to $\varphi(M_{(P)})/\psi(M_{(P)})$ by 2.$\mathbb{Z}$4. Thus it has been shown that $\mathrm{Inv}(M,\varphi,\psi) \doteq \prod \{\mathrm{Inv}(M_{(P)},\varphi,\psi) : P \in \mathcal{P}\} \doteq \mathrm{Inv}(\oplus\{M_{(P)} : P \in \mathcal{P}\},\varphi,\psi)$ (by 2.10).

Hence $M \equiv \oplus \{M_{(P)} : P \in \mathcal{P}\}$, as claimed.

To finish the proof, it must be shown that the natural map $M \longrightarrow \prod \{M_{(P)} : P$ maximal$\}$ is an elementary one: by 2.26, it is enough to show that is is pure (we have already seen that it is an embedding). So let φ be pp and let $a \in M$ be such that $M \models \neg\varphi(a)$. Let $I = (\varphi(M) : a) = \{r \in R : M \models \varphi(ar)\}$. By assumption, I is a proper ideal, so it is contained in a maximal ideal, P say. If we had $M_{(P)} \models \varphi(Q_{(P)}a)$ then, by 2.$\mathbb{Z}$4, there would be $s \in R\setminus P$ with $M \models \varphi(as)$ – contradiction. Hence $M_{(p)} \models \neg\varphi(Q_{(P)}a)$ and so the embedding is indeed pure, as required. □

<u>Corollary 2.$\mathbb{Z}$6</u> *As abelian groups,* $\mathbb{Z} \equiv \oplus_{p \text{ prime}} \mathbb{Z}_{(p)}$. □

<u>Corollary 2.$\mathbb{Z}$7</u> [Gar79; Cor1] *Let* R *be a commutative regular ring and let* M *be any module. Then* $M \equiv \oplus \{M/MI : I$ *is a maximal ideal of* $R\}$. *In particular, every module is elementarily equivalent to a direct sum of indecomposable injective modules* (*cf.* 4.36 *and also* 4.38). □

Exercise 2 [Gar79; Cor2] Suppose that R is commutative without nilpotent elements. Then every R-module is elementarily equivalent to a direct sum of cyclic modules iff R is (von Neumann) regular.

Question How much of the above goes for classical rings and modules of quotients in the non-commutative case? (see [St75; Chpts2,11]).

The next result appears in [Kap69; Exercise 65] for PID's (also see [War69; Prop12]). The general result is [Zg84; 5.4]. The localisation technique is used also in [Fis75; 2.29].

Theorem 2.ℤ8 *Suppose that R is commutative. Then, for every indecomposable pure-injective module N, there is a maximal ideal P of R such that the natural R-morphism $Q_{(P)}: N \longrightarrow N_{(P)}$ is an isomorphism. Moreover $N_{(P)}$ is indecomposable as an $R_{(P)}$-module.*

Proof Since R is commutative it acts as a ring of endomorphisms of N. Since End(N) is local (4.27), the sum of any two non-automorphisms is a non-automorphism. Hence $\{r \in R : -\times r: N \longrightarrow N$ is not an automorphism of $N\}$ is an ideal of R (proper, of course). Choose any maximal ideal, P, of R containing this ideal. Since every element of $R \setminus P$ acts automorphically on N (note: such elements need not be invertible in R - consider $\mathbb{Q}_{\mathbb{Z}}$), it follows that N already has a natural $R_{(P)}$-structure. Thus the result follows.

Notice that the last remark follows trivially, since N"="$N_{(P)}$ has "identical" structure as an R-module and as an $R_{(P)}$-module (in particular, the pp-definable subgroups are the same, since the $R_{(P)}$-action is pp-definable in terms of the R-action). □

Thus every indecomposable pure-injective R-module may be found already over some localisation $R_{(P)}$ of R at some maximal ideal. The next lemma says that one may pull back indecomposable pure-injective $R_{(P)}$-modules to R, and that essentially nothing changes in the process.

Lemma 2.ℤ9 [Zg84; 5.3] *Suppose that R is commutative. Let S be a multiplicative subset of R, and let M, N be R_S-modules.*

(*a*) *M and N are elementarily equivalent as R-modules iff they are elementarily equivalent as R_S-modules.*

(*b*) *M is an (indecomposable) (pure-)injective module over R iff it is so over R_S.*

(*c*) *N is the pure-injective hull of M as an R-module iff it is so as an R_S-module.*

(*d*) *M is an elementary substructure of (resp. pure in) N as an R-module iff it is so as an R_S-module.*

Proof This is left as an exercise using, for example, 2.ℤ4. □

Applying this together with 2.ℤ3, one obtains the following description of indecomposable pure-injectives over certain rings including Dedekind domains (in particular this applies when $R = \mathbb{Z}$). For the generalisation to Prüfer rings, see §10.V.

Theorem 2.ℤ10 [Zg84; 5.2] *Let R be a commutative ring such that each localisation of R at a maximal prime is a field or a discrete rank 1 valuation domain. Then the indecomposable R-modules are, where P ranges over the maximal primes:*

$R/P = R_{(P)}$ *if* $R_{(P)}$ *is a field;*

$R_{(P)}/R_{(P)}P^n$, $K_P/R_{(P)}$, $\overline{R_{(P)}}$, K_P *otherwise, where K_P is the field of quotients of $R_{(P)}$.* □

Corollary 2.ℤ11 *Suppose that R is a Dedekind domain. Then the indecomposable pure-injective R-modules are:*

$R_{(P)}/P^nR_{(P)} \simeq R/P^n$; $Q/R_{(P)} =$ "R_{P^∞}"; Q; $\overline{R_{(P)}}$;

where $n \in \omega$, $n \geqslant 1$, where P ranges over the maximal ideals of R, and where "Q" denotes the field of quotients of R (= the field of quotients of each localisation). □

In particular, the indecomposable pure-injective abelian groups are: the finite groups $\mathbb{Z}_{p^n}$; the prüfer groups $\mathbb{Z}_{p^\infty}$; the p-adic integers $\overline{\mathbb{Z}(p)}$; the rationals $\mathbb{Q}$.

The argument of this section may also be made to show that there are no continuous pure-injectives (i.e., pure-injectives without indecomposable direct summands) over a Dedekind domain (see [Fis75; 2.29)]).

Corollary 2.$\mathbb{Z}$12 *Suppose that R is a Dedekind domain. Then R has no more than $|R|+\aleph_0$ indecomposable pure-injective modules. If R is countable and not a field, then there are exactly $\aleph_0$ indecomposable pure-injectives.*

Proof Since R is noetherian, there are no more than $|R|+\aleph_0$ prime ideals so, by 2.$\mathbb{Z}$11, this is an upper bound. If R is countable and if P is any non-zero prime, then the modules R/P^n are all non-isomorphic (being of different lengths) indecomposable pure-injectives, so $\aleph_0$ is a lower bound. □

Ziegler [Zg84; 5.7] classifies the indecomposable pure-injective pairs, (U,V), where $U \geqslant V$ are torsionfree modules over a Dedekind domain. The list is: $(Q,0)$; (Q,Q); $(\overline{R(P)},\overline{R(P)}.P^n)$; $(\overline{R(P)},0)$, $(Q(\overline{R(P)}),\overline{R(P)})$, where P is an arbitrary non-zero prime, $n \geqslant 0$, $Q=Q(R)$ is the quotient field of R and $Q(\overline{R(P)})$ is the quotient field of $\overline{R(P)}$.

In [Zg84; 9.8] he shows that the CB-rank of the space that they form (§4.7) is 2. From his explicit description of the space, decidability of the theory of such "torsionfree pairs" follows ([Zg84; 9.10]) over any suitably recursive Dedekind domain (cf. 17.15).

2.L Other Languages

I refer the reader to the various papers for the definitions of the quantifiers mentioned in this section.

The first results concern eliminability of the Magidor-Malitz quantifiers, $Q_\alpha{}^m$: the quantifiers $Q_0{}^m$ are also known as Ramsey quantifiers.

It was proved by Baldwin and Kueker [BK80] and (later) by Rothmaler and Tuschik [RT82] that if T is a countable stable theory then the $Q_0{}^m$ are eliminable (i.e., formulas involving them have first-order equivalents) iff T does not have the finite cover property (fcp).

Baudisch [Bd84a] showed that in theories of modules, the Magidor-Malitz quantifiers $Q_\alpha{}^m$ $(n \geqslant 2)$ all have the same expressive power as $Q_\alpha{}^2$ (and that the elimination of the quantifiers $Q_\alpha{}^m$ in favour of $Q_\alpha{}^2$ is relatively effective). He also showed that the Ramsey quantifiers are eliminable in any complete theory of modules. Previously, Rothmaler [Rot81] had shown the eliminability of the quantifier Q_0 ("there exists infinitely many") in any complete theory of modules. So, by Baudisch's result, it followed that every module fails to have the fcp. This was given another proof by Hodges using Poizat's results on "beautiful pairs" [Poi84] and a direct proof was also given by Rothmaler [Rot82]. Rothmaler also has some partial results showing eliminability of Q_0 in certain incomplete theories of modules [Rot84]. Baudisch [Bd83] shows decidability of the theory of abelian groups with Magidor-Malitz quantifiers and gives an elimination procedure.

For all this, and more, see [BSTW85], especially §1.3. On p.249 there is a useful table summarising what is known about the various languages (as above, also Hartig's quantifiers and L_{aa}).

As for stationary logic: Eklof and Mekler [EkMk79; §4] show that modules have an elimination of quantifiers in the language L_{aa} and that, moreover, the L_{aa}-theory of abelian groups is decidable. The results on abelian groups were obtained independently by Baudisch

[Bd81]. Furthermore, if R is countable, then the theory of R-modules has an L_{aa}-model-companion iff R is right coherent (cf. **15.35**).

Eklof [Ek74] and, independently, Kueker (unpublished) consider conditions for an abelian group to be $L_{\infty\kappa}$-equivalent to a free group or to a direct sum of cyclic groups. Also Ruyer [Ru84; Thm 2] has a criterion for a module over a Dedekind domain to be $L_{\kappa\omega}$-equivalent to a direct sum of finitely presented modules. For classification of abelian groups up to $L_{\infty\omega}$-equivalence, see [BE70] and the survey article [Ek85].

Sabbagh and Eklof [SE71] have various results on $L_{\infty\omega}$-equivalence of direct sums and products and on classes of modules definable in the language $L_{\infty\omega}$. They show [SE71; Prop 10] that if, for every cardinal κ, one has $R^{(\kappa)}$ $L_{\infty\omega}$-equivalent to R^{κ}, then R is left coherent: furthermore, if R is left noetherian then, for every κ, $R^{(\kappa)}$ is an $L_{\infty\omega}$-substructure of R^{κ}. This contrasts sharply with the finitary ($L_{\omega\omega}$) case (§2.5).

In the same paper they asked whether the class of commutative noetherian rings is definable in $L_{\omega_1\omega}$. They observed that, if so, then there would be a fixed countable bound on the length of any chain of ideals in a commutative noetherian ring (i.e. the "depth" of the zero ideal). Bass [Bas71] had showed that any such chain has to be countable, but his proof did not give an upper bound. In fact, he showed that if there exist countable noetherian rings of arbitrarily high countable Krull dimension then there could be no such bound. Gordon and Robson [GR73; 9.8] gave examples of countable commutative noetherian domains of arbitrarily high countable Krull dimension, so they were able to conclude that the class of countable commutative noetherian rings is not definable in $L_{\omega_1\omega}$.

For classification of abelian groups up to $L_{\infty\omega}$-equivalence, see [BE70] and the survey article [Ek85].

The model theory of modules in a certain topological language has been considered. The relevant language has two sorts: one for elements and the other for sets which are to be interpreted as forming a topology (the module operations have to be continuous with respect to the topology) and there are certain restrictions on the occurrences of quantifiers applying to set variables. This language allows one to say significant things about the topology but is also quite close to being first-order and is reasonably well-behaved. Garavaglia [Gar78; §5] obtained a criterion for elementary equivalence of topological modules in terms of satisfying the same sentences which are $\forall\exists$ with respect to individual variables (this restricts to the criterion seen in §2.4). In his thesis ([Kuc84; Chpt.IV], see [Kuc86]), Kucera considers stability in these topological languages, for modules in particular. Cherlin and Schmitt [CS83], generalising some of [EF72], classify saturated torsionfree locally pure topological abelian groups and obtain decidability for these groups in the appropriate topological language.

CHAPTER 3 STABILITY AND TOTALLY TRANSCENDENTAL MODULES

Stability theory is a relatively new area of model theory which has developed rapidly. It is much concerned with types: with their properties and the relations between them, and with how this is reflected in terms of the structure of models. It is perhaps not surprising then, that one finds notions from stability theory relevant to modules (sometimes even identical with already existing algebraic notions). For we have seen that every type reduces essentially to a collection of pp formulas and negations of pp formulas, and pp formulas are not far from being "algebraic" since they express solvability of systems of linear equations.

I will state definitions and results from stability theory in various chapters, as I need them. A short introduction which covers most of what I will need is Pillay's book [Pi83]. A concise continuation of this is Makkai's article [Mak84]: Rather more inclusive are the books of Baldwin [Bal8?] and Lascar [Las8?]. The already-mentioned model theory texts of Poizat [Poi85] and Hodges [Ho??] contain all the stability theory that we will need. Of course there is also Shelah's book [She78], which contains a vast amount of material (though much is implicit rather than explicitly pointed out and the presentation is not designed to facilitate "dipping into" the book).

Stability theory divides complete theories into two major classes: those which are stable (where there is some possibility of developing a structure theory, at least for sufficiently saturated models); and those which are unstable (in the sense that they contain an infinite definable linear order and are, in some senses, less well-behaved than stable theories - though see [PiSt86]). Every theory of modules is stable. Since an arbitrary stable (non-superstable) theory may yet be very complicated it is not immediately clear how useful one should expect this to be. But, in fact, modules have pleasant model-theoretic properties which go beyond mere stability, and we will see that reasonable classification theorems exist even in the non-superstable case.

Within the class of stable theories there is a much better-behaved class - the superstable theories - and beyond these there are the totally transcendental (or more prosaically, t.t.) theories. For modules (and even in general) the totally transcendental theories have good structure theorems for their models. Rather suggestive for the general case is that for theories of modules the non-superstable case may be refined considerably (see Chapter 10).

I now define all these terms and immediately establish their algebraic characterisations in modules (which are what I will usually, but not invariably, use).

A complete theory T is **κ-stable** (κ being an infinite cardinal) if $|S_1{}^T(A)|\leqslant\kappa$ whenever $|A|\leqslant\kappa$ (A denotes an arbitrary subset of a model of T). Thus T is κ-stable if it has no more than the minimum possible number of types defined over sets of cardinality κ.

Exercise 1

(a) Show that in the definition of κ-stability one may replace $S_1(A)$ by $S_n(A)$ for any $n\in\omega$.

(b) Show that for any set A, $|S_1(A)|\geqslant|A|$.

(c) Show that the (theory of the) rationals with the usual ordering "<" is not $\aleph_0$-stable. Rather more difficult to show is that this theory is not κ-stable for any (infinite) κ.

It turns out that there are surprisingly few patterns of behaviour with regard to this notion. The basic classification theorem (due to Shelah) is as follows (see [Pi83; 5.3], [Poi85; 13.10, 17.12, 17.19]).

<u>Theorem 3.A</u> *Let T be any complete theory. Then either:*

(i) T is unstable - that is, T is κ-stable for no infinite $\kappa\geqslant|T|$; or

(ii) T is **stable** *- that is, T is κ-stable for some $\kappa\geqslant|T|$ - in which case T is κ-stable for all $\kappa\geqslant2^{|T|}$ satisfying $\kappa^\lambda=\kappa$, for all $\lambda<\kappa(T)$, where $\kappa(T)$ is an infinite cardinal which depends only on T (so T is stable in "most" cardinals).*

In the stable case there are essentially three possibilities:

(*a*) *T is merely stable - that is, stable but not superstable - in which case T is* κ-*stable for a given* $\kappa \geq 2^{|T|}$ *exactly if* κ *is as described in* (*ii*) *above*; *or*

(*b*) *T is* **superstable** - *that is*, κ-*stable for all* $\kappa \geq 2^{|T|}$; *or*

(*c*) *T is* **totally transcendental** (**t.t.**) - *this is usually defined in terms of Morley rank* (*see* §5.2) *but is equivalent to every reduct*, T_0, *of T to a countable language being* **ω-stable** (= $\aleph_0$-**stable**), *and this implies that* T_0 *is* κ-*stable for every infinite* κ (*and that T is* κ-*stable for every* $\kappa \geq |T|$ - *this condition characterises T being totally transcendental*). *If T is totally transcendental then T is superstable.* □

Given any infinite cardinal κ, there exists a cardinal $\kappa_1 \geq \kappa$ such that $\kappa_1^{\aleph_0} > \kappa_1$ (take $\kappa_1 \geq \kappa$ to have cofinality ω). It follows that, in order to prove T superstable, it is sufficient to show that there is some cardinal κ_0 such that T is κ-stable for all $\kappa \geq \kappa_0$.

In some sense the totally transcendental theories are the simplest: they have fewest types, so there is less possibility for "complexity" of models. This is, indeed, reflected in structure theorems (e.g., see [Pi83a]). Of course cardinality is a very coarse measure, and one might argue that the real point of total transcendence (and superstability) is the existence of rank functions on types. These ranks are very useful in the analyses of models. The model-theoretic rank functions will be considered in §5.2. Algebraic and topological dimensions, related to the model-theoretic ranks, are used at various points. The use of ranks and dimensions is one of the main strands running through these notes.

Every module is stable. This is shown in the first section, which also contains "algebraic" characterisations of totally transcendental and superstable modules. It is seen that totally transcendental modules are precisely the Σ-pure-injective modules of §2.3. Various corollaries are deduced with the aid of the invariants of §2.4. In particular, if $M \prec N$ are superstable then the quotient N/M is totally transcendental.

In the second section it is shown that every totally transcendental module is a direct sum of indecomposable submodules. This theorem encompasses a number of well-known algebraic direct-sum decompositions. A countable theory all of whose models are pure-injective must be totally transcendental: this, and related results, are considered at the beginning of the section.

3.1 Stability for modules

The main theorem here states that all modules are stable and characterises superstable and totally transcendental modules (recall that to say that a module is stable,... is to say that its complete theory is so). Stability of modules was proved independently by Fisher ([Fis75; 5.5], it also follows from [Fis72]) and Baur ([Bau75; Thm 1]) (there are partial results in [Brt75]) before pp-elimination of quantifiers was available: the original proof used pure-injective extensions (see below). The characterisation of totally transcendental modules is due to Garavaglia and Macintyre (see [Mac71; Lemma3], [Gar79; Lemma5] and [Gar80; Thm 1]) and that of superstable modules is due to Garavaglia [Gar79; Lemma 7]. In another guise, totally transcendental modules were characterised in rather similar terms ("p-functors" and "subgroups of finite definition" take the place of pp formulas) by Gruson and Jensen [GJ76; Thm] and Zimmermann [Zim77; 3.4]. In giving upper bounds for numbers of types, I follow the proof of [Zg84; 2.1].

<u>Theorem 3.1</u> *Let T be a complete theory of modules.*

(*a*) *T is stable.*

(*b*) *The following conditions on T are equivalent*:

(*i*) *T is superstable*;

(*ii*) *for any set* $\{\varphi_n : n \in \omega\}$ *of pp formulas in one free variable, there is* $n \in \omega$ *such that for all* $m \geq n$, $\mathrm{Inv}(T, \bigwedge_0^n \varphi_i, \bigwedge_0^m \varphi_i)$ *is finite*;

(iii) *in any, equivalently every, model* M *of* T *if* $\varphi_0(M) \geqslant \varphi_1(M) \geqslant \ldots$ *is a descending chain of pp-definable subgroups, then there exists* $n \in \omega$ *such that for every* $m \geqslant n$ *the index* $[\varphi_m(M) : \varphi_{m+1}(M)]$ *of* $\varphi_{m+1}(M)$ *in* $\varphi_m(M)$ *is finite*;

(ii)′, (iii)′ *as* (ii), (iii), *but for formulas in any finite number of free variables.*

(c) *The following conditions on* T *are equivalent*:

(i) T *is totally transcendental*;

(ii) *for any set* $\{\varphi_n : n \in \omega\}$ *of pp formulas in one free variable there is* $n \in \omega$ *such that, for all* $m \in \omega$, $\bigwedge_1^n \varphi_i \to \varphi_m$;

(iii) *any, equivalently every, model of* T *has the descending chain condition on pp-definable subgroups*;

(iv) *any set of pp formulas in one free variable is equivalent to a finite subset*;

(ii)′, (iii)′, (iv)′ *as* (ii), (iii), (iv), *but for formulas in any finite number of free variables.*

Proof Let A be any set of parameters. By **2.20** one has $|S_1(A)| = |S_1{}^+(A)|$ so it is sufficient when counting types to count pp-types. Now, if $\varphi(\bar{v},\bar{a}) \in p(\bar{v})$ with φ pp then, for any tuple $\bar{b}$ with $\varphi(\bar{v},\bar{b}) \in p(\bar{v})$, one has that $\varphi(\bar{v},\bar{a})$ and $\varphi(\bar{v},\bar{b})$ are equivalent (since p is consistent and cosets of the same subgroup are equal or disjoint). Thus if p is a pp-type then p is equivalent to a set of pp formulas which contains just one instance of each pp formula represented in p (precisely: this set, together with the pp-diagram of the set A of parameters suffices to prove p).

In particular, a pp-type may be given by a partial function from the set $\mathcal{P}$ of pp formulas (in the appropriate number of free variables) to the set $A^{<\omega} = \bigcup\{A^n : n \in \omega\}$ of finite tuples from A. Now, there are just $|A^{<\omega}|^{|\mathcal{P}|} = |A|^{|T|}$ of these, where we may as well assume from now on that $|A| \geqslant 2$. Thus there are no more than $|A|^{|T|}$ pp-types over A.

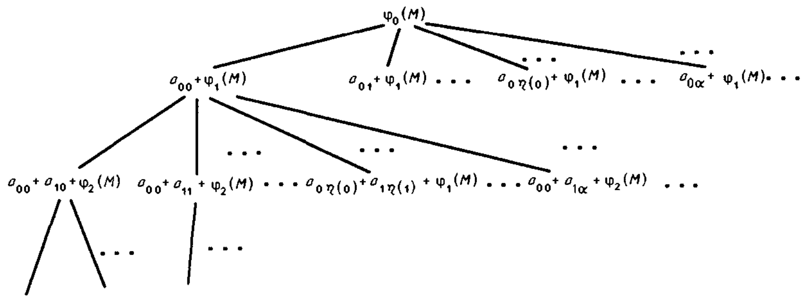
Therefore $|S_1(A)| = |S_1{}^+(A)| \leqslant |A|^{|T|}$. So if $|A|^{|T|} = |A|$ then $|S_1(A)| \leqslant |A|$; therefore, by definition, T is stable (since it is κ-stable for some κ) and (a) is proved.

(b) Clearly (ii) and (iii) are equivalent, as are (ii)′ and (iii)′. So I will prove (i)$\Rightarrow$(iii)′ and finish by showing that (ii)$\Rightarrow$(i) since (iii)′$\Rightarrow$(ii) is obvious.

(i)$\Rightarrow$(iii)′ Suppose that (iii)′ fails. Then there is a descending chain of pp-definable subgroups $\varphi_0(M) \geqslant \varphi_1(M) \geqslant \ldots$, in k free variables for some $k \in \omega$, such that each index of one group in that above it is infinite.

Let κ be a cardinal satisfying $\kappa \geqslant 2^{|T|}$ and $\kappa^{\aleph_0} > \kappa$ (if $2^{|T|} = \aleph_\alpha$ then $\aleph_{\alpha+\omega}$ will do for κ). If T were superstable then (3.A) T would have to be κ-stable. Let M be a κ-saturated model for T. Then (exercise!) $[\varphi_n(M) : \varphi_{n+1}(M)] \geqslant \kappa$ for each n. So choose elements $\bar{a}_{n,\alpha}$ $(\alpha < \kappa)$ of $\varphi_n(M)$ which lie in distinct cosets of $\varphi_{n+1}(M)$. If $A = \{\bar{a}_{n,\alpha} : n \in \omega, \ \alpha < \kappa\}$ then $|A| = \kappa \aleph_0 = \kappa$: so we show that T is not superstable by producing strictly more than κ distinct types defined over A (for then T will not be κ-stable).

Let $\eta \in \kappa^\omega$. Define the pp-type $p_\eta(\bar{v})$ to be (the pp-deductive closure of) $\{\varphi_{n+1}(\bar{v} - \sum_{i=0}^{n} \bar{a}_{i,\eta(i)}) : n \in \omega\}$. It is perhaps easiest to think of the pp-types, p_η, as defining branches through a κ^ω-tree:



With this picture in mind, it is an easy exercise to check that the p_η are consistent and that if $\eta,\mu\in\kappa^\omega$, $\eta\neq\mu$ then p_η and p_μ are contradictory. But there are $\kappa^{\aleph_0}>\kappa$ of these pp-types and so, extending these to (necessarily distinct) complete types over A, one sees that T is not superstable.

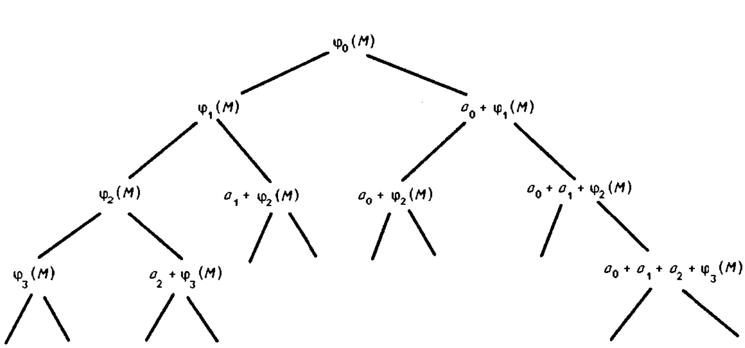
(ii)$\Rightarrow$(i) Let $p\in S_1{}^+(A)$. By assumption there is $\varphi(\bar v,\bar a)\in p$ such that for all $\psi(\bar v,\bar b)\in p$ the index $[\varphi(\bar v,\bar 0):\varphi(\bar v,\bar 0)\wedge\psi(\bar v,\bar 0)]$ is finite. Since there are only finitely many cosets of $\varphi(\bar v,\bar 0)\wedge\psi(\bar v,\bar 0)$ in $\varphi(\bar v,\bar a)$, once $\varphi(\bar v,\bar a)$ has been determined to be in p (and there are just $|A|.|T|$ inequivalent possibilities for such a formula), there are only finitely many inequivalent choices left for a representative of $\psi(\bar v,\bar y)$. Since there are only $|T|$ pp formulas, one therefore obtains at most $|A|.|T|.\aleph_0{}^{|T|}=|A|.2^{|T|}$ possibilities for p.

So if $|A|\geqslant 2^{|T|}$ then $|S_1(A)|=|S_1{}^+(A)|\leqslant|A|.2^{|T|}=|A|$ and hence T is superstable, as required.

(c) As with (b) it is enough to show (i)$\Rightarrow$(iii)$'$ and (ii)$\Rightarrow$(i), for (iv)$'$ is clearly equivalent to (ii)$'$.

(i)$\Rightarrow$(iii)$'$ Again, a tree of cosets is constructed. Suppose that there is an infinite properly descending chain $\varphi_0(M)>\varphi_1(M)>\ldots$ of pp-definable subgroups (of some model M of T). It must be shown that T is not totally transcendental: what is shown is that the reduct of T to a certain countable language is not ω-stable.

For each $n\in\omega$ choose $\bar a_n\in\varphi_n(M)\setminus\varphi_{n+1}(M)$, and set $\bar a_{n,1}=\bar a_n$, $\bar a_{n,0}=\bar 0$. For each $\eta\in 2^\omega$ define the (partial) pp-type p_η to be $\{\varphi_{n+1}(\bar v-\sum_{i=0}^{n}\bar a_{i,\eta(i)}):n\in\omega\}$. Thus the p_η define paths through a 2^ω-tree of pp-definable cosets. In particular it is easily seen that the p_η are consistent and are mutually contradictory.



Let L_0 be a countable sublanguage of L which contains all ring "elements" which appear in at least one of the formulas φ_n $(n \in \omega)$ (and make sure to include the identity $1 \in R$). Thus, in effect, we consider modules as being over the corresponding countable subring of R. Extend the p_η $(\eta \in 2^\omega)$ to, necessarily distinct, complete types over the countable set $\{\bar{a}_n : n \in \omega\}$ in this language L_0: there are $2^{\aleph_0}$ of these. So the reduct of T to L_0 fails to be ω-stable. Thus T is not totally transcendental, as required.

(ii)⇒(i) By (ii)/(iv) every $p \in S_1{}^+(A)$ is equivalent to a single pp formula over A. There are only $|T|.|A|$ of these. Hence $|S_1(A)| = |S_1{}^+(A)| = |A|.|T| = |A|$ provided $|A| \geqslant |T|$, as required (by 3.A). □

Example 1

(a) Consider the indecomposable pure-injective abelian groups (after 2.$\mathbb{Z}$11). All the $\mathbb{Z}_{p^n}$, $\mathbb{Z}_{p^\infty}$ and $\mathbb{Q}$ are ω-stable. The $\overline{\mathbb{Z}_{(p)}}$ are superstable, not ω-stable.

(b) Referring back to Example 2.1/5: all of (v)-(viii) are totally transcendental.

(c) The ring $\mathbb{Z}_{(p)}$ as a module over itself, is superstable but not totally transcendental.

(d) The ring $K[X]$, where K is a field, is noetherian but not artinian, so is not t.t. as a module over itself. Explicitly, an infinite descending chain of pp-definable subgroups (i.e., (left) ideals) is $K[X] > \langle X \rangle > \langle X^2 \rangle > \dots$. If K is infinite then this module fails to be superstable. If K is finite then it is superstable (by 3.1, since every proper factor is artinian).

Exercise 1 The cardinal $\kappa(T)$ which appears in the statement of 3.A(ii) is the smallest infinite cardinal strictly greater than the length of any properly descending chain in the fundamental order (cf. §5.1). Show that for modules it is the smallest infinite cardinal strictly greater than the length of any descending chain of pp-definable subgroups where, within the chain, the index of each subgroup in the preceeding is infinite. [Hint: this is essentially contained in the proof above.]

Exercise 2 Deduce that any $\kappa(T)$-saturated module is pure-injective (an example where $\kappa(T)$ is significantly less than $|T|^+$ is provided by any non-zero vectorspace over an uncountable field).

Modules, then, are stable in the languages which fix the ring. A more powerful kind of language has one sort for module elements and another sort for ring elements (so quantification over ring elements is allowed). In such a language one may express a good deal more than in the languages we are considering. A heavy price is paid, however, and one reflection of this is that complete theories of (ring, module) pairs in such languages are not in general stable.

Exercise 3 Investigate (ring, module) pairs (R, M_R) using this language (see [Ol70], [Ol71] and [Sab71a] first).

Perhaps it would be fruitful to consider bimodules $(S, {}_SM_R, R)$ using a 3-sorted language, since endomorphism rings can then be brought into the picture (although it might be possible to subsume this under the 2-sorted case, since an (S,R)-bimodule is just an $S^{op} \otimes R$-module).

I now outline the original proof of Baur that modules are stable [Bau75; Thm 1] (also [Fis72] contains all the ingredients). Let $\lambda \geqslant 2^{|T|}$ satisfy $\lambda^{|T|} = \lambda$: it will be sufficent (3.A) to show that T is λ-stable. So suppose that $A \subseteq M \models T$ and $|A| = \lambda$: it must be shown that $|S_1{}^T(A)| \leqslant \lambda$.

It may be supposed that M is a $|T|^+$-saturated model of cardinality λ [CK73; 5.1.4] - so M is pure-injective. Let $p \in S_1(A)$, and realise p in an elementary extension, M', of M. Say $M' = M \oplus C'$ and (m,c) realises p. Let C be an elementary substructure of C' of cardinality $\leqslant |T|$ and containing the element $c \in C'$. By the Fefermann-Vaught Theorem ([FV59; 5.1]), one has that $M \oplus C$ is an elementary extension of M (since $M' = M \oplus C$ is).

Thus every 1-type over A is realised in a module of the form $M \oplus C$ where C has cardinality $\leq |T|$. There are at most $2^{|T|} \leq \lambda$ non-isomorphic modules of cardinality $\leq |T|$, so there are at most λ possible isomorphism types for $M \oplus C$. Each such module contains λ elements. Therefore there are at most $\lambda = \lambda \times \lambda$ different realisations of types in $S_1(A)$, as required.

Exercise 4 A module M is said to be **saturated** if it is λ-saturated, where $|M| = \lambda$. Let $\kappa(T)$ be as in Exercises 1, 2 and let $\kappa < \kappa(T)$. Show that, if $\lambda^{\kappa} > \lambda$, then there cannot be a saturated model of the theory of M of cardinality λ [Hint: cf. proof of **3.1**]. (For instance, show that there is no non-superstable saturated module of cardinality $\aleph_\omega$ (use the fact that if κ is any cardinal then $\kappa^{\mathrm{cf}(\kappa)} > \kappa$, where $\mathrm{cf}(\kappa)$ is the cofinality of κ, plus the fact that the cofinality of $\aleph_\omega$ is $\aleph_0$ (for $\aleph_\omega$ is the union of $\aleph_0$ smaller cardinals: $\aleph_\omega = \bigcup \{\aleph_n : n \in \omega\}$. By way of contrast, $\aleph_1$ is not the union of countably many countable sets).

Show that there is a saturated model in each cardinality $\geq 2^{|T|}$ iff T is superstable (in fact, if T is λ-stable then there is a saturated model of cardinality λ, see [Poi85; 5.19]). Deduce from this a characterisation of the rings of finite representation type (see Chapter 11).

For a description of saturated modules as products of indecomposables, see [Zg84; 6.15].

As a first corollary of **3.1**, one sees that the totally transcendental modules coincide with the Σ-pure-injective modules of §2.3. This was proved independently by Gruson and Jensen [GJ76; Thm], Zimmermann [Zim77; 3.4]: the word "totally transcendental" does not appear in the first two papers, but the relevance of those papers was noted by Garavaglia [Gar80a; §6], the author [Pr80c] and Rothmaler [Rot83; Thm].

Corollary 3.2 *Let M be any module. Then M is totally transcendental iff M is Σ-pure-injective.*

Proof This is immediate by **2.11** and **3.1**. □

Then there are some corollaries to **3.1** which are immediate on using the invariants of §2.4.

Corollary 3.3 *If $T = T^{\aleph_0}$ and if T is superstable then T is totally transcendental.*

Proof This follows by **3.1** since (**2.23**) each of the invariants is 1 or ∞. □

Corollary 3.4 *If T is totally transcendental then $T^{\aleph_0}$ is totally transcendental.*

Proof This follows since T and $T^{\aleph_0}$ have isomorphic lattices of pp-definable subgroups (by **2.10**) so have, or fail to have, the dcc together. □

Corollary 3.5 *If M and N are totally transcendental then so is $M \oplus N$ (this is true of general structures - see* [Mac71; Lemma1]).

Proof If $\varphi(M \oplus N) > \psi(M \oplus N)$ then **2.10** yields $\varphi(M) \oplus \varphi(N) > \psi(M) \oplus \psi(N)$, so either $\varphi(M) > \psi(M)$ or $\varphi(N) > \psi(N)$. Therefore if both M, N have dcc on pp-definable subgroups, so has $M \oplus N$. □

Example 2 Infinite direct sums of t.t. modules need not be t.t.. Take $R = \mathbb{Z}$ and note that each $\mathbb{Z}_n$ $(n \geq 2)$, being finite, is t.t. but that $M = \bigoplus \{\mathbb{Z}_n : n \geq 2\}$ is not t.t. (for consider the formulas $2|v$, $2^2|v$, ..., $2^n|v$,...). M is not even superstable (since $\mathrm{Inv}(M, 2^n|v, 2^{n+1}|v)$ is infinite for each n). A more subtle example is $M' = \bigoplus \{\mathbb{Z}_p : p$ is prime$\}$, which may be seen to be superstable but not totally transcendental. Of course $\bigoplus \{\mathbb{Z}_p^{(\aleph_0)} : p$ is prime$\}$ is not superstable.

Corollary 3.6 *If M and N are superstable then so is $M \oplus N$.*

Proof Since $\mathrm{Inv}(M\oplus N,\varphi,\psi)=\mathrm{Inv}(M,\varphi,\psi).\mathrm{Inv}(N,\varphi,\psi)$ (2.23), this follows as in 3.5. □

Corollary 3.7 *If M is pure in N and N is superstable (respectively totally transcendental) then M is superstable (resp. totally transcendental).*

Proof Since purity of M in N implies $\varphi(M)=M\cap\varphi(N)$ for each pp φ, this follows by 3.1. □

Corollary 3.8 [Zg84; 2.3] *If M is an elementary substructure of N and N is superstable then the factor module N/M is totally transcendental.*

Proof One has (2.23) that $\mathrm{Inv}(N,\varphi,\psi)=\mathrm{Inv}(M,\varphi,\psi).\mathrm{Inv}(N/M,\varphi,\psi)$ for each φ,ψ. Now $M\equiv N$ implies that $\mathrm{Inv}(M,\varphi,\psi)\doteq\mathrm{Inv}(N,\varphi,\psi)$. Therefore, if $\mathrm{Inv}(N/M,\varphi,\psi)>1$, then $\mathrm{Inv}(N,\varphi,\psi)$ must be infinite. There are, in N, no infinite descending chains of pp-definable subgroups each of infinite index in the preceding one; hence in N/M there is no infinite descending chain of pp-definable subgroups, as required. □

Exercise 5 Let N_R be a pure-injective module and let $S=\mathrm{End}(N_R)$ be the endomorphism ring of N - so N is a natural left S-module.
(i) If N is totally transcendental then ${}_SN$ has dcc on finitely generated submodules.
(ii) If ${}_SN$ is artinian then N_R is t.t. (hence any left artinian ring is right t.t.).
(iii) If ${}_SN$ is noetherian, then N_R is t.t. iff ${}_SN$ is artinian.

Exercise 6 ([Mac71; Thm 1]) Suppose that R is commutative, and let M be totally transcendental. Then there is a divisible submodule M' of M (that is, for c a non-zero-divisor of R, $M'c=M'$), such that M/M' is c-torsion for some non-zero-divisor $c\in R$ (i.e., $(M/M')c=0$ - that is, $Mc\leqslant M'$).

In particular if R is a Dedekind domain (e.g. $R=\mathbb{Z}$) and if M is t.t., then $M=E\oplus B$ for some injective module E and some module B of "bounded" torsion (meaning $Bc=0$ for some $c\in R$, $c\neq 0$). Conversely, any module of this form is totally transcendental.

Exercise 7 A module M is said to be **compressible** if it embeds in each of its non-zero submodules. For example: any simple module is trivially compressible; $\mathbb{Z}_{\mathbb{Z}}$ is compressible, as are the $(\mathbb{Z}_{(p)})_{\mathbb{Z}}$ for p prime.

Show that if R is commutative and if M is a compressible module which is not simple, then M is not totally transcendental.

[Hint: treat the case of a cyclic module mR first, noting that since the ring is commutative each endomorphism is induced by a multiplication; then choose a proper embedding $f:M\longrightarrow mR$ and consider
$(mR,m)\leqslant(M,m)\simeq(fM,fm)\leqslant(mR,fm)\leqslant(M,fm)\simeq(fM,f^2m)\leqslant\ldots$.]

Is there any generalisation to the non-commutative case?

3.2 A structure theorem for totally transcendental modules; part I

The main object of this section is to present a broad structure theorem for totally transcendental modules and to observe that it covers a number of well-known structure theorems for modules. Part II, in Chapter 4, will investigate in more detail the indecomposable factors which occur, and will delineate the restrictions on how often they occur. A consequence of this will be Vaught's Conjecture for ω-stable modules; a more direct proof of this, due to Garavaglia, is outlined below as an exercise.

First, we look at some aspects of the relationship between pure-injectivity and total transcendentality.

Proposition 3.9 [Gar80; Thm 2] *If M is totally transcendental then M is pure-injective.*

Proof This follows from 3.2; but it also follows directly from 3.1. For, let $\Phi(v)$ be any set of pp formulas with parameters from M which is finitely satisfied in M. By 3.1, if M is t.t. then there is a single pp formula $\varphi\in\Phi$ such that $\varphi\leftrightarrow\Phi$. Since Φ is finitely satisfied in M, φ is satisfied in M and hence is realised in M. Thus M is pure-injective by 2.8. □

Let us record part of 3.1 separately.

<u>Proposition 3.10</u> *Suppose that T is totally transcendental and let $p\in S_n{}^T(A)$. Then there is some $\varphi\in p^+$ with $\varphi\leftrightarrow p^+$.* □

The next result barely counts as a weak converse to 3.9.

<u>Proposition 3.11</u> [Gar80; Footnote 2], [Zim82; Prop 3] *If M is pure-injective and if $|M|<2^{\aleph_0}$ (in particular (but see* [MS74]), *if M is a countable pure-injective), then M is totally transcendental.*

Proof Consider the proof of 3.1(c)(i)⇒(iii)′, assuming that M were not totally transcendental. Since M is pure-injective, the $2^{\aleph_0}$ distinct pp-types would all be realised in M: but this would contradict $|M|<2^{\aleph_0}$. □

The same argument gives the following ([Zim82; Prop 3]; also see [Pa77; 3.2.11] and [Law77]). If M is a (D,R)-bimodule, with D a division ring, and if $\dim_D M$ is countable, then M_R is totally transcendental (so this applies if R is an algebra over a field and M is countable-dimensional over the field). Zimmermann uses this to prove the following result: [Zim82; Thm 1] the group ring $R[G]$ is pure-injective (as a module over itself) iff R is pure-injective (as a module over itself) and G is finite.

From 3.11 one obtains a reasonable converse to 3.9.

<u>Theorem 3.12</u> [Gar79; Lemma 6] *If T is countable (or at least if $|T|<2^{\aleph_0}$) and if every model of T is pure-injective, then T is totally transcendental.*

Proof The assumption on $|T|$ implies, by the downward Lowenheim-Skolem Theorem, that T has a model of cardinality less than $2^{\aleph_0}$. By 3.11 such a model must be totally transcendental. Hence T is totally transcendental. □

If T has cardinality at least $2^{\aleph_0}$ then T may have only pure-injective models yet not be totally transcendental.

Example 1 [Pr80d] Let $R=\overline{\mathbb{Z}(p)}$ (as a ring) and take $T=\mathrm{Th}(R_R)$. Then every model of T has the form $R\oplus Q^{(\lambda)}$ for some λ, where Q is the quotient field of R (this can be proved using the invariants (cf. §2.$\mathbb{Z}$)). Thus every model is pure-injective.

But, since $R>pR>p^2R>\ldots>p^nR>\ldots$, T is not totally transcendental.

One does at least have the following.

<u>Proposition 3.13</u> [Rot83; (3)] *If every model of T is pure-injective then T is superstable.*

Proof Let $\kappa\geqslant 2^{|T|}$ be such that $\kappa^{\aleph_0}>\kappa$.

It is not difficult to see that there is $M\models T$ with $|M|=\kappa$ and with $[\varphi(M):\psi(M)]=\kappa$ for all pp formulas φ,ψ with $\mathrm{Inv}(T,\varphi,\psi)$ infinite (go to a κ^+-saturated model and then cut down).

Then if M were not superstable one could construct, as in 3.1(b)(i)⇒(iii)′, $\kappa^{\aleph_0}$ distinct pp-types defined with parameters in M. Therefore M, being pure-injective, would have to realise all these pp-types. This would imply $|M|\geqslant\kappa^{\aleph_0}>\kappa=|M|$ - the required contradiction. □

Exercise 1 Suppose that R is countable, or simply that $|R|<2^{\aleph_0}$. If N is pure-injective with no proper non-zero pure submodule then N is t.t. (cf. [Ok77]).

<u>Theorem 3.14</u> [Gar80; Lemma1] *If M is a totally transcendental module then M is a direct sum of indecomposable (pure-injective) submodules.*

Proof First it is shown that M has an indecomposable direct summand. Choose $a \in M$, $a \neq 0$ (I am ignoring the trivial case $M = 0$). Let N be a pure submodule of M which does not contain a and which is maximal such. Note that, although the union of a chain of direct summands need not be a direct summand, the union of a chain of pure submodules is pure, so N does exist by Zorn's Lemma.

Since N is pure in M, **3.7** implies that N is totally transcendental. So by **3.9**, N is pure-injective. Hence $M = N \oplus N_0$ for some N_0; it will be shown that N_0 is indecomposable.

If $N_0 = N_1 \oplus N_2$ then, since $(N \oplus N_1) \cap (N \oplus N_2) = N$ it may be supposed that $a \notin N \oplus N_1$ say (since $a \notin N$). But then, by maximality of N, one has that $N = N \oplus N_1$. So $N_1 = 0$. Thus N_0 is indeed indecomposable.

Now let $\mathcal{F}$ be the collection, ordered by inclusion, of all sets of the form $\{N_i\}_i$ with $\Sigma_i N_i = \oplus_i N_i$ pure in M and with the N_i all indecomposable. Since increasing chains of pure submodules are pure (and directness of the sum is no problem, being a finitary property), Zorn's Lemma implies that there is a maximal family in $\mathcal{F}$. Let N_1 be the (direct) sum of the (indecomposable) members of this maximal family.

As above, one has $M = N_1 \oplus N'$ for some N'. Since N' is pure in M, N' is (**3.7**) t.t. so, by the above, N' has an indecomposable direct summand if it is non-zero. But this would contradict maximality of the chosen family. Hence $N' = 0$.

That is: $M = N_1$ is a direct sum of indecomposable submodules. □

It will follow from **4.A14** that this representation of M (t.t.) as a direct sum of indecomposable submodules is essentially unique (in the sense discussed in §4.3). Garavaglia ([Gar80; Lemma3]) proved directly that the endomorphism ring of an indecomposable totally transcendental module is local (cf. **4.27**) and so deduced uniqueness. Kucera ([Kuc87; §4]) derives uniqueness by using dimension theory for regular types, so avoiding **4.A14** and the need to show that the endomorphism ring of an indecomposable pure-injective is local.

Exercise 2

(i) Show that **3.14** does not characterise the totally transcendental theories - that is, find a non-t.t. theory, each of whose models is a direct sum of indecomposable pure-injective modules.

(ii) What if we assume that R is countable? [Hint: count types; cf. proof of **3.1**.]

(iii) What if we then drop the requirement that the indecomposables be pure-injective? [Hint: cf. §7.2.]

Exercise 3 [Gar80; Thm6] Show (without assuming the continuum hypothesis!) that if T is countable and ω-stable then the number of non-isomorphic models of T is either countable or is $2^{\aleph_0}$ (Vaught's Conjecture holds for t.t. modules).
[Hint: $|\bigcup\{S_n{}^T(0) : n \in \omega\}|$ is either $\aleph_0$ or $2^{\aleph_0}$. If the latter, then there must be $2^{\aleph_0}$ countable models. If the former, then there is a countable prime model M_0, and a countable saturated model M_1. One has $M_1 = M_0 \oplus N$ for some N. Then show that the number of countable models is either no more than $\aleph_0$ or is $2^{\aleph_0}$ according as the number of non-isomorphic factors of N is finite or $\aleph_0$. Use **3.14** and the uniqueness stated just above and the invariants of §§2.4, 2.5. Also use **2.24**.]

Example 2 Recall that an injective module E is said to be Σ-injective if $E^{(\aleph_0)}$ is injective. An injective is Σ-injective (i.e., by **3.2**, is t.t.) iff the ring has acc on annihilators of elements of E (exercise). In particular, if R is right noetherian then every injective is Σ-injective. Thus **3.14** includes the theorem of Matlis [Mat58; 2.5] which states that every injective over a noetherian ring has an (essentially unique) decomposition as a direct sum of indecomposable injectives.

Exercise 4 [Gar80a; §6] Suppose that M is a finite module. Show that every power M^{κ} of M is a direct sum, $M^{(\lambda)}$ $(\lambda = 2^{\kappa})$, of copies of M.

[Hint: by **3.14**, M^{κ} is a direct sum of indecomposable modules, so it need only be shown that all the indecomposable summands are isomorphic to M. It is probably easiest to use some of our later results on the space $\mathcal{I}(M)$, but a proof is possible at this stage.

1. M, hence $M^{\aleph_0}$, has only finitely many pp-types, hence only finitely many types; so $M^{\aleph_0}$ is $\aleph_0$-categorical.

2. If there were an indecomposable factor $N \not\cong M$ of a model of the theory of $M^{\aleph_0}$, then $M^{\aleph_0} \oplus N$ would be a countable model of this theory but not isomorphic to $M^{\aleph_0}$ (by **4.A14**) - contradicting $\aleph_0$-categoricity.]

Exercise 3 Let $|R| + \aleph_0 = \kappa$. Suppose that E is an indecomposable injective.

(i) If E is Σ-injective then $|E| \leqslant \kappa$.

(ii) In particular if R is countable and if E is an indecomposable injective then E is Σ-injective iff E is countable.

(Of course the same goes more generally for pure-injectives.)

Example 3 An abelian group is ω-stable iff it has the form $E \oplus B$ where E is injective and B is torsion of bounded exponent (see Exercise 3.1/6). The decomposition of the injective is covered by Ex 2 above. Also **3.14** gives that B has a direct sum decomposition, so this includes (with a bit more work describing the possible indecomposable factors) the decomposition for abelian groups of bounded exponent.

Example 4 Projective modules over right perfect, left coherent rings may be shown to be t.t. (see **14.19**, **14.22**) - so they are direct sums of indecomposable projectives.

Example 5 [Gar79; Cor 3] Suppose that R is commutative and let M_R be artinian. Then M is totally transcendental. Hence each power, M^{κ}, of M is t.t., so is a direct sum of indecomposable submodules. This argument applies to any power of a t.t. module.

Example 6 Rings over which each module is totally transcendental (so has a direct sum decomposition) are considered in Chapter 11 (they are the right pure-semsimple rings).

3.A Abelian Structures

I think that the main omission from the body of the text is a treatment of Fisher's "abelian structures". This is especially unfortunate since Fisher's account of them was produced as long ago as 1975, yet they have been unduly neglected since. Abelian structures include modules as a very special case and it has long been folklore that "if it goes for modules then it goes for abelian structures".

Abelian structures are "many-sorted versions of modules, with predicates" (for a more precise definition, see below). For instance, if one wishes to discuss the model theory of pairs of R-modules with a specified morphism between them $(M \xrightarrow{f} N)$, then one sets up a two-sorted language (one sort for the domain, another for the codomain) and then adds a function symbol which takes arguments of the first sort and has values in the second sort (alternatively, introduce a predicate for the graph of the function). Then one specifies that the domain and the codomain are R-modules and that the function is an R-morphism between them.

Again, if one wishes to discuss pairs $M \leqslant N$ of R-modules, then one may use the usual 1-sorted language for R-modules, together with a predicate for a submodule. Alternatively, one may use the previous set-up, adding the extra axiom that the function is monic. One could also talk about pairs of R-modules and the groups of R-morphisms between them by using a 3-sorted language (two sorts for elements; one sort for morphisms).

My reasons for not developing this in the text are, first, that to develop a substantial part of the theory in this generality would involve a considerable amount of work and would make this book rather dificult to read (*vide* the comparative neglect of [Fis75]). Another reason is that, with no clear major applications in mind, the exercise would be of doubtful value: It seems more sensible to let applications of abelian structures build up, before acknowledging the necessity (as opposed to desirability) of a careful development.

There are applications - even in the text (namely, representations of quivers: but they may be treated as modules over the appropriate path algebra). For instance, Baur in his work on the four-subspace problem [Bau80] used abelian structures, in that he employed the language of vectorspaces with four predicate symbols, rather than that of modules over the path algebra of $\tilde{D}_4$. Also, the representations of posets that I discuss in Chapter 17 are abelian structures - but I don't need any theory there. It is a little unsatisfactory that authors tend to call on results about abelian structures which may not have been written down carefully (at least in print), but which "clearly" are proved just as in the modules case. At least the pp-elimination of quantifiers has been treated properly by Weispfenning [Wei83a] and, of course, there is already a great deal in [Fis75]. Kucera [Kuc84] has made the effort to develop some of his results in this generality, and Piron's thesis [Pir87] is largely set in the context of certain kinds of abelian structure.

Apart from the ways in which abelian structures allow one to extend to scope of the "model theory of modules", there are interesting developments in pure model theory. In the course of investigating superstable structures of U-rank 1, Buechler and others have found that abelian structures arise in a natural way from the geometry of realisations of types (see comments in §5.1, §7.2).

I describe abelian structures and then outline the main results of Fisher's work [Fis75], the first three sections of which are published as [Fis77] (unfortunately, the remainder is unpublished).

The language L has a symbol for addition, one for subtraction and, for each sort, a zero element of that sort. Addition and subtraction are applied only to arguments of the same sort. There may (and usually will) be additional relation and function symbols in L. The following requirements must be met (they are easily expressed as axioms): the elements of each sort form an abelian group (under the restriction of "+" and "-", with the zero of that sort being interpreted as the zero element); each function is a group morphism; each relation is a group. Let λ be the cardinality of the language L.

One works within an **abelian class** K: a class which is closed under substructures and products, which is "compact" and which satisfies the HEP. Fisher's definition of a "compact" class includes elementary classes but also allows, for example, reducts of elementary classes (for "concreteness", one may think in terms of axiomatisable classes). The HEP ("Homomorphism Extension Property) is: every diagram as given has a completion as shown.

(note that this excludes certain universal Horn classes of modules - cf. Ex15.3/1).

For some of the results, it is not necessary to assume that the language is the usual finitary $L_{\omega\omega}$ (i.e., no expressions of infinite length) but, <u>from now on we assume that K is an abelian class and L is finitary.</u>

Examples are: R-modules; ($\mathbb{Z}$-indexed) chain complexes of R-modules (one function symbol suffices for the boundary maps, since functions are not restricted to having domains entirely within one sort); representations of a quiver; small additive categories (there is more than one way of treating these within the framework of abelian structures); if $\mathcal{S}$ is a small additive category, the functor category $(\mathcal{S},\mathrm{Ab})$; representations of posets; if $R\leqslant S$ is an inclusion of rings, the category of pairs (M,N) where $M\leq N$, N is an R-module and M is an S-module which is an R-submodule of N when given its induced R-structure.

Also, given a compact class K, closed under substructures and products, but not necessarily having the HEP, one may expand the language L to L^* by adding a relation symbol R_φ for each pp formula φ. Then let K^* be the class of L^*-structures with L-reduct in K and which satisfy $\forall\bar{x}(R_\varphi(\bar{x})\leftrightarrow\varphi(\bar{x}))$. So every member of $\mathcal{K}$ has a unique expansion to an L^*-structure and the L^*-embeddings are just the morphisms whose reduct to L are pure embeddings. Then define $\mathcal{K}^p$ to be the class of substructures of members of $\mathcal{K}^*$: $\mathcal{K}^p$ is an abelian class. Note that $\mathcal{K}^p$ and $\mathcal{C}_T{}^*$ (§5.4) are essentially the same - each structure in $\mathcal{K}^p$ carries with it information about how it embeds into an "honest" member of the class (one where $R_\varphi(\bar{x})$ holds iff $\varphi(\bar{x})$ does). The absolutely pure members of $\mathcal{K}^p$ are just those of $\mathcal{K}^*$.

The use of $\mathcal{K}^p$ and $\mathcal{K}^*$ is similar to our use of various functor categories in Chapter 12: the pure-injective objects of $\mathcal{K}$ are exactly those whose L^*-expansions are injective objects of $\mathcal{K}^*$ (equivalently, of $\mathcal{K}^p$) (3.5, 3.6 - these refer to numbering in [Fis75], as do all numbered references below). Also, in any abelian class the pure-injectives are precisely the algebraically compact objects (3.7). (I should say at this point that I often quote Fisher's results under stronger hypotheses than those in [Fis75].)

The third section of [Fis75] is much concerned with injectivity: in particular, there is a generalisation (3.10) of Eklof and Sabbagh's result which says that every $\aleph_0$-injective module is injective iff the ring is right noetherian (cf. §15.4). As a corollary of this generalisation, every abelian class has enough injectives (3.11). Essential embeddings are considered, and it is shown that if $\mathcal{K}$ has enough injectives then it has injective hulls (3.17). Also (3.25), in this generality, injectives, each embeddable in the other, are isomorphic (the proof is as in **1.8**). Of course, all this is applicable to $\mathcal{K}^p$, so we obtain corollaries about pure-injectives and, more to the point, we get the hulls of §4.1 which are such an essential part of the model theory of modules. These corollaries on "pp-essential morphisms" were quoted by Garavaglia (who also gives them direct proofs) in [Gar80a]. The existence of hulls was not sufficiently appreciated for some time, though see [Che76; ChptVI], [EkMk79; §4], [Gar80], [Gar80a].

The fourth section begins with a lengthy discussion of why the use of saturated extensions is preferable to the use of ultraproducts (the main point being that ultraproducts give more than one really needs or wants). The first result of the section is the proof, in this general setting, of Sabbagh's result for modules: $M\equiv N$ iff there are pure embeddings $M\prec^*_1 N'\equiv N$ and $N\prec^*_1 M'\equiv M$. Various corollaries follow (as in the modules case): in particular, every abelian structure is an elementary substructure of its pure-injective hull (4.5). It is also shown that, for any abelian class, the existentially complete, finitely generic and infinitely generic structures coincide (4.7). Then there are various results on elimination of quantifiers (note that elimination of quantifiers in $\mathcal{K}^p$ corresponds to pp-elimination of quantifiers in $\mathcal{K}$!). In particular, it is shown (4.13) that every formula is equivalent to an $\forall\exists\forall$ formula (cf. after **2.19**). Fisher also finds a structural criterion (4.14), in terms of cancellation in saturated models, for every formula to be equivalent to a boolean combination of $\forall\exists$ sentences and pp formulas (cf. **2.13**). He is not, however, able to establish this structural property in general (though he does show it for boolean rings - i.e., commutative regular rings with every factor field having two elements).

His fifth section discusses homogeneous-universal structures and, in particular, he shows (by two different arguments) that abelian structures are stable (5.5) (cf. **3.1**). One argument uses the previous results of the section to show that if $\kappa=\kappa^\lambda$ and if $A\in\mathcal{K}$ then there is a saturated structure elementarily equivalent to A of cardinality κ: recall that λ is the cardinality of the language. The other argument is more direct, and is essentially that which is also given by Baur (see §3.1).

There is a natural notion of a member of $\mathcal{K}$ being presented by generators and atomic relations. Let γ be the least infinite regular cardinal such that every cyclic (= 1-presented) structure is γ-presentable (the definitions of these terms are just as one should expect). It is shown (5.7) that the class of injectives is axiomatisable in $L_{\kappa\kappa}$ iff $\kappa\geqslant\gamma$.

The sixth section generalises results of Eklof and Sabbagh. Say that the abelian class $\mathcal{K}$ is noetherian if $\gamma=\aleph_0$. Then, by the above, $\mathcal{K}$ is noetherian iff the class of injectives is axiomatisable (for modules this is [ES71; 3.19]): a number of other equivalents are given as 6.1. Say that $\mathcal{K}$ is coherent if every finitely generated substructure of a finitely presented structure is finitely presented. Then (6.2) $\mathcal{K}$ is coherent iff the class of absolutely pure structures is axiomatisable, and this is so iff the class $\mathcal{K}$ has a model-completion (equivalently, a model-companion). The proof of that also shows that if $\mathcal{K}$ is a coherent abelian class then the $\aleph_0$-injectives are just the absolutely pure structures (it is not known if this is so over non-coherent rings). It is shown that if $\mathcal{K}$ is abelian then $\mathcal{K}^{\mathrm{p}}$ is coherent (6.4): Fisher also shows (6.5) that if $\mathcal{S}$ is a small abelian category then the abelian class $(\mathcal{S},\mathrm{Ab})$ of all abelian-group valued functors on $\mathcal{S}$ is a coherent class (cf. [Aus66]). From this is derived a proof of the Mitchell embedding theorem: if $\mathcal{S}$ is a small abelian category then there is a ring R and a full exact embedding of $\mathcal{S}$ into the category of R-modules. There is a criterion (6.9) for coherence which generalises the usual one in terms of right ideals of the ring ([St75; I.13.3], also cf. §15.4).

The seventh section considers subdirect representation. It is shown (7.10) that if there is a bound on the cardinality of subdirectly irreducible structures then 2^{λ} is such a bound (cf. [Tay71], [MS74]). Fisher also introduces a notion of irreducible structure: A is irreducible if, given morphisms $A \longrightarrow B$ and $A \longrightarrow C$, if the induced morphism $A \longrightarrow B \times C$ is an embedding then at least one of $A \longrightarrow B$, $A \longrightarrow C$ is an embedding. He relates this to irreducibility of presentations and hence sees it as a generalisation of irreducibility of right ideals (cf. §8.1). Thus his 7.12 is very closely related to (perhaps includes?) the fact (8.2) that a pp-type is irreducible iff it has indecomposable hull. His 7.14 (combined with 7.12) shows that an indecomposable pure-injective has local endomorphism ring. Fisher says that $\mathcal{K}$ is Azumayan if every injective is discrete. He obtains the criterion (7.15) that $\mathcal{K}$ is Azumayan if every cyclic presentation of a non-trivial structure can be written as the intersection of a presentation of an irreducible structure and a presentation strictly larger than the first: this generalises the usual criterion in terms of the right ideal structure of the ring for every injective to be discrete (1.12), and also includes the obvious criterion in terms of pp-types (namely, that every pure-injective model is discrete iff every pp-type p can be written as $q \cap q'$ where q is irreducible and $q' \supset p$ - for this says exactly that $N(p)$ has an indecomposable summand. He shows (7.16) that noetherian implies Azumayan and hence, as a special case, derives Matlis's result that an injective over a noetherian ring is the injective hull of a direct sum of indecomposables.

Fisher then goes on to derive the general structure theorem for injective objects in his "abelian classes". He uses the term "impregnable" where I have used "continuous". His structure theorem (7.21) is the result which I quoted as **4.A10**, but set in the context of abelian structures. There is a strong elimination of quantifiers for injective members of noetherian abelian classes (7.24). He shows that over a Dedekind domain, there are no continuous pure-injectives (7.29) (cf. after **2.ℤ11**): he uses localisation - cf. Ziegler's more general results in §2.ℤ. He also obtains **16.26** for the special case of boolean rings (7.31).

In the eighth section Fisher is concerned with artinian and completely reducible classes (cf. modules over artinian rings and over semisimple artinian rings), obtaining a Krull-Schmidt theorem (8.7) for objects in an abelian class which are both "noetherian" and "artinian". He also derives, under the strong hypothesis of finiteness of the endomorphism ring, pp-elimination of quantifiers (8.9). Then there is a generalisation of the Wedderburn characterisation of semisimple artinian rings (8.18). There are results on those abelian classes which are "dual" to the completely reducible ones. There is a generalisation of the Faith-Walker Theorem (8.26): if $\mathcal{K}$ is an abelian class and there exists a cardinal κ such that arbitrarily large homogeneous-universal models are direct sums of substructures of cardinality $\leqslant \kappa$, then $\mathcal{K}$ is noetherian (cf. **11.4**, **11.6**).

CHAPTER 4 HULLS

It was shown in the last chapter that every totally transcendental module is a direct sum of indecomposable submodules. The proof of this was short - in a sense too short, since it tells us little about the indecomposable factors which occur. For instance, if a is an element of the totally transcendental module M and if N is a minimal direct summand of M containing a, then what is the relationship between N and a? Is N uniquely determined by a? Does N depend on a or just on the pp-type of a? These questions will be answered in this chapter.

In section 1 it is shown that, given any pure-injective module M and any element (or subset) of M, there is a minimal direct summand of M containing the element (or subset). We will call this the hull, $N(A)$, of the element or subset A and the terminology is justified by showing that this hull is unique up to isomorphism over A. Furthermore, it is shown that the hull of A depends only on the pp-type of A.

The terminology is reminiscent of that for injective hulls: indeed, the above hulls can be seen as injective hulls in an appropriate (functor) category. I don't, however, take that approach to them, preferring to work on a more "concrete" level. The injective hull of a module A is characterised by the fact that every element in it is "linked" in a non-trivial way to A by an atomic relation (equation). There is an analogy for hulls: every element of the hull of A is linked in a non-trivial way to A by a pp-relation.

Injective hulls are examples of hulls in the sense of section 1. There are other examples. For instance, pure-injective hulls lie at an opposite extreme and are special cases of the hulls of §1. The second section considers there special cases and some others, and provides some examples.

A major theme of these notes is the representation of pure-injective modules in terms of simpler components (best of all, as pure-injective hulls of sums of indecomposable factors). There is a general decomposition theorem for pure-injective modules: in fact it is equivalent to the better-known corresponding result for injective objects in Grothendieck abelian categories, since there are functors which convert pure-injective modules into injective objects. The general result says that every pure-injective module is the direct sum of a discrete pure-injective and a continuous pure-injective. A pure-injective is discrete if it is the pure-injective hull of a direct sum of indecomposable summands. A pure-injective is continuous if it has no indecomposable factor. This representation is essentially unique. All this material is presented in the third section. The discussion takes place in the context of Grothendieck abelian categories: proofs are omitted, but I intend that the section should allow the reader to feel fairly comfortable with the ideas discussed.

One may study the indecomposable pure-injectives by looking at the pp-types which are realised in them: we say that a pp-type is irreducible if its hull is (direct-sum-) indecomposable. A syntactic criterion is given for a pp-type to be irreducible. The fourth section contains various result on indecomposable (and continuous) pure-injectives. It is shown, for example, that every continuous pure-injective is a direct summand of a product of indecomposable pure-injectives and (hence) every module is elementarily equivalent to a discrete pure-injective. The section also contains a general method for producing irreducible types that allows some control over their properties.

The fifth section follows up the observation that, in complete theories of modules, certain indecomposable summands may be constrained to appear only a limited (finite) number of times in the decomposition of any pure-injective model. The factors so constrained are identified, as are the types whose hulls they are, and the "unlimited part" of a complete theory of modules is defined. When it comes to describing forking, ranks, regularity and other model-theoretic notions (Chapters 5 and 6), it is this unlimited part which is important.

The models of a totally transcendental theory of modules are described in §6. It is shown that certain indecomposable factors must occur in every model; that certain of these occur a fixed, finite number of times in every model; others may occur any number of times, but must appear; others must occur infinitely often; then there are some factors for which there is no restriction on the number of occurrences.

With this description to hand, various points about these theories become fairly trivial: these include characterising the prime model, proving Vaught's Conjecture, describing saturated models and characterising the $\aleph_0$-categorical and $\aleph_1$-categorical theories (the last is done in a supplementary section). Almost all of the material of this section goes through in much wider circumstances, though rather more work is needed to show this (cf. Chapters 9 and 10).

If T is a complete theory of modules then the set of (isomorphism types of) indecomposable pure-injective factors of models of T carries a natural topology, under which it is a compact space. It is shown that the closed sets of this topology are in natural correspondence with those component theories of T (in the sense of §2.6) which are closed under products. Examples relate this topology to the Zariski topology and to the Pierce spectrum. The section (the seventh) closes with a criterion, in terms of this space, for existence of prime and/or minimal models.

4.1 pp-essential embeddings and the construction of hulls

In this section hulls are constructed. We will see that given any subset A of a pure-injective module, there is a minimal direct summand containing that subset: moreover this "hull" does deserve the name, in that it is in some sense a smallest extension of A and it is unique up to A-isomorphism. Even the existence of such minimal direct summands is surprising and seems to have been unknown until Fisher's work [Fis75] (and insufficiently appreciated for some while afterwards). I do not approach hulls by the most direct route (for that, one may consult [Zg84]), but rather take a somewhat meandering path which allows us to view them from various angles. In this, I follow [Pr81]. Let us proceed with the construction.

The embedding of an element in its hull is a generalised version of an essential embedding. Turning back to §1.2 one sees that there were three ways of defining essential embeddings: in terms of elements; in terms of morphisms; and in terms of ($\bigwedge$-atomic) types. Each of these methods is also represented here.

First I consider the element-wise definition. In the injective case the central type of relation has the form $ar=b$ where a,b are non-zero elements of a module and $r\in R$. In view of our replacement of atomic formulas by pp formulas the following definition is a natural one.

Let $A,B\subseteq M$. Say that A and B are linked if there are $\bar{a}$ in A, $\bar{b}$ in B, and φ a pp formula, such that $M\models\varphi(\bar{a},\bar{b})\wedge\neg\varphi(\bar{a},\bar{0})$: by linearity of pp formulas, equivalent to this requirement is $M\models\varphi(\bar{a},\bar{b})\wedge\neg\varphi(\bar{0},\bar{b})$, so the condition is symmetric. Thus: A and B are linked iff there is a tuple $\bar{a}$ from A whose pp-type over B is not that of the zero tuple. The context M is quite essential in the definition (in contrast to the $\bigwedge$-atomic case). The terminology is extended to elements and tuples in the obvious way.

Example 1

(i) If E is a non-zero indecomposable injective and if a and b are non-zero elements of E, then a and b are linked. For there is a relation $ar=bs\neq 0$ for some $r,s\in R$ (by 1.7). So, setting $\theta(v,w)$ to be $vr-ws=0$, we have $E\models\theta(a,b)\wedge\neg\theta(a,0)$.

(ii) $R=\mathbb{Z}$; $M=\mathbb{Z}_4$; $a=2\in\mathbb{Z}_4$, $b=1\in\mathbb{Z}_4$. The fact that a and b are linked is witnessed by the fact that $b2=a$ and $b2\neq 0$.

(iii) To contrast with the injective case, consider $R=\mathbb{Z}$, $M=\overline{\mathbb{Z}_{(p)}}$. Since each indecomposable injective $\mathbb{Z}$-module is countable and since $|E(\overline{\mathbb{Z}_{(p)}})|\geq|\overline{\mathbb{Z}_{(p)}}|=2^{\aleph_0}$, the module M is not uniform. In particular there are (many pairs of) elements $a,b\in M$ with $a\mathbb{Z}\cap b\mathbb{Z}=0$. Such elements are not "linked" in a purely algebraic sense. It will follow, however, from 4.11 below, that a,b are linked in the sense just defined. This may also be shown directly - exercise (use the description of elements of M given in §2.$\mathbb{Z}$).

(iv) $R=K[X,Y]/\langle X,Y\rangle^2$ (see Ex2.1/6(vi)). The elements $x,y\in R$ (being respectively the images of X,Y) satisfy $xR\cap yR=0$, but they are linked. For let $\varphi(v,w)$ be $\exists u\,(v=ux\wedge w=uy)$. Then $R\models\varphi(x,y)$ (take $u=1$) but $R\models\neg\varphi(x,0)$. To justify the latter assertion, suppose otherwise: then there is $c \in R$ with $cy=0$ and $cx=x$; but $\mathrm{ann}_R y = J$ so $c \in J$; also $Jx=0$, so $x=cx=0$ - contradiction as required.

Exercise 1 A little care is needed in using the notion of linking: from the fact that A and B are linked it cannot be concluded that there is a single $a \in A$ and $\overline{b}$ in B with a linked to $\overline{b}$.

It will be seen in Chapter 5 that A and B being linked is a property closely connected with A and B being dependent (in the sense of stability theory) over 0 (in general it is slightly weaker, but for theories closed under product it is precisely this). On the other hand, as has been noted already, it generalises the situation where $AR \cap BR \neq 0$ (by AR I mean the submodule generated by A): for if $\Sigma_1^n a_i r_i = \Sigma_1^m b_j s_j \neq 0$ then take $\theta(\overline{v},\overline{w})$ to be the formula $\Sigma_1^n v_i r_i = \Sigma_1^m w_j s_j$ and observe that $\theta(\overline{a},\overline{b}) \wedge \neg\theta(\overline{a},\overline{0})$ holds.

It was noted that the relation "linked" is symmetric. For convenience I will regard the zero tuple as being linked to any other tuple. Also if $\overline{b}$ is linked to $\overline{a}$ and if $\overline{c}$ is arbitrary then, trivially, $\overline{b}\frown\overline{c}$ is linked to $\overline{a}$.

Exercise 2 Suppose that N is pure in N' and that $\overline{a},\overline{b}$ are in N. Then $\overline{a},\overline{b}$ are linked in N iff they are linked in N'.

This notion may be described in terms of types (compare with 1.9).

Say that $p \in S^+(A)$ is maximal if it is maximal with respect to inclusion in $S^+(A)$. In the case of $S^+(0)$ there is a unique maximal element, namely the pp-type of the zero tuple of appropriate length. In general, realised types are maximal (for then $G(p)=0$) but there may well be non-realised maximal pp-types. We say that a type is maximal if its pp-part is so.

Clearly (note!) if $\mathrm{pp}(\overline{c}/A)$ is maximal then either $\overline{c}=\overline{0}$ or $\overline{c}$ is linked to A. The converse is not true.

Exercise 3 $R=\mathbb{Z}$; $M=\mathbb{Z}_8 \oplus \mathbb{Z}_2$. Take $a=(0,1)$, $b=(4,1)$, $c=(2,1)$, all in $\mathbb{Z}_8 \oplus \mathbb{Z}_2$. The endomorphism $(-\times 2) \oplus \mathrm{Id}_{\mathbb{Z}_2}$ takes $c\frown a$ to $b\frown a$, so by **2.7** $\mathrm{pp}(c\frown a) \leq \mathrm{pp}(b\frown a)$ and hence $\mathrm{pp}(c/a) \leq \mathrm{pp}(b/a)$. Also $2 | (c-a)$ but $2 \nmid c$ so c is linked to a. But $4 | (b-a)$ whereas $4 \nmid (c-a)$, so $\mathrm{pp}(c/a)$ is not maximal in $S^+(A)$.

<u>**Lemma 4.1**</u> *Suppose that p is a type over A such that p^+ is maximal. Then p^+ proves p (modulo T).*

Proof It must be shown that every formula in p is a consequence of some pp formula in p. By **2.20** it is enough to show that if φ is pp and not in p then p^+ proves $\neg\varphi$. By maximality of p^+ it must be that $p^+ \cup \{\varphi\}$ is inconsistent - as required. □

Thus a type is maximal (i.e, its pp-part is maximal) iff it is proved by its pp-part (the converse to **4.1** is immediate).

It will be seen that the hulls constructed in this section are independent of the over-theory T and depend only on pp-type. So, in order to compare pp-types in modules which are not necessarily elementarily equivalent, we consider the way in which the posets of the form $S^+(A)$ are connected by morphisms.

Suppose $A \leq M$ and let $f: A \longrightarrow A' \leq M'$ be a bijection such that $\mathrm{pp}^M(A) = \mathrm{pp}^{M'}(A'=fA)$. Then, for each α, there are natural maps

$$S^+(A) \underset{f_*}{\overset{f^*}{\rightleftarrows}} S^+(A')$$

where $S^+(A)$ is the set of pp-α-types over A modulo $\mathrm{Th}(M)$ and $S^+(A')$ is the set of pp-α-types over A' modulo $\mathrm{Th}(M')$. These maps are defined by:

$f^*p = \langle \{ \varphi(\overline{v}, f\overline{a}) : \varphi(\overline{v},\overline{a}) \in p \} \rangle$ and

$f_*q = \langle \{ \varphi(\overline{v}, f^{-1}\overline{a}') : \varphi(\overline{v},\overline{a}') \in q \} \rangle$, where "$\langle - \rangle$" denotes pp-deductive closure modulo the appropriate theory.

These order- (and topology-) preserving maps need be neither onto nor 1-1, although it is clear that $f_*f^*p \geq p$. However, when restricting to maximal pp-types, one does have a bijection (homeomorphism even).

<u>Lemma 4.2</u> *Let $A \leq M$ and suppose that $f: A \longrightarrow A' \leq M'$ is a bijection such that $\text{pp}^M(A) = \text{pp}^{M'}(A')$. Then the induced maps $f^*: S^+(A) \longrightarrow S^+(A')$ and $f_*: S^+(A') \longrightarrow S^+(A)$, when restricted to the respective sets of maximal pp-types, are inverse bijections.*

Proof This follows easily since f^* and f_* are order-preserving and since f^*f_* and f_*f^* are increasing maps. ▫

Exercise 4 Show that f^* need be neither onto nor 1-1. When is $f_*f^*p = p$?.

The point of the lemma above is that the hull of A will depend only on its pp-type, and not on any over-theory. That hulls should show some such independence is reasonable. For example, the module $\mathbb{Z}_2$ occurs as a direct summand of (pure-injective) models of many theories of $\mathbb{Z}$-modules.

Given a set or module of parameters, A, the hull of A will be realised by successively realising maximal pp-types. Let us see where such a process can stop.

<u>Lemma 4.3</u> [Pr81; 2.3] *Let A be a subset of the pure-injective module M. Then A realises every maximal pp-type in $S^+(A)$ iff A is a direct summand of M (that is, if A is pure-injective and is pure in M). It is equivalent to require that A realise every maximal pp-1-type in $S^+(A)$.*

Proof $\Rightarrow$ Every pp-type $p_0 \in S^+(A)$ is contained in a maximal pp-type in $S^+(A)$. By assumption such a type is realised in A so, in particula p_0 is realised in A. Thus, by 2.8, A is pure-injective.

Now suppose that $M \models \varphi(\overline{a})$ where φ is pp and $\overline{a}$ is in A. It must be shown that $A \models \varphi(\overline{a})$. The formula $\varphi(\overline{v})$ has the form $\exists \overline{w}\, \theta(\overline{v}, \overline{w})$ where θ is $\bigwedge$-atomic. Then $\theta(\overline{a}, \overline{w})$ is consistent (with $\text{Th}(M)$ in L_A) so extends to some maximal pp-n-type p over A, where $n = l(\overline{w})$. By assumption $p(\overline{w})$ is realised in A, say by $\overline{b}$. Thus one has $A \models p(\overline{b})$, in particular $A \models \theta(\overline{a}, \overline{b})$ - hence $A \models \varphi(\overline{a})$. So A is indeed pure in M.

$\Leftarrow$ If $M = A \oplus B$ and p is a pp-type over A, then, using that M is pure-injective, realise p by, say, $\overline{c} = (\overline{a}, \overline{b}) \in A \oplus B$. Since pp formulas are preserved by morphisms (in particular by projections), from $p(\overline{c})$ one concludes $p(\overline{a})$ (since the parameters A are fixed by the projection). Therefore p is realised in A, as required.

To see that realising every maximal pp-1-type is enough to imply that A is a summand of M, note first that, by 2.8(ii), this does imply that A is pure-injective. Also, the proof that A is pure needs only a little modification: instead of extending $\theta(\overline{v}, \overline{w})$ to a maximal pp-n-type, remove the quantification from only one variable, w_1, say, and then proceed as before. Then, having replaced w_1, by an element of A, go on to w_2, and so on (cf. proof of 2.8(ii)$\Rightarrow$(i)). ▫

For the remainder of this section let us suppose that we are working inside some very saturated model $\tilde{M}$: so all sets and parameters mentioned are in $\tilde{M}$, and all models are elementary submodels of $\tilde{M}$. Thus "pure in a model" is equivalent to pure in $\tilde{M}$.

An embedding $A \hookrightarrow C$ (both sets in $\tilde{M}$) is **pp-essential** if for all morphisms $f \in \text{End}(\tilde{M})$ one has that $\text{pp}(A) = \text{pp}(fA)$ entails $\text{pp}(C) = \text{pp}(fC)$ (in the terminology of [Zg84], C is "small" over A). Since, given such a morphism f there is $g \in \text{End}(\tilde{M})$ which reverses the action of f on A, it is enough to suppose that $f \upharpoonright A = \text{Id}_A$. Thus $A \hookrightarrow C$ is a pp-essential embedding iff for all $f \in \text{End}(\tilde{M})$, if $f \upharpoonright A = \text{Id}_A$ then $\text{pp}(C) = \text{pp}(fC)$.

Notice how this generalises the notion of essential embedding, which says that if the Λ-atomic type of A is preserved then so is that of C.

The next result is an analogue (actually a generalisation - exercise) of **1.9**.

<u>Proposition 4.4</u> [Pr81; 2.6], [Zg84; 3.7] *The embedding* $A \longrightarrow C$ *is pp-essential iff* $\mathrm{pp}(\bar{c}/A)$ *is maximal for all* $\bar{c}$ in C.

Proof $\Rightarrow$ If $\mathrm{pp}(\bar{d}/A) \geq \mathrm{pp}(\bar{c}/A)$ then, by **2.8**, there is a morphism $f(\in \mathrm{End}(\tilde{M}))$ taking $A\frown\bar{c}$ to $A\frown\bar{d}$. Since f fixes A, the assumption gives that $\mathrm{pp}(A\frown\bar{c}) = \mathrm{pp}(A\frown\bar{d})$ - that is, $\mathrm{pp}(\bar{c}/A) = \mathrm{pp}(\bar{d}/A)$. So $\mathrm{pp}(\bar{c}/A)$ is maximal, as required.

$\Leftarrow$ Suppose that f is the identity on A. Let $\bar{c}$ be in C and suppose that $\varphi(f\bar{c})$ holds, where φ is pp. Since $\mathrm{pp}(\bar{c}/A)$ is maximal and since necessarily $\mathrm{pp}(f\bar{c}/A) \geq \mathrm{pp}(\bar{c}/A)$, one has $\mathrm{pp}(f\bar{c}/A) = \mathrm{pp}(\bar{c}/A)$. Thus one concludes that $\varphi(\bar{c})$ holds. That is $\mathrm{pp}(fC/A) = \mathrm{pp}(C/A)$, as required. ▫

Thus we have a "local" definition of pp-essential embedding which corresponds to the criterion $cR \cap A \neq 0$ for essential embeddings.

Notice that by **4.2** the use of the word "maximal" in **4.4** is unambiguous (has the same meaning in any context where the pp-type of A is as given).

Exercise 5 Take $R = \mathbb{Z}$, $B = \mathbb{Z}_2 \oplus \mathbb{Z}_4$ pure in (so a direct summand of) $\tilde{M}$ (the over-theory could be that of $B^{(\aleph_0)}$ for example). Show that the embedding $0 \oplus \mathbb{Z}_2 \hookrightarrow 0 \oplus \mathbb{Z}_4$ is pp-essential, but that $0 \oplus \mathbb{Z}_2 \hookrightarrow B$ is not. The non-zero map $\mathbb{Z}_2 \oplus 0 \longrightarrow 0 \oplus \mathbb{Z}_4$ is not even an embedding in our sense since it is strictly pp-type-increasing.

Now we are ready to construct hulls. Given $A (\leq \tilde{M})$ set $A_0 = A$. If there is $p \in S_1^+(A_0)$ maximal but not realised in A_0 then choose a realisation a_0, say, of p and set $A_1 = A_0\frown a_0$. Continue inductively: given A_α, if there is $p \in S_1^+(A_\alpha)$ maximal but unrealised in A_α, then choose a realisation a_α of p and set $A_{\alpha+1} = A_\alpha\frown a_\alpha$. For limit ordinals λ, set $A_\lambda = \bigcup\{A_\alpha : \alpha < \lambda\}$.

If N is a direct summand of $\tilde{M}$ containing A then, since any (maximal) pp-type in $S^+(A)$ is (by **2.8**) realised in N, the realisation may always be taken to be inside N. So the construction must stop and, moreover, stop with some direct summand $N(A)$ of N by **4.3** (the notation, which implies essential uniqueness, will soon be justified). In [Che76; Chpt.6], Cherlin gives the same construction of hulls, as do Eklof and Fisher [EF72; 4.4] for abelian groups (also see [Gar81; Lemma 2]).

Exercise 6 Referring to the above construction, show that, in any case, if κ is strictly greater than $|A| + |T|$ and if T is κ-stable, then the construction stops at some $N \geq A$ of cardinality at most κ.

The next result, taken from [PP83], is due to Pillay and, in essence, independently to Ziegler [Zg84]. It is also implicit in [Fis75].

<u>Proposition 4.5</u> ("pp-constructible $\Rightarrow$ pp-atomic") *If* $\bar{c}$ *is in* $N(A)$ *then* $\mathrm{pp}(\bar{c}/A)$ *is maximal in* $S^+(A)$.

Proof Set $N(A) = \bar{a} = (a_\alpha)_{\alpha<\beta}$, where A itself is an initial segment in the enumeration (so the indexing here is shifted from that used in the construction).

It is claimed that $\mathrm{pp}(\bar{a}/A)$ is maximal in $S_\beta^+(A)$. So suppose that $\bar{c} = (c_\alpha)_{\alpha<\beta}$ is such that $\mathrm{pp}(\bar{c}/A) \geq \mathrm{pp}(\bar{a}/A)$. I show inductively that for $\gamma < \beta$, $\mathrm{pp}((a_\alpha)_{\alpha<\gamma}/A) = \mathrm{pp}((c_\alpha)_{\alpha<\gamma}/A)$.

Since A occurs initially, the induction can start. Suppose then that the induction hypothesis holds at γ. Since $\mathrm{pp}(\bar{c}/A) \geq \mathrm{pp}(\bar{a}/A)$ there is a morphism f with

$a_\gamma \frown (a_\alpha)_{\alpha<\gamma} \mapsto c_\gamma \frown (c_\alpha)_{\alpha<\gamma}$. The induction hypothesis and construction implies that 4.2 can be applied, and we conclude that $\mathrm{pp}((a_\alpha)_{\alpha\leqslant\gamma}) = \mathrm{pp}((c_\alpha)_{\alpha\leqslant\gamma})$.

The limit case is obvious.

Therefore the claim has been established.

Now let $\bar{a}'$ be any sub-sequence of $\bar{a}$. Note that $\mathrm{pp}(\bar{a}'/A)$ contains that pp-type which is obtained from $\mathrm{pp}(\bar{a}/A)$ by existentially quantifying out all variables, $\bar{w}$ say, corresponding to $\bar{a}\backslash\bar{a}'$. Now this latter pp-type is maximal, since if $\bar{b}'$ realises it, then clearly $\bar{b}'\frown(\bar{a}\backslash\bar{a}')$ realises $\mathrm{pp}(\bar{a}/A)$ which is maximal (hence $\mathrm{pp}(\bar{b}'/A) = \mathrm{pp}(\bar{a}'/A)$). That is, $\mathrm{pp}(\bar{a}'/A)$ is maximal for any sub-sequence $\bar{a}'$ of $\bar{a}$, as required. ◻

Proposition 4.6 *A pp-type $p \in S^+(A)$ is maximal iff it is realised in $N(A)$.*

Proof The direction "⇐" is by above. For "⇒", we note that if $A \subseteq B$, then any (maximal) pp-type in $S^+(A)$ extends to a maximal pp-type in $S^+(B)$, so this direction follows by construction (alternatively, start the construction by realising p, then use 4.15).

Corollary 4.7 *The embedding $A \hookrightarrow N(A)$ of any set into its hull (the context being understood) is pp-essential.*

Proof This is immediate from 4.5 and 4.4. ◻

Lemma 4.8 *The embedding $A \hookrightarrow N(A)$ is a minimal extension of A to a direct summand of $\tilde{M}$.*

Proof Suppose that $A \subseteq N_1 \prec^{\oplus} \tilde{M}$ with N_1 pure-injective and $N_1 \leqslant N(A)$.

From the fact that N_1 is pure in $\tilde{M}$ it is immediate that N_1 is pure in $N(A)$. Hence (2.8) N_1 is a direct summand of $N(A)$, say $N(A) = N_1 \oplus N_2$, and note that $A \subseteq N_1$.

Let $\pi: N(A) \twoheadrightarrow N_1$ be the canonical projection. Since N_1 is pure in $N(A)$ one has $\mathrm{pp}(A) = \mathrm{pp}(\pi A)$. Then $A \hookrightarrow N(A)$ pp-essential (4.7) yields $\mathrm{pp}(N(A)) = \mathrm{pp}(\pi N(A) = N_1)$. Hence $N_2 = 0$, as required. ◻

An alternative approach to the construction of hulls is based on the following definition and lemma. Given $A (\subseteq \tilde{M})$ set $\mathrm{Li}(A) = \{\bar{c} : \bar{c}$ is linked to $A\}$: if the entries of $\bar{c}$ are to be restricted to some set then use a superscript to indicate this.

Exercise 7

(i) $A \subseteq B$ implies $\mathrm{Li}(A) \subseteq \mathrm{Li}(B)$.

(ii) $\mathrm{Li}(\underrightarrow{\bigcup}_\lambda A_\lambda) = \underrightarrow{\bigcup}_\lambda \mathrm{Li}(A_\lambda)$.

(iii) If $A \subseteq M \prec^{\oplus} N$ then $\mathrm{Li}^M(A) = M$ "$\cap$" $\mathrm{Li}^N(A)$.

(iv) $\mathrm{Li}(A)$ may be of arbitrarily large size.

Proposition 4.9 [Pr81; 2.2], [Zg84; 6.5] *Suppose that $\mathrm{pp}(\bar{c}/A)$ is maximal. Then $\mathrm{Li}(A\frown\bar{c}) = \mathrm{Li}(A)$.*

Proof (After Fisher - see [Gar80a; Lemma2]) Suppose, for a contradiction, that there is $\bar{b} \in \mathrm{Li}(A\frown\bar{c}) \backslash \mathrm{Li}(A)$. Since $\bar{b} \notin \mathrm{Li}(A)$ the definition gives $\mathrm{pp}(\bar{b}/A) = \mathrm{pp}(\bar{0}/A)$. Thus $\mathrm{pp}(\bar{b}\frown A) = \mathrm{pp}(\bar{0}\frown A)$, so by 2.8 there is a morphism f (in $\mathrm{End}(\tilde{M})$) fixing A and taking $\bar{b}$ to $\bar{0}$.

Since $\bar{b} \in \mathrm{Li}(A\frown\bar{c})$ there is φ pp, and $\bar{a}$ in A such that $\varphi(\bar{b},\bar{c},\bar{a}) \wedge \neg\varphi(\bar{0},\bar{c},\bar{a})$ holds. Applying f to $\varphi(\bar{b},\bar{c},\bar{a})$ yields $\varphi(\bar{0},f\bar{c},\bar{a})$. From the assumed maximality of $\mathrm{pp}(\bar{c}/A)$ and 2.7 it follows that $\mathrm{pp}(f\bar{c}/A) = \mathrm{pp}(\bar{c}/A)$. Hence $\varphi(\bar{0},f\bar{c},\bar{a})$ yields $\varphi(\bar{0},\bar{c},\bar{a})$ - contradiction, as required. ◻

Then one may construct hulls by defining $N(A)$ to be a set maximal with $\mathrm{Li}(A) = \mathrm{Li}(N(A))$.

The next theorem summarises the basic properties of hulls. Since, for purposes of precise references, I have used [Pr81] and [Zg84], it should be emphasised that the material of this section is largely due to Fisher ([Fis75; §3], where he constructs hulls as injective hulls in a suitable category (see §3.A)).

Theorem 4.10 *Let $N(A)$ be constructed as above. Then:*
(a) *$N(A)$ is a minimal direct summand of $\tilde{M}$ containing A;*
(b) *$N(A)$ is a maximal pp-essential extension of A;*
(c) *every $\bar{c}$ in $N(A)$ has maximal pp-type over A;*
(d) *$N(A)$ is maximal with $\mathrm{Li}(A)=\mathrm{Li}(N(A))$; in particular, if $\bar{b}$ is in $N(A)$ then there is φ pp and $\bar{a}$ in A with $\varphi(\bar{b},\bar{a})\wedge\neg\varphi(\bar{0},\bar{a})$.*

Proof We have (a) and (c) already (4.8, 4.5), and (b) is clear from 4.4 and the fact that $N(A)$ is a direct summand of $\tilde{M}$ (exercise - compare proof of 4.8).

For (d) we note first that by 4.9 one has $\mathrm{Li}(A)=\mathrm{Li}(N(A))$. Next set $\tilde{M}=N(A)\oplus M$ for some M. Then suppose that $\bar{m}_1=(\bar{n},\bar{m})\in N(A)\oplus M$ has $\bar{m}\neq 0$ but is such that $\mathrm{Li}(N(A)^\frown\bar{m}_1)=\mathrm{Li}(N(A))=\mathrm{Li}(A)$. Now, $\bar{m}_1-\bar{n}=\bar{m}$: regard this equation as the formula $\theta(\bar{m},\bar{m}_1,\bar{n})$ and note that, since $\bar{m}\neq 0$, one therefore has $\bar{m}\in\mathrm{Li}(N(A)^\frown\bar{m}_1)$.

On the other hand if $\varphi(\bar{m},\bar{n}_1)$ holds with φ pp and $\bar{n}_1$ in $N(A)$, then projecting yields $\varphi(\bar{0},\bar{n}_1)$ (in $N(A)$, equally, since $N(A)$ is pure in $\tilde{M}$, in $\tilde{M}$). Thus $\bar{m}\notin\mathrm{Li}(N(A))$. This contradiction completes the proof. □

The next result is due to Fisher [Fis75; 7.18] but was not published. Therefore Garavaglia included a direct proof in [Gar80a] (Cor 2); also see [Gar80; Lemma2] for the t.t case.

Corollary 4.11 *Suppose that a and b are elements of the indecomposable pure-injective module N. Then there is a pp formula φ such that $N\models\varphi(a,b)\wedge\neg\varphi(a,0)$.*

Exercise 8 Show that an indecomposable pure-injective module cannot contain a pure submodule of the form $A\oplus B$ with A and B non-zero.

Exercise 9 Given $A\leq\tilde{M}$ show that any module satisfying one of the three conditions (a), (b), (d) of 4.10 is a hull of A.

Exercise 10 Suppose that the hull of $\bar{a}$ decomposes as $N(\bar{a})=N'\oplus N''$ and that, accordingly, $\bar{a}=(\bar{a}',\bar{a}'')$. Show that N' is the hull of $\bar{a}'$.

It has now been shown that, given $A(\leq\tilde{M})$, a "hull" $N(A)$ may be constructed within any summand of $\tilde{M}$ which contains A or even (in view of 4.2) within any pure-injective module which contains a copy of A sitting in it with the appropriate pp-type. To what extent is $N(A)$ unique?

Example 2 Take $R=\mathbb{Z}$, $N=\mathbb{Z}_4^{(\aleph_0)}$, and let a be any element of order 2. Then clearly a hull of a is any copy of $\mathbb{Z}_4$ in N which contains a. So certainly $N(A)$ is not in general unique as a submodule of $\tilde{M}$. Of course in some cases (e.g. if the hull is definable over A) the hull may actually be unique: replace a by an element b of order 4 in this example, so that the hull of b is just the module it generates (already a direct summand). If T is the theory of the abelian group $\mathbb{Z}_{(2)}$, then the hull of any non-zero element which is divisible by all prime powers is a copy of $\mathbb{Q}$. Such a hull is unique since it is definable over the element (for every model is torsionfree).

Theorem 4.12 [Fis75], [Pr81], [Zg84] *Suppose that A is contained in the pure-injective module N and let $A\xhookrightarrow{f} fA\leq N'$ be such that $\mathrm{pp}^N(A)=\mathrm{pp}^{N'}(fA)$. Suppose also that N' is pure-injective: so there is an extension $N\xrightarrow{g}N'$ of f (by 2.8). Let $N(A)$ be as above. Then the restriction of g to $N(A)$ is an embedding of*

$N(A)$ as a direct summand of N', and pp$(gN(A)/fA)$ *is maximal. In particular $gN(A)$ is a hull of fA.*

Proof Let $h:N'\longrightarrow N$ extend f^{-1}. Since $A\longrightarrow N(A)$ is pp-essential (4.7) one has $\mathrm{pp}^N(N(A))=\mathrm{pp}^N(hgN(A))$ (for hg fixes A). Therefore $\mathrm{pp}^N(N(A))=\mathrm{pp}^{N'}(gN(A))$ - since $\mathrm{pp}^N(N(A))\le\mathrm{pp}^{N'}(gN(A))\le\mathrm{pp}^N(hgN(A))$. So by **4.2** the maximal pp-types over $N(A)$ are in natural bijective correspondence, *via* g, with those over $gN(A)$.

Now, the maximal pp-types over $N(A)$ are realised in $N(A)$ by construction. Hence those over $gN(A)$ are (clearly) realised in $gN(A)$. So by **4.3**, $gN(A)$ is a direct summand of N'.

Also pp$(N(A)/A)$ is maximal so, by **4.2**, pp$(gN(A)/fA)$ also is maximal. Therefore by what was shown above and (the exercise following) **4.10**, $gN(A)$ is a hull of fA. ▫

Corollary 4.13 *Let N be an indecomposable pure-injective and let a be a non-zero element of it. Suppose that the endomorphism f of N is such that* pp$(fa)=$pp(a). *Then f is an automorphism of N.* ▫

Corollary 4.14 [Pr81; 2.10] *Let $A\le M\prec^{*}N(A)$ and suppose that $M\xrightarrow{f}M'$ is such that* pp$(A)=$pp(fA). *Then f is a pure embedding of M into M'.*

Proof Extend $M\xrightarrow{f}M'$ to $N(A)\xrightarrow{g}N'$ (by **2.8**) where N' is some pure-injective module purely extending M' (e.g., a sufficiently saturated elementary extension of M'). Then $gN(A)$ is pure in N' and f purely embeds M into $gN(A)$ (by **4.12**). So f purely embeds M in M', as desired. ▫

Corollary 4.15 [Fis75] (see [Gar80a; Lemma2]) *Suppose that $A^{(\prime)}\le N^{(\prime)}$ and that $N^{(\prime)}$ is pure-injective with $\mathrm{pp}^N(A)=\mathrm{pp}^{N'}(A')$. If $N(A)$ and $N(A')$ are constructed as above, then $(N(A),A)\simeq(N(A'),A')$ and in particular $\mathrm{pp}^N(N(A))=\mathrm{pp}^{N'}(N(A'))$.*

Proof Apply **4.12** to $A\le N$ and $A'\le N(A')$ to obtain a morphism g with $gN(A)$ a direct summand of $N(A')$ containing A'. Since $N(A')$ is a minimal direct summand containing A' (**4.10**) g is therefore an isomorphism from $N(A)$ to $N(A')$ taking A to A'. That is, $(N(A),A)\simeq(N(A'),A')$. Since $N(A)$, $N(A')$ are direct summands one actually has $\mathrm{pp}^N(N(A))=\mathrm{pp}^{N'}(gN(A)=N(A'))$. ▫

Thus $N(A)$ is indeed unique up to isomorphism over A and, moreover, depends only on the pp-type of A. Call $N(A)$ the **hull** of A. If p is a pp-type over 0 then denote by $N(p)$ the **hull** of p - that is, the hull of any exact realisation of p in some (pure-injective) module: by an **exact realisation** of a pp-type p I mean a tuple whose pp-type is p (whereas a realisation is just a tuple whose pp-type contains p). Extend this definition of, and notation for, hulls to types *via* their pp-parts.

The hulls that we have constructed are examples of Taylor's "minimal compact" structures [Tay71]; also see [Ki67].

Corollary 4.16 *Let g be an endomorphism of $N(A)$ fixing A. Then g is an automorphism of $N(A)$.*

Proof This is immediate by **4.12**. ▫

The process of constructing hulls should be compared with the general notion of construction in [She78]. The difference lies in the fact that maximal pp-types are dense only with respect to positive formulas: therefore one should not expect $N(A)$ to be a model but rather the "positive primitive") version thereof.

Beware that $N(A)$ will not in general be a model but it can be thought of as the minimal positively saturated extension of A. Modules of the form $N(A)$ are the building blocks of pure-injective models and decomposition of models in terms of hulls is an important theme here. After giving, in the second section, some examples of hulls I consider, in §4, those pp-types which have indecomposable hulls. We will see that such hulls have local endomorphism rings. This gives uniqueness of decompositions expressed in terms of indecomposable pure-injectives.

Finally, let us note that any type realised in the hull of A is isolated by its pp-part.

Corollary 4.17 *Let* $\overline{b}$ *be in* $N(A)$. *Then* $\mathrm{pp}(\overline{b}/A)\vdash\mathrm{tp}(\overline{b}/A)$. *In particular, if* $\mathrm{pp}(\overline{b}/A)$ *is finitely generated then* $\mathrm{tp}(\overline{b}/A)$ *is isolated.*

Proof This is immediate by **4.5** and **4.1**. □

One says that right ideals I, J of the ring are **related** if $E(R/I)\simeq E(R/J)$. Generalising this, we say that two (pp-)types over 0, p and q, are **related** if $N(p)\simeq N(q)$, and then write $p\sim q$. This may not at first look like a generalisation, but take the point of view that pp-types generalise right ideals and note that I, J are related iff there is an injective module which is the injective hull of both an element with annihilator exactly I and an element with annihilator exactly J.

In [Dei77], Deissler introduced a rank which measures how far an element is from being definable. His rank is defined by: $\mathrm{rk}(b,A,M)=0$ (M a model, b and A in M) iff b is definable with parameters from A; $\mathrm{rk}(b,A,M)=\alpha$ iff $\mathrm{rk}(b,A,M)$ is not less than α and if there exists a formula φ with parameters from A such that $\varphi(M)$ is non-empty and, for every $c\in\varphi(M)$, one has $\mathrm{rk}(b,A^\frown c,M)<\alpha$.

Deissler showed [Dei77] that every element of the structure M is assigned a rank iff M is a **minimal** model of its own theory (i.e., has no proper elementary submodel).

Kucera studies this rank for modules in his thesis [Kuc84], also see [Kuc8?], [Kuc8??]. He begins in [Kuc8?] by defining Deissler rank relative to a given set of formulas (which has to satisfy certain conditions), and generalises Deissler's characterisation of minimal models [Kuc8?; 2.6]. Specialising to modules, he defines rk^+ ("positive Deissler rank") to be Deissler rank relative to the set of pp formulas and shows that, if M is a module and if b, A are in M, then $\mathrm{rk}^+(b,A,M)<\infty$ iff b belongs to every pure submodule of M which also contains A. So, for example, if M is totally transcendental and if A is a subset of M, then M is the hull of A iff $\mathrm{rk}^+(M/A)<\infty$ (i.e., $\mathrm{rk}^+(b,A,M)<\infty$ for every $b\in M$) [Kuc8?; 2.7].

In [Kuc8?; §§3,4], Kucera sets up some machinery for computing (relative) Deissler rank and also establishes some relation with U-rank.

He goes on in [Kuc8??] to look at, from this point of view, injective modules over commutative noetherian rings. First it is shown [Kuc8??; 1.1] that, if $N=N(A)$ is t.t. and $b\in N$, then $\mathrm{rk}^+(b,A,N)=\mathrm{rk}(b,A,N)$. Then he obtains the following estimates for the positive Deissler rank of injective modules over a commutative noetherian ring R:

[Kuc8??; 2.4] if E is an indecomposable injective - so $E=E(R/P)$ for some prime ideal P - then $\mathrm{rk}^+(E/(R/P))\leqslant 2$, with $\mathrm{rk}^+(E/(R/P))=1$ iff E is just the quotient field of R/P (one defines $\mathrm{rk}^+(M/A)$ to be $\sup\{\mathrm{rk}^+(b,A,M)+1: b\in M\}$);

[Kuc8??; 2.9] if $A=(R/P)^{(\kappa)}$ and $E=E(A)$ then $\mathrm{rk}^+(E/A)\leqslant\omega$ (Kucera conjectures that if κ is infinite then the actual value is either 1 or ω);

[Kuc8??; 2.12] if A is a direct sum of modules of the form R/P (P prime), if $E=E(A)$ and if $\mathcal{P}$ is the poset of all primes appearing, then $\mathrm{rk}^+(E/A)\leqslant\omega^{\mathrm{dp}(\mathcal{P})+1}$, where $\mathrm{dp}(\mathcal{P})$ is the depth of the poset $\mathcal{P}$ (i.e., the foundation rank of its opposite) - thus positive Deissler rank is related to classical Krull dimension (see §10.5).

4.2 Examples of Hulls

1. **Injective hulls** Let A be a subset of the injective module E (to accord with the usual algebraic usage one would replace A by the module it generates - a point of no consequence). By 4.10 and 4.15 there is a unique-to-isomorphism-over-A direct summand of E, usually written $E(A)$, which is minimal such containing A. This module $E(A)$ is of course injective and is called the **injective hull** of A (see §1.1).

Of what pp-type is this the hull? Having fixed the algebraic isomorphism type (i.e., the $\bigwedge$-atomic type) of A we may ask in what ways this may be completed to a pp-type. In general there are many ways, but there are two canonical ones: that which adds as few pp formulas as possible is discussed in 2. below; that which adds the maximum number of pp formulas is the pp-type of A sitting inside its injective hull. Since every module containing A has an A-fixing morphism to $E(A)$ it is obvious that this latter is the largest pp-type 'completing' the $\bigwedge$-atomic type of A.

Those readers acquainted with injective hulls may be interested to see how 4.10 is reconciled with §1.2, so I spend the rest of the sub-section on this. The main point is that injective modules have a "local" elimination of quantifiers. Let us say that a module N is **locally substructure complete** if whenever $\bar{a},\bar{b}$ are in N with $\mathrm{tp}_0(\bar{a})=\mathrm{tp}_0(\bar{b})$ then $\mathrm{tp}^N(\bar{a})=\mathrm{tp}^N(\bar{b})$ (i.e., $\mathrm{pp}^N(\bar{a})=\mathrm{pp}^N(\bar{b})$) - recall that "$\mathrm{tp}_0$" denotes $\bigwedge$-atomic type. It is immediate from the definitions that any injective module is locally substructure complete.

Now, if N has complete elimination of quantifiers then N is locally substructure complete (by 16.1), but the converse fails even for injective modules. For, over any non-coherent ring there exist an injective E and some module M elementarily equivalent to E which is pure-injective but not injective (15.42). So if one ensures that E and M are large enough then there will be $a,b\in M$ with $aR\simeq bR$ but with $\mathrm{tp}^M(a)\neq\mathrm{tp}^M(b)$. Thus the theory of E does not admit complete elimination of quantifiers; yet E is locally substructure complete. This again points up the local aspect of the developments of the preceding section.

<u>Lemma 4.18</u> *Suppose that* E *is injective and that* $\bar{a},\bar{b}$ *are in* E.
(a) *If* $\mathrm{tp}_0(\bar{a})=\mathrm{tp}_0(\bar{b})$ *then* $\mathrm{pp}^E(\bar{a})=\mathrm{pp}^E(\bar{b})$;
(b) $\bar{a}$ *and* $\bar{b}$ *are linked iff there is a* $\bigwedge$*-atomic formula* θ *such that* $\theta(\bar{a},\bar{b})\wedge\neg\theta(\bar{0},\bar{b})$ *holds, and this is so iff* $\bar{a}R\cap\bar{b}R\neq 0$.

Proof (a) This has already been noted.

(b) Suppose that there is no such $\bigwedge$-atomic θ linking $\bar{a}$ and $\bar{b}$. Let $\bar{a}'$ in E' be a copy of $\bar{a}$ in E, and consider the direct sum $E\oplus E'$. The absence of such a formula θ means that $\mathrm{tp}_0(\bar{a}\frown\bar{b})=\mathrm{tp}_0(\bar{a}'\frown\bar{b})$ (each being equivalent to $\mathrm{tp}_0(\bar{a})\cup\mathrm{tp}_0(\bar{b})$).

Since $E\oplus E'$, being injective, is locally substructure complete, it follows that $\mathrm{tp}(\bar{a}\frown\bar{b})=\mathrm{tp}(\bar{a}'\frown\bar{b})$. So if φ is pp with $\varphi(\bar{a},\bar{b})$ then also $\varphi(\bar{a}',\bar{b})$ holds. Projecting this to E one obtains $\varphi(\bar{0},\bar{b})$. Thus the first equivalence follows.

It has been noted after Exercise 4.1/1 that if $\bar{a}R\cap\bar{b}R\neq 0$ then $\bar{a}$ and $\bar{b}$ are linked. Conversely, if there is a $\bigwedge$-atomic formula θ with $\theta(\bar{a},\bar{b})\wedge\neg\theta(\bar{0},\bar{b})$ then clearly one has some relation $\sum a_i r_i=\sum b_j s_j$ with $\sum a_i r_i\neq 0$, as required. □

An easy corollary of this is the following.

<u>Corollary 4.19</u> *If* A *is embedded in* $C\leqslant E$ *with* E *injective, then this embedding of* A *in* C *is essential iff it is pp-essential (with respect to* $\mathrm{Th}(E)$*).* □

It is left to the reader to finish reconciling the previous section with §1.2.

Example 1 Some injective hulls of abelian groups are: $E(\mathbb{Z})=\mathbb{Q}$; $E(\mathbb{Z}_{(p)})=\mathbb{Q}$; $E(\mathbb{Z}_{p^n})=\mathbb{Z}_{p^\infty}$; $E(\mathbb{Z}_6)=E(\mathbb{Z}_2\oplus\mathbb{Z}_3)=\mathbb{Z}_{2^\infty}\oplus\mathbb{Z}_{3^\infty}$; $E(\overline{\mathbb{Z}_{(p)}})=\mathbb{Q}^{(2^{\aleph_0})}$ (for $\overline{\mathbb{Z}_{(p)}}$ is torsionfree and has cardinality $2^{\aleph_0}$).

It is not difficult to check that all injective $\mathbb{Z}$-modules are ω-stable: this is not true of all pure-injective $\mathbb{Z}$-modules but it is a property of injective modules over any right noetherian ring.

2. **Pure-injective hulls** Let A be any module and let A' be a pure-injective elementary extension of A. Then the hull of A in A' is called the **pure-injective hull** of A and is denoted by $\overline{A}$. The pp-type of A in A' is just the pp-type of A in itself, which is of course the minimal possible pp-type, given the isomorphism type of A.

A point about notation: rather than place bars above very long expressions I use pi(-) to denote the pure-injective hull of "-"; for example I may write "pi($\oplus\{N_\lambda : \lambda\in\Lambda\}$)".

From the previous section one has the following description of the pure-injective hull (see [War69; Prop 6 and preceding comments]).

<u>Theorem 4.20</u> *Let M be any module. Then there is a pure-injective module $\overline{M}$ with M purely embedded in $\overline{M}$ such that, whenever $M \xrightarrow{f} N$ is a pure embedding with N pure-injective, there is an extension of f to an embedding of $\overline{M}$ as a direct summand of N.*

$\overline{M}$ is unique up to M-isomorphism. □

Retaining the notation of the first paragraph, one has $A \prec^{*} \overline{A} \prec^{*} A'$ and $A\equiv A'$, so 2.25 yields the next result (already recorded as 2.27) which says that, in some sense, pure-injective modules are more typical than injective modules.

<u>Theorem 4.21</u> (see 2.27) *Every module is an elementary substructure of its pure-injective hull.* □

We have already encountered the example $\mathbb{Z}_{(p)}\prec\overline{\mathbb{Z}_{(p)}}$ (and generalisations to Dedekind domains in §2.Z). Injective hulls and pure-injective hulls coincide for absolutely pure modules - in particular for all modules over regular rings. We will continually use the concept of pure-injective hull, though there are relatively few explicit (non-injective, non-Σ-pure-injective) examples to hand, so this use may be compared with that of injective hulls in module theory and that of saturated extensions in model theory. Many results in these notes do, however, help to elucidate the structure of a general pure-injective module.

Now I digress to discuss pure-essential embeddings: these form the basis of the more usual approach to pure-injective hulls (another approach makes use of dualities).

One says that an embedding $A\leqslant B$ is **pure-essential** if A is pure in B and for all non-zero submodules C of B, if $A\cap C=0$ then $(A\oplus C)/C$ is not pure in B/C. One may see that these are just the pp-essential embeddings provided B is taken as the context (in which to measure pp-types).

<u>Proposition 4.22</u> [Pr81; 1.2] *If the embedding of A in B is pure-essential then $B\subseteq \mathrm{Li}(A)$ (we are working in $\mathrm{Th}(B)$).*

Proof Choose $\overline{c}$ in B, $\overline{c}\neq 0$. It must be shown that $\overline{c}$ is linked to A. It may as well be supposed that $\overline{c}$ enumerates a submodule C of B.

If $C\cap A\neq 0$ then certainly $\overline{c}$ is linked to A. So suppose that $C\cap A=0$, and let $\pi:B\twoheadrightarrow B/C$ be the canonical projection.

By definition of pure-essential πA is not pure in πB. So there is a conjunction, $\theta(\overline{w},\overline{v})\equiv \bigwedge_j t_j(\overline{w})+t'_j(\overline{v})=0$, of atomic formulas, and there is $\overline{a}$ in A with

$\pi B \models \exists \overline{w}\, \theta(\overline{w}, \pi\overline{a})$ but with πA not satisfying this formula. Choose $\overline{b}$ in B with $\pi B \models \theta(\pi\overline{b}, \pi\overline{a})$.

Lifting back to B and noting that $C = \ker\pi$, one sees that there are $c_j \in C$ with $B \models \bigwedge_j t_j(\overline{b}) + t'_j(\overline{a}) = c_j$. Regard this formula as $\theta_0(\overline{b}, \overline{a}, \overline{c})$ and let $\varphi(\overline{v}, \overline{u})$ be $\exists \overline{w}\, \theta_0(\overline{w}, \overline{v}, \overline{u})$. So $B \models \varphi(\overline{a}, \overline{c})$.

If one had $B \models \varphi(\overline{a}, \overline{0})$ then, since A is pure in B, one would have $A \models \varphi(\overline{a}, \overline{0})$ also. Therefore there would be $\overline{a}'$ in A with $A \models \theta_0(\overline{a}', \overline{a}, \overline{0})$: that is $A \models \theta(\overline{a}', \overline{a})$. Applying the projection π to this gives $\pi A \models \theta(\pi\overline{a}', \pi\overline{a})$ - contrary to choice of θ.

Thus $B \models \varphi(\overline{a}, \overline{c}) \wedge \neg\varphi(\overline{a}, \overline{0})$ - so $\overline{c} \in \mathrm{Li}(A)$ as required. □

In the other direction one has the following.

<u>Proposition 4.23</u> [Pr81; 1.3] *Suppose that* $A \leq B$ *with* $B \leq \mathrm{Li}(A)$ *(in* $\mathrm{Th}(B)$*). Then for all* $C \leqslant B$, $C \neq 0$, *if* $A \cap C = 0$ *then* $(A \oplus C)/C$ *is not pure in* B/C.

Proof Suppose that C is as stated, with $A \cap C = 0$. Let $\pi: B \twoheadrightarrow B/C$ be the canonical projection. Choose $c \in C$, $c \neq 0$: by hypothesis there is some pp formula $\varphi(v, \overline{w})$ and $\overline{a}$ in A with $B \models \varphi(c, \overline{a}) \wedge \neg\varphi(0, \overline{a})$.

From $B \models \varphi(c, \overline{a})$ follows $\pi B \models \varphi(\pi c, \pi\overline{a})$: that is $\pi B \models \varphi(0, \pi\overline{a})$.

If πA were pure in πB then one would have $\pi A \models \varphi(0, \pi\overline{a})$. The formula $\varphi(v, \overline{w})$ may be taken to have the form $\exists \overline{u} \bigwedge_j v r_j + t_j(\overline{w}) + t'_j(\overline{u}) = 0$ for suitable terms t_j, t'_j and suitable $r_j \in R$. Lifting $\pi A \models \varphi(0, \pi\overline{a})$ back to A yields some $\overline{a}'$ in A and $c_j \in C$ with $A \models \bigwedge_j t_j(\overline{a}) + t'_j(\overline{a}') = c_j$. But $A \cap C = 0$, so it must be that $A \models \bigwedge_j t_j(\overline{a}) + t'_j(\overline{a}') = 0$ and hence $A \models \varphi(0, \overline{a})$. This certainly would imply $B \models \varphi(0, \overline{a})$ - contrary to choice of φ. □

<u>Corollary 4.24</u> *Suppose that* A *is pure in* B. *Then* A *is pure-essential in* B *iff* $B \leq \mathrm{Li}(A)$ *(in* $\mathrm{Th}(B)$*).* □

Actually, the above development, 4.22-4.24, contains a flaw: since pure-injective hulls may be obtained as maximal pure-esssential extensions the conclusion of 4.22 should be stronger (namely, that $A \leqslant B$ be pp-essential) and 4.24 should have the (in general stronger) equivalent that $A \leqslant B$ be pp-essential in place of $B \leq \mathrm{Li}(A)$.

This is related to a point made by Sabbagh. Under the definition of pure-essential embedding given above (and this is the usual one), it is easy to prove that if $A \longrightarrow B$ is pure-essential and if $A \longrightarrow C$ is a pure embedding, then there is an extension of $A \longrightarrow C$ to an embedding of B into C (see [Ki67; Lemma3]). In fact, any such extension is a pure embedding of B into C (by 4.14 and 4.10), but this does not seem to follow so easily: indeed, I do not know how to establish it without resorting to hulls. With this stronger conclusion, it becomes obvious, for example, that a composition of pure-essential embeddings is pure-essential, and one also may prove the stronger versions of 4.22 and 4.24 (exercise).

Exercise 1 [Sab70; Prop3] Show that $E(M) \simeq \overline{M}$ iff M is **absolutely pure** (i.e., is a pure submodule of every extension).

Exercise 2 Injective and pure-injective hulls may be wildly different. Take, for example, the base ring to be the ring of integers and show: that $\overline{\mathbb{Z}_2} = \mathbb{Z}_2$ but $E(\mathbb{Z}_2) = \mathbb{Z}_{2^\infty}$; that $|\overline{\mathbb{Z}_{(p)}}| = 2^{\aleph_0}$ but $E(\mathbb{Z}_{(p)}) = \mathbb{Q}$; that $\overline{\mathbb{Z}} \simeq \prod \{\overline{\mathbb{Z}_{(p)}} : p \text{ prime}\}$ but $E(\mathbb{Z}) = \mathbb{Q}$.

Exercise 3 If M and N are modules, then $\overline{M \oplus N} \simeq \overline{M} \oplus \overline{N}$. This point is just a little bit subtle, because the context should be taken to be that of the theory of $M \oplus N$. For instance, consider $A = \mathbb{Z}_2 \oplus \mathbb{Z}_4$ and take $a = (1,0)$, $b = (1,2)$. Each of the subgroups $\langle a \rangle$ and $\langle b \rangle$ is pure in A and $\langle a \rangle \cap \langle b \rangle = 0$, so $\langle a \rangle \oplus \langle b \rangle$ is contained in A. But the hull of $\langle a \rangle \oplus \langle b \rangle$, *as a submodule of* A, is A, *not* a copy of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. The point is that a and b are not independent in A.

A more spectacular example is the following. Take the p-adic integers $\overline{\mathbb{Z}(p)}$, considered as an abelian group. Then (exercise) there are pure submodules M, N of $\overline{\mathbb{Z}(p)}$ with $M \cap N = 0$. So, considered as a subgroup of $\overline{\mathbb{Z}(p)}$, the hull of $M \oplus N$ is just $\overline{\mathbb{Z}(p)}$.

Injective hulls and pure-injective hulls lie at opposite extremes of a spectrum of hulls. Various intermediate and special cases have been considered: see, for example, the survey [Skl78] and references therein.

Other points and examples

3. If A is pure in M then the hull of A (in Th(M)) will be its pure-injective hull: on the other hand, if $A \leqslant M$ is not a pure embedding then this certainly will not be the case.

A specific example over the ring of integers is: $A = \mathbb{Z}$, $M = \mathbb{Q}$. Then the hull $N(\mathbb{Z})$ of $\mathbb{Z}$ in $\mathbb{Q}$ is $\mathbb{Q}$, not $\overline{\mathbb{Z}}$. Another example over the same ring is: $A = \mathbb{Z}_2$, $M = \mathbb{Z}_4{}^{(\aleph_0)}$. Then the hull of any copy of $\mathbb{Z}_2$ embedded in any model of Th(M) is a copy of $\mathbb{Z}_4$ (for the pp-type modulo Th(M) of the non-zero element of $\mathbb{Z}_2$ says that that element is divisible by 2, and so this pp-type implies the existence of a copy of $\mathbb{Z}_4$ surrounding the copy of $\mathbb{Z}_2$).

It seems that this point (the dependence on context) is often not grasped at first, and so I will emphasise it: *the theory of hulls is not simply the theory of pure-injective hulls.*

4. If the subset A of M is "large enough" in M then it will be the case that $N(A)$ is a model of Th(M), but the difference between hulls and prime models may be seen clearly in the totally transcendental case. In that case, to obtain a prime model, all isolated types must be realised, but to obtain a hull, only those types (*sic*) isolated by a pp formula need be realised.

A specific example is given by $T = \text{Th}(\mathbb{Z}_2{}^{(\aleph_0)} \oplus \mathbb{Z}_4{}^{(\aleph_0)})$. Any model must contain infinitely many copies of each of $\mathbb{Z}_2$ and $\mathbb{Z}_4$, but the hulls of single elements are 0, $\mathbb{Z}_2$, $\mathbb{Z}_4$. Do note, however, that these are the "building blocks" of the models.

5. If N is an indecomposable pure-injective module then N is the hull of each of its non-zero elements. In particular (**4.11**), if $a, b \in N$ are non-zero then there is a pp formula φ with $N \models \varphi(a,b) \wedge \neg\varphi(0,b)$.

Exercise 4 Take N to be the pure-injective hull (equivalently, completion in the p-adic topology) of $\mathbb{Z}_{(p)}$ and let a, b be non-zero. Describe (in terms of suitable representations of a and b) a pp formula which links a and b in N.

6. Take $R = \mathbb{Z}$; $N = \overline{\mathbb{Z}(p)}$. It was pointed out earlier that there are non-zero submodules A and B of N with $A \cap B = 0$. Thus the intersection of two pp-essential embeddings need not be pp-essential. This contrasts with essential embeddings, and may be blamed on the fact that we are not really working in the right category (and lack "witnesses" to important relations). This may be remedied by going to appropriate functor categories (where the pure-injective modules become the injective objects) - see Chapter 12.

7. ($R = \mathbb{Z}$) It has already been noted that if $T = \text{Th}(\mathbb{Z}_2{}^{(\aleph_0)} \oplus \mathbb{Z}_4{}^{(\aleph_0)})$ then the possible hulls of single elements are 0, $\mathbb{Z}_2$ (as a direct summand of course) and $\mathbb{Z}_4$. The module $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ could be obtained as the hull of a pair of elements. If a is an element of order 2 which is divisible by 2 then the hull of a is a copy of $\mathbb{Z}_4$ (and is, in particular, not the module generated by a). If, on the other hand, b is an element of order 2 which is not divisible by 2 then the hull of b is just the module it generates - a copy of $\mathbb{Z}_2$. Thus one sees how hulls depend on pp-type and not simply on isomorphism type.

Exercise 5 Verify the list of hulls of single elements above and also show that:
in $\text{Th}(\mathbb{Z}_2{}^{(\aleph_0)} \oplus \mathbb{Z}_3{}^{(\aleph_0)})$ the possible hulls of single elements are: 0, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_6 = \mathbb{Z}_2 \oplus \mathbb{Z}_3$;
in $\text{Th}(\mathbb{Z}_2{}^{(\aleph_0)} \oplus \mathbb{Z}_8{}^{(\aleph_0)})$ the possible hulls of single elements are: 0, $\mathbb{Z}_2$, $\mathbb{Z}_8$, and $\mathbb{Z}_2 \oplus \mathbb{Z}_8$.

8. If $p \in S^T(0)$ and the hull $N(p)$ of (a realisation of) p is the sum of n indecomposable pure-injectives then the weight (see after 6.27) of p is no more than n and, in the case that T is closed under products, it equals n. Thus in 8.7 above, all non-zero 1-types in $S^T(0)$ have weight one but, for each of the theories in the exercise above, there is a 1-type over the empty set of weight two.

4.3 Decomposition of injective and pure-injective modules

One of our main concerns in these notes is the representation of pure-injective modules in terms of simple (especially indecomposable) components. The broad existence and uniqueness theorems are really quite general and belong in a text on abelian categories. Although it would not be inappropriate to prove what I need here, there would be little point in doing so since the material is adequately covered in standard texts. Therefore I refer to texts on abelian categories for proofs.

Nevertheless I feel that these is some value in presenting the main definitions and results so as to give the reader an indication of what kind of decomposition theorems may be expected.

Therefore I state some generalities about Grothendieck categories (these will also be referred to in Chapter 12) and then go on to describe the main decomposition theorems. I have used [Pop73] and [St75] as my main sources (the latter is perhaps more accessible).

Readers allergic to categories will not suffer seriously if they skip on to §4 after reading the statement of **4**.A14.

The definition of injective module (= injective object in the category $\mathcal{M}_R$) that is given in §1.2 makes sense in any category. The definitions of essential embedding and injective hull also make sense in, for example, an additive category. The existence of injective hulls (uniqueness is a formal consequence) is guaranteed in any Grothendieck abelian category (see, e.g., [St75; V§2]). An abelian category is **Grothendieck** if it has a *set* G of generators (so every object is an epi image of a coproduct (= direct sum) of objects of G) and satisfies the "AB5" condition - that is, has arbitrary coproducts and, given a subobject C and a Σ-directed system $\{C_\lambda\}_\lambda$ of subobjects (of some object), one has $C \cap \Sigma_\lambda C_\lambda = \Sigma_\lambda C \cap C_\lambda$. The latter is a kind of finiteness condition: note that if $C_{(\lambda)}$ is given by its "elements" then this condition is satisfied.

Examples of Grothendieck (abelian) categories are Module categories: let $\mathcal{C}$ be a **small** (just a set of objects and morphisms) **additive** (morphism sets are abelian groups under "+", which is respected by composition) category. Then the category $(\mathcal{C}^{op}, \mathrm{Ab})$ of functors from the opposite category $\mathcal{C}^{op}$ to the category Ab of abelian groups is a Grothendieck abelian category with a generating set of projectives consisting of the representable functors $(C,-)$ for C an object in $\mathcal{C}^{op}$. This functor category $(\mathcal{C}^{op}, \mathrm{Ab})$ is also denoted $\mathcal{M}_\mathcal{C}$ and may be thought of as a generalised module category: think of the ring R as a category consisting of one object with endomorphism ring R to see that the module category $\mathcal{M}_R$ is "really just" the functor category (R^0, Ab). The analogue of the category of left modules is $(\mathcal{C}, \mathrm{Ab}) = \mathcal{M}_{\mathcal{C}^{op}}$. Such Module categories have properties very similar to those of module categories (cf. [Mit72], [Mit78]; also [Aus66], [Aus74]). Also, any Grothendieck abelian category may be realised as a Giraud subcategory of a Module category (see [St75; X.4.1]).

In this section I will present the standard decomposition theorems for injective objects in Grothendieck abelian categories. Then I will show how these imply general decomposition results for pure-injective modules.

First one introduces the spectral category of an abelian category ([GO66]; see also [NP66] and [War69a]).

Proposition 4.A1 *For any abelian category* $\mathcal{C}$ *the following conditions are equivalent*:

(*i*) *every object of* $\mathcal{C}$ *is injective*;

(*i*)^0 *every object of* $\mathcal{C}$ *is projective*;

(*ii*) *every monomorphism in* $\mathcal{C}$ *is split*;

(*ii*)^0 *every epimorphism in* $\mathcal{C}$ *is split*. □

A category $\mathcal{C}$ as in 4.A1 is said to be **spectral**. Clearly $\mathcal{M}_R$ is spectral iff R is semisimple artinian.

Every Grothendieck abelian category $\mathcal{C}$ gives rise in a natural way to a spectral abelian category Spec$\mathcal{C}$ together with a functor Spec: $\mathcal{C} \longrightarrow$ Spec$\mathcal{C}$ which is given on objects by Spec$(C) = E(C)$ - the injective hull of C. For the action on morphisms, note that, given an object C of $\mathcal{C}$, the essential subobjects of $E(C)$ form a system directed by intersection so, given objects C, C' of $\mathcal{C}$ and any $f \in (C, C')$, the germ defined by f is a unique morphism in (Spec(C),Spec(C')) (see [St75; V §7] for details).

Theorem 4.A2 (see [St75; V.7.2]) *Let $\mathcal{C}$ be Grothendieck abelian. Then there is a left exact functor* Spec: $\mathcal{C} \longrightarrow$ Spec$\mathcal{C}$ *which takes objects to their injective hulls;* Spec$\mathcal{C}$ *is a spectral Grothendieck abelian category.* □

There is a structure theorem for spectral Grothendieck categories. An object C of a spectral category is **discrete** if it is a coproduct (direct sum) of simple subobjects. A spectral category is **discrete** if every object in it is discrete.

Theorem 4.A3 [St75; V.6.8, 6.7], [Pop73; 5.2.2, 5.2.3] *Let $\mathcal{C}$ be a spectral Grothendieck abelian category.*

(*a*) *$\mathcal{C}$ is a product* Dis$\mathcal{C}$×Cont$\mathcal{C}$ *of a discrete spectral category* Dis$\mathcal{C}$ *and a continuous spectral category* Cont$\mathcal{C}$.

(*b*) *Any discrete spectral Grothendieck category has (i.e., is naturally equivalent to one of) the form $\times_I \mathcal{M}_{D_i}$, where D_i is a division ring for $i \in I$. Conversely, every such category is discrete.* □

Here, one says that an object C of a spectral category is **continuous** if C has no simple subobject: the category is **continuous** if every object in it is continuous (the zero object is regarded as discrete, continuous, neither or both, as convenient).

Since injective hulls are "absorbed" by the functor Spec, the next result follows.

Proposition 4.A4 *Let $E \in \mathcal{M}_R$ be injective.*

(*a*) Spec(E) *is discrete iff $E = E(\bigoplus E_i)$ for certain indecomposable injective submodules E_i.*

(*b*) Spec(E) *is simple iff E is indecomposable.*

(*c*) Spec(E) *is continuous iff E has no indecomposable direct summand.* □

Corollary 4.A5 *Let $E \in \mathcal{M}_R$ be injective. Then there are unique to isomorphism submodules E_c and E_d of E such that $E = E_d \oplus E_c$, where E_c has no indecomposable direct summand and where $E_d = E(\bigoplus_I E_i)$ for suitable indecomposable injective submodules E_i.* □

A ring R is local if the set of elements which do not have an inverse form an ideal. If this condition is satisfied then this unique maximal ideal is also the unique maximal right (and left) ideal; hence it is the Jacobson radical, $J(R)$, of R and $R/J(R)$ is a division ring. To check that a ring is local, it is necessary and sufficient to show that for all $r, s \in R$, if $r+s$ is invertible then r or s is invertible.

Proposition 4.A6 *Let C be an object of the abelian category $\mathcal{C}$. Then* End(C) *is a local ring iff for all $f \in$* End(C), *either f or $1-f$ is an automorphism. If* End(C) *is local, then C must be indecomposable.* □

The importance of indecomposable objects having local endomorphism rings is shown by the next result.

Theorem 4.A7 (Krull-Remak-Schmidt-Azumaya) see [Pop73; 5.5.13] *Let* $\mathcal{C}$ *be an abelian category and suppose that* C, *an object of* $\mathcal{C}$, *has decompositions* $\bigoplus_I C_i$, $\bigoplus_J D_j$ *where the* C_i, D_j *are indecomposables with local endomorphism rings. Then there is a bijection* $\pi : I \longrightarrow J$ *such that* $C_i \simeq D_{\pi i}$ *for all* $i \in I$ (***uniqueness of decomposition***). □

Theorem 4.A8 [Mat58; 2.6] *If* E *is an indecomposable injective object of the abelian category* $\mathcal{C}$ *then* End$E$ *is a local ring.* □

A proof which covers even the pure-injective case in module categories is given in 4.27.

Theorem 4.A9 *Let* E *be a discrete object of the spectral category* $\mathcal{C}$. *Then the decomposition of* E *as a direct sum of indecomposables is essentially unique (in the sense of* 4.A.7).

Proof This is by 4.A8 and 4.A7. □

Corollary 4.A10 *Let* E *be an injective object of the Grothendieck abelian category* $\mathcal{C}$. *Then* $E = E_d \oplus E_c$ *where* E_c *has no indecomposable direct summands and where* $E_d = E(\bigoplus_I E_i)$ *for suitable indecomposable direct summands,* E_i, *of* E. *If moreover* $E_d = E(\bigoplus_J E'_j)$ *with the* E'_j *indecomposable injectives, then there is a bijection* $\pi : I \longrightarrow J$ *such that* $E_i \simeq E'_{\pi i}$ *for all* $i \in I$.

Proof Pull back 4.A9 from Spec$\mathcal{C}$ to $\mathcal{C}$ using 4.A2, and this is what one obtains. □

A broader version of 4.A10 (which follows from that result) is the following.

Theorem 4.A11 (Krull-Remak-Schmidt-Azumaya-Gabriel) [GO66; 3.3], [War69a; 4.2], also see [NP66; Prop1] *Let* $\mathcal{C}$ *be an abelian category and let* $\{C_i\}_I$, $\{D_j\}_J$ *be sets of objects such that each injective hull* $E(C_i)$, $E(D_j)$ *is indecomposable.*
Suppose that $E(\bigoplus_I C_i) \simeq E(\bigoplus_J D_j)$. *Then there is a bijection* $\pi : I \longrightarrow J$ *such that* $E(C_i) \simeq E(D_{\pi i})$ *for all* $i \in I$.
Suppose also that F *is a direct summand of* $\bigoplus_I E(C_i)$. *Then there is* $I' \subseteq I$ *such that* $\bigoplus_I E(C_i) = F \oplus \bigoplus_{I'} E(C_i)$ (***exchange property***). □

What does all this have to do with pure-injective modules? (which are of more concern to us than the special case of injectives).

Theorem 4.A12 [GJ81; §1] (also cf. [Fac85] and [Fis75]) *Let* R *be a ring. There is a Grothendieck abelian category* $\mathcal{C}$ *and a functor* $F : \mathcal{M}_R \longrightarrow \mathcal{C}$ *which is full, faithful, and pure-exact, such that* M_R *is pure-injective iff* FM *is an injective object of* $\mathcal{C}$. □

To say that F is pure-exact is to say that whenever the sequence $0 \longrightarrow M \rightarrowtail N \longrightarrow N/M \longrightarrow 0$ is exact with $M \longrightarrow N$ a pure embedding, then the image sequence $0 \longrightarrow FM \longrightarrow FN \longrightarrow F(N/M) \longrightarrow 0$ also is exact.

Actually there are a number of ways, found by various authors, of turning the pure-injective modules into injective objects of a Grothendieck abelian category. I will describe some of these ways, in more or less detail, in §12.3.

The next result has a number of independent sources: [Fis75; 7.14], [Z-HZ78; Thm 9], [GJ81; 1.3]. For a description of the radical, see 4.27.

Corollary 4.A13 *Let* N_R *be an indecomposable pure-injective. Then* End(N) *is a local ring.*

Proof I will include a direct proof of this (4.27) but it follows immediately from 4.A12 since, with notation as there, $(N,N)\simeq(FN,FN)$ (F being full and faithful), and then one has 4.A8. □

Corollary 4.A14 [Fis75; 7.21] *Let N_R be pure-injective. Then there are unique direct summands N_c and N_d of N such that $N=N_d\oplus N_c$ where N_c has no indecomposable direct summands and where $N_d=\mathrm{pi}(\bigoplus_I N_i)$ with the N_i indecomposable direct summands of N. Moreover, this decomposition is essentially unique (in the sense of 4.A10) and, if N' is a direct summand of N_d, then there is $I'\le I$ such that $N=N'\oplus\mathrm{pi}(\bigoplus_{I'} N_i)$.*

Proof This follows by 4.A12, 4.A10 and pure-exactness of the functor F in 4.A12. □

Ziegler gives a different proof of this ([Zg84; §6]) by developing a "dimension theory " for factors: I say a little about this.

Let M be a module and let $\mathfrak{F}$ be the set of all its direct summands. Ziegler defines $M'\in\mathfrak{F}$ to be dependent on $\mathfrak{F}'\le\mathfrak{F}$ if there is a finite subset, $\mathfrak{F}_0$, of $\mathfrak{F}'$ such that there is no decomposition of M of the form $M'\oplus M''$, with $\bigcup\mathfrak{F}_0\le M''$. He notes that this is a kind of dependence relation (in particular, the exchange property holds). Then he shows that, if $\mathfrak{F}$ is cut down to factors with local endomorphism rings, a weak transitivity axiom is satisfied; from this, he deduces the existence of bases as well as the fact that bases have the same cardinality. So, fixing a summand with local endomorphism ring, one may define the dimension of M with respect to that indecomposable. Finally, Ziegler makes the assumption that M is pure-injective, develops some material relating to hulls and then completes the proof of 4.A14.

Given a complete theory, T, of modules let N be a sufficiently saturated (weakly saturated is enough) model of T and express it as $N=N_d\oplus N_c$ as above. Define the continuous part of T, T_c, to be the complete theory of N_c.

4.4 Irreducible types

A major theme of these notes is the representation of pure-injective modules as (pure-injective hulls of) direct sums of indecomposable pure-injectives. The general sort of decomposition which one may expect has been described in the previous section (4.A14), but we wish to know, for example, under what conditions the continuous factor is zero. Moreover one might wish to have much more detailed information about the decompositions - what indecomposable factors can occur and how often they may occur.

This section is concerned with indecomposable pure-injectives and with the pp-types whose hulls they are.

The first task is to give a direct proof that indecomposable pure-injectives have local endomorphism rings, *via* a useful description of the radical of this ring.

It is known that if E is an injective module and if $S=\mathrm{End}E$, then the Jacobson radical $J(S)$ of S consists of precisely those $f\in S$ with $\ker f$ essential in E. In order to give another illustration of the kind of generalisation one should expect in going from injective to general (pure-injective) modules, I insert an intermediate step to the correct generalisation: one may skip over this to the definition just before 4.26.

Given N pure-injective, set $S=\mathrm{End}N$. One might define, in an effort to characterise the elements of the radical of S, the set $H(S)=\{f\in S : \ker f$ is pp-essential in $N\}$.

Lemma 4.25 *Let N be pure-injective, and set $S=\mathrm{End}N$. Then $H(S)\le J(S)$.*

Proof Let $f \in H(S)$ and note that for any $g \in S$ one has $\ker(gf) \geq \ker f$. So $Sf \leq H(S)$. Recall (e.g. [He68; 1.2.3]) that if every element r of a certain left ideal of a ring is such that $1+r$ is invertible, then that left ideal is contained in the Jacobson radical. So it will be enough to show that $1+f$ is invertible.

Let $A = \ker f$. Since $A \longrightarrow N$ is assumed to be pp-essential one has $N = N(A)$. Moreover $1+f$ fixes A (pointwise). So by **4.16** $1+f$ is an automorphism of N, as required. □

It is not difficult to see, however, that one should not expect $H(S)$ to equal $J(S)$. It has already been noted that the intersection of pp-essential submodules may be zero, so it is not even clear that one should expect $H(S)$ to be closed under addition.

Example 1 Consider $R = \mathbb{Z}$, $N = \mathbb{Z}_2 \oplus \mathbb{Z}_4$, and let $g \in S = \mathrm{End} N$ be defined by $(1,0) \mapsto (0,2)$ and $(0,1) \mapsto 0$. Since $\ker g = \mathbb{Z}_4$ is not pp-essential in N, g does not lie in $H(S)$. Nevertheless $g \in J(S)$ since $g^2 = 0$.

The point in this example is that, although the algebraic kernel is not large enough, g is strictly pp-type-preserving on a large (pp-essential) subset of N. That is: it is too much to expect to find many elements which have been annihilated; rather one should look at those elements whose pp-type has been increased - $\overline{\mathbb{Z}(\overline{p})}$ with the endomorphism multiplication-by-p provides an even better example.

So the definition of $H(S)$ did not take full account of the way in which pp formulas replace $\bigwedge$-atomic formulas.

Therefore given a pure-injective N with endomorphism ring S, we consider $K(S) = \{f \in S : f$ is strictly pp-type-increasing on a pp-essential subset of $N\}$. (Clearly $H(S) \leq K(S)$.)

Proposition 4.26 *Let N be pure-injective with endomorphism ring S. If $f \in J(S)$ then f strictly increases the pp-type of every non-zero element. In particular $J(S) \leq K(S)$.*

Proof Suppose that $f \in J(S)$. Then $1-gf$ is an automorphism of N for all $g \in S$ (see [He68; 1.2.3]). Suppose then that $b \in N$ has $\mathrm{pp}(fb) = \mathrm{pp}(b)$. By **2.8** there is $g \in S$ with $gfb = b$. Thus $(1-gf)b = 0$, and so it must be that $b = 0$, as required. □

Now we will see that $J(S) = K(S)$ provided N is indecomposable. Actually, by making use of **4.A12** and what is known of the injective case one may see that this equality holds without any restriction on N. I do not see a direct proof of this, so confine myself to the indecomposable case (which is all that we will need).

The first part of the next result characterises those endomorphisms of an indecomposable pure-injective which are in the radical (it may be found in [Pr81; 4.4, 4.8] and [Zg84; 4.3]): the fact that the endomorphism ring is local then follows (cf. **4.A13**).

Theorem 4.27 *Let N be an indecomposable pure-injective with endomorphism ring S.*

(a) $J(S) = \{f \in S : f$ strictly increases the pp-type of some element of $N\}$
$= \{f \in S : f$ strictly increases the pp-type of every non-zero element of $N\}$.

(b) S is a local ring.

Proof (a) In view of **4.26** it remains to show that if f strictly increases the pp-type of some element a of N then f must be in $J(S)$. So given such f and a let $g \in S$. It will be enough to show that $\mathrm{pp}(a - gfa) = \mathrm{pp}((1-gf)a) = \mathrm{pp}(a)$: for then, by **4.13**, $1-gf$ will have an inverse and so f will be in $J(S)$.

Suppose, for a contradiction, that $\mathrm{pp}((1-gf)a) \supset \mathrm{pp}(a)$: say φ is pp with $\varphi(a-gfa) \wedge \neg\varphi(a)$. Let $p(v) \in S_1^+(0)$ be maximal extending $\mathrm{pp}(gfa)$ and consistent with $\varphi(a-v)$. Since N is pure-injective, $p \wedge \varphi(a-v)$ is realised in N, say by c. Since $\mathrm{pp}(c) \geq \mathrm{pp}(gfa)$ there is (by 2.8) some $h \in S$ with $hfga = c$.

Since $(hgf+1)(a-c) = hgfa - hgfc + a - c = a - hgfc$, one has $\mathrm{pp}(a-hgfc) \geq \mathrm{pp}(a-c)$ (2.7). Since, by choice of c, $\varphi(a-c)$ holds, one concludes $\varphi(a-hgfc)$. Thus one has $\mathrm{pp}(hgfc) \geq \mathrm{pp}(c)$ and $\varphi(a-hgfc)$ so, by maximality of $\mathrm{pp}(c) = p$, it must be that $\mathrm{pp}(hgfc) = p = \mathrm{pp}(c)$.

Therefore by **4.13** hgf is an automorphism of N. Certainly then f is an automorphism - for it cannot increase the pp-type of any element, contradicting $\mathrm{pp}(fa) \supset \mathrm{pp}(a)$.

(b) This now follows since (by the first paragraph of the proof of (a)) it has just been shown that if $f \in J(S)$ then $1-f$ is invertible. ▫

Exercise 1

(i) Show directly that, supposing $f \in S = \mathrm{End}N$ where N is pure-injective, if $f \in K(S)$ then f strictly increases the pp-type of every non-zero element of N. [Hint: use 4.5]

(ii) If every non-zero pp-definable subgroup of the pure-injective module N contains a minimal non-zero pp-definable subgroup (e.g., if N is Σ-pure-injective = totally transcendental) deduce that $J(S) = K(S) = H(S)$.

(iii) Find conditions for $J(S)$ to be the nilradical of S, and for $J(S)$ to be nilpotent.

(iv) Identify S, $J(S)$ and $S/J(S)$ for $S = \mathrm{End}N$, N being an indecomposable pure-injective abelian group (see §2.$\mathbb{Z}$).

Now we come to a key notion. The pp-type $p \in S^+(0)$ is **irreducible** if its hull is indecomposable. This generalises the terminology for right ideals where, for the definition, the following equivalent conditions may be used: (i) I is $\cap$-irreducible in the lattice of right ideals; (ii) the injective module $E(R/I)$ - the injective hull of an element with annihilator precisely I - is indecomposable. There will be a similar equivalence for pp-types (8.2).

The result which follows the next lemma is an extremely useful syntactic characterisation of irreducible pp-types, due to Ziegler. The term "irreducible" is extended to types *via* their pp-parts.

<u>Proposition 4.28</u> [Zg84; proof of 11.4] *Let $N = N(\bar{a})$ be decomposed as $N = N_1 \oplus N_2$ with $N_1, N_2 \neq 0$. Set $\bar{a} = (\bar{a}_1, \bar{a}_2) \in N_1 \oplus N_2$. Then the pp-type of $\bar{a}$ is strictly contained in both* $\mathrm{pp}(\bar{a}_1)$ *and* $\mathrm{pp}(\bar{a}_2)$.

Proof Of course $\mathrm{pp}(\bar{a}) \leq \mathrm{pp}(\bar{a}_i)$ by 2.7. Since $\bar{a}_2$ is in $N(\bar{a})$, there is (4.10(d)) some pp formula φ with $\varphi(\bar{a}, \bar{a}_2) \wedge \neg\varphi(\bar{a}, \bar{0})$. On projecting $\varphi(\bar{a}, \bar{a}_2)$ to the first coordinate one obtains $\varphi(\bar{a}_1, \bar{0})$. Now, were $\mathrm{pp}(\bar{a}_1) = \mathrm{pp}(\bar{a})$ this would imply $\varphi(\bar{a}, \bar{0})$ - contrary to choice of φ. ▫

<u>Theorem 4.29</u> [Zg84; 4.4, 4.5] *Let $p \in S(0)$ be any non-zero type. Then p is irreducible iff for all $\psi_1, \psi_2 \in p^-$ there exists $\varphi \in p^+$ such that $(\varphi \wedge \psi_1) + (\varphi \wedge \psi_2) \in p^-$.*

More generally, p is irreducible iff for all $\psi_1, \dots, \psi_n \in p^-$ there exists $\varphi \in p^+$ such that $\Sigma_1^n \varphi \wedge \psi_i \in p^-$.

Proof $\Rightarrow$ Suppose that p is irreducible but that the right-hand side of the equivalence fails. Let $\bar{a}$ realise p in $N(p)$ and let $\psi_1, \psi_2 \in p^-$ provide a counterexample to the right hand side.

Note that $p^+(\bar{v}_1) \wedge p^+(\bar{v}_2) \wedge \psi_1(\bar{v}_1) \wedge \psi_2(\bar{v}_2) \wedge (\bar{a} = \bar{v}_1 + \bar{v}_2)$ is a consistent set of formulas (in $\mathrm{Th}(N(p))$ with $\bar{a}$ as parameters): for otherwise there would be $\varphi_1, \varphi_2 \in p^+$ with $(\varphi_1 \wedge \psi_1)(\bar{v}_1) \wedge (\varphi_2 \wedge \psi_2)(\bar{v}_2) \wedge (\bar{v}_1 + \bar{v}_2 = \bar{a})$ not satisfied in $N(p)$. But this would mean, setting $\varphi = \varphi_1 \wedge \varphi_2 (\in p^+)$, that $\varphi \wedge \psi_1 + \varphi \wedge \psi_2$ was not in p^+, contrary to choice of ψ_1 and ψ_2. So realise this set of pp formulas by $\bar{b}_1, \bar{b}_2$ (say) in $N(p)$.

Since $p^+(\bar{b}_i)$ holds there is $f_i \in \mathrm{End}N(p)$ with $f_i\bar{a}=\bar{b}_i$ $(i=1,2)$. Since $\psi_i \notin p$, f_i is not an automorphism $(i=1,2)$. But $\bar{a}=\bar{b}_1+\bar{b}_2=(f_1+f_2)\bar{a}$. So by **4.13** f_1+f_2 is an automorphism of $N(p)$. This contradicts $\mathrm{End}N(p)$ being local (**4.27**), as required.

$\Leftarrow$ If p is not irreducible then take $\bar{a}$ realising p in $N(\bar{a})\simeq N(p)$ and set $N(\bar{a})=N_1\oplus N_2$ with $N_1,N_2\neq 0$. Let π_1,π_2 be the canonical projections. Now, $\bar{a}=\pi_1\bar{a}+\pi_2\bar{a}$ and, by **4.28**, $\mathrm{pp}(\pi_i\bar{a})\supset\mathrm{pp}(\bar{a})$ - say ψ_i is pp with $\psi_i(\pi_i\bar{a})\wedge\neg\psi_i(\bar{a})$ $(i=1,2)$. Then, clearly, the syntactic criterion of the statement of the result fails, as required.

The more general statement follows by an easy induction (exercise). □

Corollary 4.30 *Suppose that the pp-type of* $\bar{a}$ *is irreducible. Then* $\bar{a}$ *cannot be a sum* $\Sigma_1^n\bar{b}_i$ *of tuples* $\bar{b}_i$ *with strictly larger pp-types. In particular if* $\mathrm{pp}(\bar{b}),\mathrm{pp}(\bar{c})>\mathrm{pp}(\bar{a})$ *then* $\bar{a}\neq\bar{b}+\bar{c}$. □

Exercise 2 Here is another way of expressing **4.29**. Let N be an indecomposable pure-injective and let S be its endomorphism ring. Take any non-zero element a of N. Then Sa strictly contains $\Sigma\{Sb : b\in N \text{ and } \mathrm{pp}(b)>\mathrm{pp}(a)\}$.

Proposition 4.31 *Suppose that* $\bar{a}$ *and* $\bar{b}$ *are linked. Then the hull of* $\bar{a}$ *and the hull of* $\bar{b}$ *have isomorphic non-zero direct summands.*

Proof Let p be the type of $\bar{a}$. Suppose that φ is a pp formula linking $\bar{a}$ and $\bar{b}$: $\varphi(\bar{a},\bar{b})\wedge\neg\varphi(\bar{a},\bar{0})$. Let $q(\bar{v},\bar{b})=\mathrm{pp}(\bar{a}/\bar{b})$. The pp-type q is finitely satisfied in $N(\bar{b})$ (which is pure in $\tilde{M}$). Since $N(\bar{b})$ is pure-injective, q is therefore realised in $N(\bar{b})$. Working modulo the implicit over-theory, let us take a realisation $\bar{c}$ which is such that $q_0=\mathrm{pp}(\bar{c})$ is maximal possible (there is such - let $q'(\bar{v})$ be a maximal pp-type such that $q(\bar{v},\bar{b})\wedge q'(\bar{v})$ is consistent, then extend to a maximal pp-type over $\bar{b}$ - this is realised in $N(\bar{b})$ by **4.6**). If $\mathrm{pp}(\bar{c})=\mathrm{pp}(\bar{a})$ then we are finished.

Otherwise $\mathrm{pp}(\bar{c})>\mathrm{pp}(\bar{a})$ (since q certainly contains the type of $\bar{a}$ over 0). Since both $\bar{a}$ and $\bar{c}$ satisfy p, certainly $p(\bar{a}-\bar{c})$ holds. From $\varphi(\bar{a},\bar{b})\wedge\varphi(\bar{c},\bar{b})$ one also has $\varphi(\bar{a}-\bar{c},\bar{0})$. Let p_1 be the pp-type of $\bar{a}-\bar{c}$. Since $\bar{a}=(\bar{a}-\bar{c})+\bar{c}$, the following pp-type over $\bar{a}$ is consistent: $\bar{a}=\bar{v}+\bar{w}\wedge p_1(\bar{v})\wedge q_0(\bar{w})$. Let $(\bar{d},\bar{e})$ realise this in $N(\bar{a})$.

The pp-type of $\bar{e}$ over $\bar{b}$ extends that of $\bar{a}$. For if $\psi(\bar{a},\bar{b})$ holds then so does $\psi(\bar{c},\bar{b})$ (by definition of $\bar{c}$) and hence $\psi(\bar{v},\bar{0})\in p_1$. Therefore $\psi(\bar{d},\bar{0})$ holds. Since $\psi(\bar{a},\bar{b})$ is just $\psi(\bar{d}+\bar{e},\bar{b})$, one concludes $\psi(\bar{e},\bar{b})$, as required. Therefore $\bar{e}$ satisfies $\varphi(\bar{v},\bar{b})$ and also $\mathrm{pp}(\bar{e})\geq q_0=\mathrm{pp}(\bar{c})$. So, by maximality of $\mathrm{pp}(\bar{c})$, $\mathrm{tp}(\bar{e})=\mathrm{tp}(\bar{c})$. Hence the hull of $\bar{e}$ is isomorphic to that of $\bar{c}$, and so $N(\bar{a})$ and $N(\bar{b})$ have a non-zero direct summand in common. □

Corollary 4.32 *Suppose that* $\bar{a}$ *and* $\bar{b}$ *are linked and that the pp-type of* $\bar{a}$ *is irreducible. Then the hull of* $\bar{b}$ *contains, as a direct summand, a copy of the hull of* $\bar{a}$. □

One may note that to prove the corollary, one may terminate the above proof at the stage when one has the element $\bar{a}$ expressed as the sum of two elements with strictly greater pp-type (contradicting **4.30**).

The next corollary of **4.29** is a rather general method for constructing irreducible pp-types while retaining some control over their properties. This sort of construction was used first for pp-types in [Pi84a] and [Zg84] and in fact generalises a reasonably familiar argument from ring theory (in which right ideals replace pp-types). Recall that an ideal of a lattice is a subset which: contains the least element of the lattice; does not contain the largest element; is downwards closed; is closed under finite unions.

Theorem 4.33 [Zg84; 4.7], [PP87; 6.6] *Let* Ψ *be a collection of pp formulas which defines an ideal of subgroups (in models of* T*). Let* $p_0 \in S^+(0)$ *be a pp-type (i.e., a filter of pp-definable subgroups) such that* $p_0 \cap \Psi = \emptyset$.

Then there is at least one pp-type $p \in S^+(0)$ *maximal with respect to:* $p \geq p_0$ *and* $p \cap \Psi = \emptyset$. *Every such pp-type* p *is irreducible and, together with* $\neg\Psi$, *defines a complete type (where* $\neg\Psi = \{\neg\psi : \psi \in \Psi\}$*).*

Proof Existence of such a pp-type p is immediate by Zorn's Lemma (and the compactness theorem).

To show that p is irreducible we suppose that $\theta_1, \theta_2 \in p^-$. By maximality of p there are $\varphi_1, \varphi_2 \in p$ such that $\varphi_1 \wedge \theta_1$ and $\varphi_2 \wedge \theta_2$ are in Ψ. (For, inconsistency of $\theta_1 \wedge p \wedge \neg\Psi$ means that there is some $\varphi_1 \in p$ and $\psi_1, \ldots, \psi_n \in \Psi$ with $\varphi_1 \wedge \theta_1 \rightarrow \bigvee_1^n \psi_i$ but, identifying formulas with the sets they define, $\bigcup_1^n \psi_i \leq \Sigma_1^n \psi_i$; whence $\varphi_1 \wedge \theta_1 \in \Psi$ (for Ψ is closed under "+", and is downwards closed).)

Replacing φ_1 and φ_2 by their conjunction φ, one has that $\varphi \wedge \theta_1$ and $\varphi \wedge \theta_2$, hence $\varphi \wedge \theta_1 + \varphi \wedge \theta_2$, are in Ψ. In particular $\varphi \wedge \theta_1 + \varphi \wedge \theta_2$ is in p^-. Hence, by 4.29, p is indeed irreducible.

Finally, observe that we showed that θ is in p^- iff there are φ in p and ψ in Ψ with $\theta \wedge \varphi \rightarrow \psi$ - that is, with $\varphi \wedge \neg\psi \rightarrow \neg\theta$. Thus $p \cup \neg\Psi$ proves $p \cup \neg p^-$ which, by 2.20, is complete. □

Corollary 4.34 [Zg84; 4.7] *Every complete theory of modules has irreducible types.*

Proof Take $\Psi = \{v = 0\}$ and $p_0 = \langle v = v \rangle$ in 4.33. □

The above result is stated under the conventions that the zero module is not indecomposable and that the trivial theory Th(0) is excluded from consideration.

For any theory T of modules we set $\mathcal{I}(T)$ to be the set of all (isomorphism types of) indecomposable pure-injective direct summands of models of T. We shorten $\mathcal{I}(T^*_R)$ to $\mathcal{I}_R$: recall (§2.6) that T^*_R is the largest complete theory of R-modules. Also we shorten $\mathcal{I}(\mathrm{Th}(M))$ to $\mathcal{I}(M)$. For example $\mathcal{I}_{\mathbb{Z}}$ consists of all the indecomposable pure-injectives listed after 2.ℤ11: $\mathcal{I}(\mathbb{Z}) = \{\overline{\mathbb{Z}(p)} : p \text{ is prime}\} \cup \{\mathbb{Q}\}$; $\mathcal{I}(\mathbb{Z}_2 \oplus \mathbb{Z}_4^{(\aleph_0)}) = \{\mathbb{Z}_2, \mathbb{Z}_4\}$; $\mathcal{I}(\mathbb{Z}_{(p)}) = \{\overline{\mathbb{Z}(p)}, \mathbb{Q}\}$. Actually $\mathcal{I}(T)$ carries the structure of a topological space, and that structure will turn out to be very important but, for the moment, $\mathcal{I}(T)$ is merely a set. Given a theory T and pp formulas φ, ψ we write $\varphi > \psi$ iff T proves $\psi \rightarrow \varphi \wedge \neg(\varphi \rightarrow \psi)$ (and so $\varphi(M) > \psi(M)$ in every model M).

Corollary 4.35 [Zg84; 4.8] *If* $\mathrm{Inv}(T, \varphi, \psi) > 1$ *then there is an irreducible pp-type* p *which contains* φ *but not* ψ: *we write* $\varphi/\psi \in p$. *Thus there is an indecomposable pure-injective module* $N \in \mathcal{I}(T)$ *with* $\varphi(N) > \psi(N)$.

Proof Take Ψ in 4.33 to be ψ (or rather the ideal of pp-definable subgroups generated by ψ) and take p_0 to be the pp-type generated by ψ. □

Corollary 4.36 [Zg84; 6.9] *Every module is elementarily equivalent to a direct sum of indecomposable pure-injective modules, equivalently, to a discrete pure-injective module.*

Proof Let M be any (non-zero) module and let M_1 be a very saturated module elementarily equivalent to M. By 4.A14 the pure-injective model M_1 has the form $M_d \oplus M_c$ where M_d is the pure-injective hull of a direct sum of indecomposable pure-injectives, and M_c has no indecomposable direct summands. It is claimed that $M_d \equiv M$.

If this were not so then, taking note of 2.23(a), there would be φ, ψ with $\mathrm{Inv}(M_d, \varphi, \psi)$ finite and $\mathrm{Inv}(M_c, \varphi, \psi) > 1$. So there is by 4.35, some pure-injective elementary extension M' of M_c realising an irreducible type p with $\varphi/\psi \in p$: say $M' = M'' \oplus N$ where N is an indecomposable pure-injective with $\mathrm{Inv}(N, \varphi, \psi) > 1$. Since M_1 was chosen to be very saturated we have $M_1 \simeq M_1 \oplus N$ and so $M_d \simeq M_d \oplus N$. Hence $\mathrm{Inv}(M_d, \varphi, \psi)$ must be infinite - the required contradiction. □

For modules over a commutative regular ring, the above was proved by Garavaglia [Gar79; Cor 1].

Corollary 4.37 *Suppose that M is a continuous pure-injective module. Then $M \equiv M^{\aleph_0}$.* □

Exercise 3 Give a simpler proof of 4.36 for the case $M \equiv M^{\aleph_0}$.

Exercise 4 Use 4.36 and 2.ℤ8 to give a proof of 2.ℤ5.

Corollary 4.36 is also immediate from the next result and 2.24.

Corollary 4.38 [Zg84; 6.14] (cf. [FS85; XI 2.8]) *Let T be complete. Every continuous pure-injective summand of a model of T is a direct summand of a direct product of members of $\mathcal{I}(T)$. In particular, every continuous pure-injective module is a direct summand of a product of indecomposable pure-injectives.*

Proof First we note that it is sufficient to treat the case that the continuous pure-injective E is the hull of a single element a: for by Zorn's Lemma there is a direct sum of hulls of single elements which is pure in E and has E as its hull (alternatively, one may add a little argument at the end of the proof).

Let p be the pp-type of a. For each ψ in p^- let q_ψ be maximal with pp-part containing p^+ and omitting ψ. By 4.33, q_ψ is irreducible: realise q_ψ by a_ψ in the indecomposable N_ψ. Consider the element $b = (b_\psi)_\psi$ in $\prod_\psi N_\psi$. Certainly $\mathrm{pp}(b) \geqslant p^+$ and, if $\psi \in p^-$ then, since $\neg\psi(b_\psi)$ holds, one has $\neg\psi(b)$. Thus $\mathrm{tp}(b) = \mathrm{tp}(a)$. Hence E, the hull of a, is isomorphic to a direct summand of $\prod_\psi N_\psi$, as required. □

Here is another corollary of 4.29. Complete theories which are closed under product are much more amenable than those which are not - for example the link between model theory and algebra is much more direct in the former case - and so we often will replace a complete theory T by $T^{\aleph_0}$. This would not be so useful if the models of these two theories were wildly different; but that is not the case. Indeed, the next result says that no new indecomposable pure-injectives appear when we move from T to $T^{\aleph_0}$.

Corollary 4.39 *Let T be any complete theory of modules. Then $\mathcal{I}(T) = \mathcal{I}(T^{\aleph_0})$.*

Proof Of course $\mathcal{I}(T) \subseteq \mathcal{I}(T^{\aleph_0})$. So take $N \in \mathcal{I}(T^{\aleph_0})$: N is the hull of some irreducible 1-type, p, over 0 (a type modulo $T^{\aleph_0}$). If we can show that there is some 1-type p_0 in $S^T(0)$ with $p_0{}^+ = p^+$ then, by 4.15, it will follow that $N(p_0) \simeq N(p) \simeq N$ and hence $N \in \mathcal{I}(T)$, as required.

If there is no such type then it must be that $p^+ \cup \neg p^-$ is inconsistent with T. Therefore there is $\varphi \in p^+$ and $\psi_1, \ldots, \psi_n \in p^-$ such that $T \vdash \varphi \to \bigvee_1^n \psi_i$. Since p is irreducible there is, by 4.29, some $\varphi_0 \in p^+$ such that $\sum_1^n \varphi_0 \wedge \psi_i \in p^-$, and it may as well be supposed (replace φ_0 by $\varphi \wedge \varphi_0$) that $\varphi_0 \leqslant \varphi$.

Now $T^{\aleph_0} \vdash \exists v\, (\varphi_0(v) \wedge \neg(\sum_1^n \varphi_0 \wedge \psi_i).(v))$ (since p is consistent for $T^{\aleph_0}$). Thus $\mathrm{Inv}(T^{\aleph_0}, \varphi_0, \sum_1^n \varphi_0 \wedge \psi_i) > 1$. Hence $\mathrm{Inv}(T, \varphi_0, \sum_1^n \varphi_0 \wedge \psi_i) > 1$. But $T \vdash \varphi \to \bigvee_1^n \psi_i$ certainly implies $T \vdash \varphi_0 \to \bigvee_1^n \varphi_0 \wedge \psi_i$. Also $\bigcup_1^n \varphi_0 \wedge \psi_i \subseteq \sum_1^n \varphi_0 \wedge \psi_i$. Therefore, in T one has $\varphi_0 \leqslant \sum_1^n \varphi_0 \wedge \psi_i$ - contradicting $\mathrm{Inv}(T, \varphi_0, \sum_1^n \varphi_0 \wedge \psi_i) > 1$. □

It should not, therefore, come as a surprise that the difference between (discrete) pure-injective models of T and those of $T^{\aleph_0}$ is simply that some indecomposables may be constrained to a fixed finite number of occurrences in the decomposition of each pure-injective model of T (4.63, 10.24).

As a corollary, no new continuous pure-injectives can appear either.

Corollary 4.40 *Let T be a complete theory of modules. Suppose that $a \in M \models T^{\aleph_0}$ is such that $N(\bar{a})$ is continuous. Then the hull of $\bar{a}$ is also a direct summand of a model of T. Therefore, any continuous summand of a model of $T^{\aleph_0}$ is also a summand of a model of T.*

Proof Let p be the type of $\bar{a}$ in M. By **4.38**, p is in the closure of the set of irreducible types in the space of 1-types modulo $T^{\aleph_0}$. By **4.39**, these irreducibles all are contained in the image of the canonical inclusion $S_1{}^T(0) \longrightarrow S_1{}^{T^{\aleph_0}}(0)$ which, by **2.33**, is a closed set. Hence $N(p)$ is a summand of a model of T.

Since every continuous pure-injective is the pure-injective hull of a direct sum of hulls, N, of single elements and, by **4.37**, these all satisfy $N \equiv N^{\aleph_0}$, the second statement follows.
□

4.5 Limited and unlimited types

Consider the following example.

Example 1 The ground ring is the ring of integers and the theory is that of $\mathbb{Z}_2{}^{(\aleph_0)} \oplus \mathbb{Z}_4$. It is easy to check (using the invariants) that the models of T are just those modules of the form $\mathbb{Z}_2{}^{(\kappa)} \oplus \mathbb{Z}_4$ with κ infinite. Now, although it is true that one may find infinitely many distinct copies of $\mathbb{Z}_4$ as direct summands of $\mathbb{Z}_2{}^{(\aleph_0)} \oplus \mathbb{Z}_4$, it is the case that, in any decomposition of any model, only one copy of $\mathbb{Z}_4$ appears. Once this has been recognised, the indecomposable direct summand $\mathbb{Z}_4$ becomes, in some sense, irrelevant to the purpose of classifying models and only the one cardinal, κ as above, is needed to parametrise them.

It will be useful, therefore, to identify those indecomposable pure-injective direct summands of models which, like $\mathbb{Z}_4$ in the example above, are "limited" in the number of their occurences. This is of course a notion which is entirely relative to the over-theory T.

Denote by $\mathcal{P}(T)$ the class of (isomorphism types of) pure-injective direct summands - **components** - of models of T.

A pure-injective (isomorphism type) $N \in \mathcal{P}(T)$ is said to be T-**unlimited**, or just **unlimited** if T is understood, if whenever M is a model of T then so is $M \oplus N$ a model. Otherwise N is (T-)**limited**. For a type or pp-type p, say that p is (T-)(un)**limited** if $N(p)$ is so. (See §5.3 for the more general notion of being unlimited over a set of parameters.)

Lemma 4.41 [Pr81; 5.8], [PP87; 1.2] *The following conditions on the module $N \in \mathcal{P}(T)$ are equivalent:*
(i) p is unlimited;
(ii) there exists $M \models T$ with $M \oplus N \models T$;
(iii) for any κ and any $M \models T$ one has $M \oplus N^{(\kappa)} \models T$;
(iv) for any pair φ, ψ of pp formulas (in one free variable) if $\mathrm{Inv}(N, \varphi, \psi) > 1$ then $\mathrm{Inv}(T, \varphi, \psi) = \infty$.

Proof (i)⇒(ii) This is trivial from the definition.

(ii)⇒(iv) Suppose that $\varphi(N)$ strictly contains $\psi(N)$. From (ii) one has $\mathrm{Inv}(T,\varphi,\psi)=\mathrm{Inv}(M\oplus N,\varphi,\psi)=\mathrm{Inv}(M,\varphi,\psi)\times\mathrm{Inv}(N,\varphi,\psi)$ and this equals $\mathrm{Inv}(T,\varphi,\psi)\times\mathrm{Inv}(N,\varphi,\psi)$. Since $\mathrm{Inv}(N,\varphi,\psi)>1$ it must be that $\mathrm{Inv}(T,\varphi,\psi)$ is infinite.

(iv)⇒(iii) This is immediate by **2.18**.

(iii)⇒(i) This is immediate from the definition. □

<u>Proposition 4.42</u> *Let* $p\in S^T(0)$. *Then the following conditions are equivalent:*

(*i*) p *is limited;*

(*ii*) *there are* $\varphi\in p^+$ *and* $\psi\in p^-$ *with* $\mathrm{Inv}(T,\varphi,\psi)$ *finite.*

Proof (i)⇒(ii) Suppose that the condition (ii) is not satisfied. Let M be a pure-injective model of T, and consider the set of formulas $\Phi(\bar{v})=p^+(\bar{v})\cup\{\neg\psi(\bar{v}-\bar{m}):\bar{m}\in p^+(M)$ and $\psi\in p^-\}$. It is claimed that $\Phi(\bar{v})$ is finitely satisfied in M.

Otherwise there would be $\bar{m}_1,\ldots,\bar{m}_k\in p^+(M)$, $\psi_1,\ldots,\psi_k\in p^-$ and $\varphi\in p^+$ such that $\varphi(\bar{v})\leqslant\bigcup_1^k\psi_i(\bar{v}-\bar{m}_i)$. By Neumann's Lemma (**2.12**) this implies that $\mathrm{Inv}(T,\varphi,\psi_i)$ is finite for some i - contrary to assumption (ii). Thus $\Phi(\bar{v})$ is indeed consistent.

Therefore there is $\bar{c}$ realising Φ in some elementary extension M' of M. Since M is pure-injective there is a decomposition $\bar{c}=(\bar{c}_0,\bar{c}_1)\in M\oplus N=M'$. We will see that $\bar{c}_1$ realises p.

Since $p^+(\bar{c})$ holds, certainly it is the case that both $\bar{c}_0$ and $\bar{c}_1$ satisfy p^+. Let $\psi\in p^-$: then $\neg\psi(\bar{c}-\bar{c}_0)$ holds by definition of Φ. That is, $\neg\psi(\bar{c}_1)$ holds. Therefore $\mathrm{tp}(\bar{c}_1)=p$.

Hence $M\oplus N(p)$ is a direct summand of M' (by construction of hulls) and, by **2.25**, it is a model of T. Hence p is indeed unlimited (by **4.41**), as required.

(ii)⇒(i) This is immediate by **4.41**(i)⇒(iv). □

Notice that it is not obvious from **4.42**(ii) that being limited is a relatedness invariant on types (i.e., depends on the hull, rather than on the particular type). The next corollary is immediate from **4.42**.

<u>Corollary 4.43</u> *A type* p *over* 0 *is unlimited iff* $G(p)=G_0(p)$.

Minimal non-zero pp-definable subgroups have a technical importance in many arguments (see for example §6; also compare **6.23**): they tend however to exist only in rather restricted situations. The next definition provides what turns out to be a useful generalisation, which has applicability in much wider circumstances. In model-theoretic terms, we are going from regular types to types of weight one (regarded in another way, we are moving to regular types in T^{eq} - see §10.T). The notion appears in essence in [Gar80a] and was heavily exploited in [Zg84]. Its main uses will be later (Chapters 9 and 10), but we note that it has some relevance here.

Suppose that φ,ψ are pp formulas. Say that φ/ψ is a T-**minimal pair** (or **minimal pair** for T) if φ strictly contains ψ but no pp-definable subgroup lies strictly between φ and ψ. That is, φ is a cover of ψ (in any, equivalently every, model of T). Similarly, φ/ψ is an N-**minimal** pair if $\varphi(N)$ covers $\psi(N)$ in the lattice of pp-definable subgroups of N. Garavaglia [Gar80a; §6] made some use of minimal pairs ("simple pairs"), but it was Ziegler who made decisive use of them.

<u>Corollary 4.44</u> *Suppose that* p *is a type in* $S^T(0)$. *Then* p *is limited iff there is a* T*-minimal pair* $\varphi/\psi\in p$ *with* $\mathrm{Inv}(T,\varphi,\psi)$ *finite.*

(*In particular such a type, if irreducible, is neg-isolated -see* §9.3.)

Proof By **4.42** it will be sufficient to establish just one direction: that if p is limited then there is such a minimal pair. Moreover, **4.42** implies that if p is limited then there is $\varphi/\psi\in p$ with $\mathrm{Inv}(T,\varphi,\psi)$ finite.

Since $\mathrm{Inv}(T,\varphi,\psi)$ is finite, the interval $[\varphi,\psi]^T$ in the lattice of pp-definable subgroups of (models of) T has finite length. So clearly there are pp formulas φ',ψ' with $\varphi(N)\geqslant\varphi'(N)>\psi'(N)\geqslant\psi(N)$, with φ'/ψ' a T-minimal pair and with $\varphi'/\psi'\in p$. Also, $\mathrm{Inv}(T,\varphi',\psi')\leqslant\mathrm{Inv}(T,\varphi,\psi)$ is finite, as required. ◻

Proposition 4.45 *Suppose that* $p\in S^T(0)$ *is limited. Then there is* $n(p)\in\omega$ *such that if* $M\models T$ *is any pure-injective model then* $N(p)$ *appears no more than* $n(p)$ *times in any direct-sum decomposition of* M.

Proof This is immediate from 4.42: for $n(p)$ one may take $\mathrm{Inv}(T,\varphi,\psi)/\mathrm{Inv}(N(p),\varphi,\psi)$ in the notation of that result. ◻

A sharper result than this is true: it will be obtained in the t.t. case in §6 (as 4.60); and in the general case as 9.5. The next result is immediate from 4.41.

Lemma 4.46 *Suppose that* $T'\leqslant T$ *and let* p' *be a* T'*-type over* 0. *If* jp' *(notation as in* 2.33) *is* T*-limited then* p' *is* T'*-limited.* ◻

Lemma 4.47 *Let* p *be any pp-type. Then* $N(p)\equiv N(p)^{\aleph_0}$ *iff* p *is unlimited in the theory of* $\mathrm{Th}(N(p))$.

Proof If $N(p)$ is not elementarily equivalent to its infinite powers, and hence (2.29) if $N(p)\not\equiv N(p)\oplus N(p)$ then it is immediate from the definition that p is $\mathrm{Th}(N(p))$-limited.

If conversely p is $\mathrm{Th}(N(p))$-limited then by 4.42 there are pp formulas φ,ψ with $\mathrm{Inv}(N(p),\varphi,\psi)$ finite, so clearly $N(p)\not\equiv N(p)^{\aleph_0}$. ◻

Note that if $T=T^{\aleph_0}$ then there are no limited types.

Example 2 (for 4.47) Take $R=\mathbb{Z}$: since $\mathbb{Q}\equiv\mathbb{Q}^{\aleph_0}$, $\mathbb{Q}$ is not limited in its own theory or, therefore, in any theory. Since $\mathbb{Z}_2\not\equiv{\mathbb{Z}_2}^{\aleph_0}$, $\mathbb{Z}_2$ is limited in its own theory: similar is $\overline{\mathbb{Z}_{(2)}}$.

Exercise 3 If N is a(n indecomposable) pure-injective all of whose invariants are 1 or ∞ then N is unlimited in every theory.

Given a complete theory T, we may define the **unlimited** part of T to be that complete theory, T_u, which is given by the condition:

$\mathrm{Inv}(T_u,\varphi,\psi)>1$ iff $\mathrm{Inv}(T_u,\varphi,\psi)=\infty$ iff $\mathrm{Inv}(T,\varphi,\psi)=\infty$.

If M is any model of T and if M' is any $|M|^+$-saturated elementary extension of M, then one sees (exercise) that M'/M is a model of T_u. This notion was defined in [Pr80a] and in [BR84a].

Let M be a $|T|^+$-saturated model of T (in particular M is pure-injective). Decompose M according to 4.A14. Note that, by 4.40, the continuous part of M is unlimited. Split the discrete part of M as the pure-injective hull, M_l, of the direct sum of all the T-limited indecomposables, plus a complement for this (the latter is unlimited). Thus, one has $M=M_l\oplus M_u$, where M_u is a model of T_u.

Given any pure-injective model M' of T, one may split M' similarly, to obtain $M'=M'_l\oplus M'_u$. It will follow from the results in Chapter 10 (cf. after 10.24) that M'_l is a factor of M_l (and of course, $M'_u\in\mathcal{P}(T_u)$). Thus, one may say that T itself decomposes as $\mathrm{Th}(M_l)\oplus T_u$ (cf. [Pr80a]) although, of course (consider, for example, $T=\mathrm{Th}(\mathbb{Z}_{(2)})$), it may happen that $\mathrm{Th}(M_l)=T$.

4.6 A structure theory for totally transcendental modules: part II

In this section we improve on 3.14 by describing the indecomposable factors which occur in the decomposition of a totally transcendental module and we relate the decomposition of a module to the decompositions of modules elementarily equivalent to it. In particular the models of a given totally transcendental theory will be classified in a way which slightly extends the stability-theoretic classification of models in terms of realisations of types: the difference is that here the prime model may be described in the same terms as the other models, so the classification is over 0 rather than over the prime model.

While the main results here are self-contained, I have also included some observations and results which relate to ideas introduced later in these notes. The reader unfamiliar with the terms used may safely skip over such points and perhaps return to them later.

The main result of this section, which comes from **[Pr81]**, was also proved independently as part of a more general result in **[Zg84]**. This more general case will be dealt with in §10.4 after the appropriate machinery has been assembled. Although things are simpler in the totally transcendental case and the results are more complete, in that they refer to all models, a surprising amount of what is developed in this section does generalise (see **9.5**, **10.24**).

Recall that, if T is the ubiquitous totally transcendental theory of this section, then any model M of T is a direct sum of indecomposable submodules (**3.14**) in an essentially unique way (**4.A14**) and every pp-n-type over any set of parameters is finitely generated (**3.1**(c)). If p is a pp-type equivalent (modulo T) to the single pp formula φ then we say that p is **(finitely) generated** by φ. Beware that this notion is relative to T; one may emphasise this by saying that p is T-**finitely generated**.

First I introduce some terminology, notions and results which are directly generalised from the (purely algebraic) Σ-injective case. The theory T is implicit, but it is not yet assumed to be totally transcendental.

Suppose that $N \in \mathcal{P}(T)$ and that p is a pp-1-type such that $N(p) \simeq N$. Say that p is (N-)**critical** if $p(N)$ is minimal among the non-zero $\bigwedge\!\!\!\bigwedge$-pp-definable subgroups of N. Extend the definition to types *via* their pp-parts. Observe that this notion does not depend on the over-theory T.

Say that $p \in S_1{}^T(0)$ is T-**critical** if p^+ defines a minimal $\bigwedge\!\!\!\bigwedge$-pp-definable subgroup in a sufficiently saturated model of T. This notion does make real reference to the over-theory T (as opposed to the local notion of "critical").

Recall that, given (pp-)types p, q over 0, one says that they are related and writes $p \sim q$ iff $N(p) \simeq N(q)$. Thus $p \in S_1(0)$ is critical iff its pp-part is maximal in its relatedness class of pp-1-types.

Example 1
(i) $R = \mathbb{Z}$, $N = \mathbb{Z}_{p^\infty}$. Then there is just one $\mathbb{Z}_{p^\infty}$-critical type - namely the type of an element of order p.
(ii) $R = K[X,Y]/\langle X,Y\rangle^2$. As a module over itself, R is totally transcendental. There are $|K|+1$ critical types for R_R (since there is this number of minimal non-zero pp-definable subgroups).
(iii) $R = \mathbb{Z}$, $N = \overline{\mathbb{Z}_{(p)}}$. There is no $\overline{\mathbb{Z}_{(p)}}$-critical type since there is no minimal non-zero $\bigwedge\!\!\!\bigwedge$-pp-definable subgroup. There is, however, a T-critical type (Exercise 2 below) as, indeed, there must be (by **4.49**). This example shows that, in the definition of T-critical, one does need the model to be sufficiently saturated.

Exercise 1 Show that if T is t.t. and if $p \in S_1{}^T(0)$ is T-critical then p has Morley rank 0 or 1.

Exercise 2 Let T be the theory of the abelian group $\overline{\mathbb{Z}_{(p)}}$. Let p be the type of the element $(0,1)$ in the model $\overline{\mathbb{Z}_{(p)}} \oplus \mathbb{Q}$ of T. Then p is a T-critical type.

<u>Lemma 4.48</u> *Let* T *be any complete theory of modules.*
(*a*) *If* $p \in S_1{}^T(0)$ *then* p *is* T*-critical iff* p^+ *is a maximal non-zero pp-type over* 0 *for* T.

(b) *Any T-critical type is critical.*
(c) *Any critical type is irreducible.*

Proof (a) This is clear.

(b) Let $p \in S_1^T(0)$ be T-critical, and let a realise p. Suppose that $b \in N(a)$ lies in some $\mathcal{M}$-pp-definable subgroup of $N(a)$ which is strictly contained in $p^+(N(a))$. Then pp(b) has pp-part strictly containing that of p. Hence $b=0$ and p is critical, as required.

(c) Let p be a critical type and let a realise p. If $N(a)=N_1 \oplus N_2$ and if $a=(a_1,a_2)$ is decomposed accordingly then, by minimality of $p^+(N(a))$, one has pp$(a_i)=p^+$ or $a_i=0$ $(i=1,2)$. By 4.28 it follows that $a_1=0$ and $a_2=0$ - so $a=0$, as required. □

Lemma 4.49 *Let T be any complete theory of modules.*
(a) *Every non-zero type in $S_1^T(0)$ is below a T-critical type.*
(b) *If T is totally transcendental and if p is an irreducible 1-type over 0 then there is a critical 1-type q over 0 related to p and with $q \geqslant p$.*
(c) *If T is totally transcendental then every T-critical type p is isolated by a formula of the form $\varphi(v) \wedge v \neq 0$ where φ is a pp formula generating p (modulo T).*

Proof (a) If p is a 1-type over 0 then, by the compactness theorem, there is some 1-type q over 0 with $q \geqslant p$ (recall that this means that $q^+ \supseteq p^+$) which does not contain the formula "$v=0$" and is maximal such.

(b) Let $X=\{q \in S_1^T(0) : q \sim p$ and $q \geqslant p\}$. Since T is t.t., so is $N(p)$. Therefore (by 3.1) X has a maximal element and such an element will be as required.

(c) Suppose that p is T-critical. Let φ be pp generating p modulo T. If $\psi \in p^-$ then clearly $T \vDash (\varphi \wedge \psi)(v) \rightarrow v=0$, and so $T \vDash \varphi(v) \wedge v \neq 0 \rightarrow \neg\psi(v)$. Thus $\varphi(v)$, together with $v \neq 0$, implies p (by 2.20). □

Part (a) of 4.49 is an improvement on the algebraic situation (cf. [LM73], [Gol75]). It generalises the fact that if a hereditary torsionfree class (in the sense of §15.2) is of finite type (equivalently, 15.9, is elementary) then every right ideal is below a critical right ideal for this class.

A type is said to be **isolated** if it is equivalent to a single formula (the notion of a pp-type being finitely generated is the positive version of this). The terminology arises because $p \in S^T(A)$ is isolated in this sense iff it is an isolated point of the topological space $S^T(A)$ (see §1.1). Observe that the notion makes sense only for types in finitely many free variables.

Example 2
(i) A type need not even be related to a T-critical type (take T to be the theory of the abelian group $\mathbb{Z}_4^{(\aleph_0)} \oplus \mathbb{Z}_8^{(\aleph_0)}$). In T^{eq} the situation improves somewhat (cf. §10.T).
(ii) Let T be the (superstable but not t.t.) theory of the abelian group $\mathbb{Z}_{(p)}$ and let p_n be the type of an element divisible by p^n but not by p^{n+1}. Then p_n is isolated by the formula $p^n | v \wedge p^{n+1} \nmid v$. Yet p_n is not even related to a critical type (Ex 1(iii) above). Again, in T^{eq}, the situation improves.

Lemma 4.50 *Suppose that T is totally transcendental. Let $A \subseteq \tilde{M}$ and let $\bar{b}$ be a finite tuple in (a copy of) $N(A)$. Then tp$(\bar{b}/A)$ is isolated by a single pp formula.*

Proof This is immediate by 4.17 and 3.1. □

Example 3 $T=\mathrm{Th}(\mathbb{Z}_{2^\infty})$. Take an element a with hull $\mathbb{Q}$ and let b be a non-zero element of some chosen copy of the hull of a: say $bm=an$ (m,n non-zero integers). Then $vm=an$ isolates the type of b over a.

The classification which I give here of the models of a t.t. theory is somewhat more detailed then one might expect from consideration of the classification theorem for non-multidimensional ω-stable theories (see [Pi83a]). There, one must be content with classifying models up to isomorphism over the prime model. Here, it will be possible to describe the prime model in the same terms as the other models. To do this, one considers an algebraic notion of independence which is rather finer than the model-theoretic one (*viz*. non-forking - see §5.1). Consider the next example.

Example 4 Consider the theory of the abelian group $\mathbb{Z}_4 \oplus \mathbb{Z}_2^{(\aleph_0)}$ - an ω-stable theory. Let $a=(1,0) \in \mathbb{Z}_4 \oplus \mathbb{Z}_2^{(\aleph_0)}$ and set p to be the type of a.

Now, p is not an algebraic type since, given any element b of order 2, the sum $a+b$ realises p (and there are infinitely many such "b"). It follows that given any cardinal κ, a model may be found containing at least κ independent realisations of p: independent in the model-theoretic sense.

Such realisations will not however be independent in a stronger sense of the term. For if a_1 and a_2 are distinct realisations of p then a_1-a_2 is an element of order 2. In particular there cannot be a direct summand (of any model) of the form $N_1 \oplus N_2$ with $a_i \in N_i$ $(i=1,2)$. In this section I will use this stronger notion of direct-sum independence. (Section 5.3 explores the relationship between these two notions of independence.)

Exercise 3 In the example above, the limited type p is at least related to an algebraic type. Show that this is not a general phenomenon, even in the t.t. case.

Let $\bar{a},\bar{b}$ be in $\tilde{M}$. Say that $\bar{a}$ and $\bar{b}$ are **direct-sum independent** (**ds-independent**) if there is $N \oplus N'$ pure in $\tilde{M}$ with $\bar{a}$ in N and $\bar{b}$ in N'. In the example above it is clear that no two realisations of p can be ds-independent. This is so because T forces there to be at most one (in fact exactly one) copy of $\mathbb{Z}_4$ appearing in the direct sum decomposition of any model of T (consider the invariant $\mathrm{Inv}(T, v=v, v2=0)$).

<u>Lemma 4.51</u> *Let* $p \in S^T(0)$. *Suppose that* $\bar{a},\bar{b}$ *are ds-independent realisations of* p. *Then* $\bar{a}-\bar{b}$ *realises* p.

Proof Take $N \oplus N'$ pure in $\tilde{M}$ with $\bar{a}$ in N and $\bar{b}$ in N'. Then $\mathrm{pp}(\bar{a}-\bar{b}) = \mathrm{pp}((\bar{a},-\bar{b})) = \mathrm{pp}(\bar{a}) \cap \mathrm{pp}(\bar{b}) = p^+ \cap p^+ = p^+$ (by 2.10). □

For purposes of the discussion which follows, say that a set of tuples is (p-) equiptype if all its elements have the same type (p). Although there will be no upper bound on the cardinality of such sets unless p is algebraic there will be some constraint if p is limited. If X is a p-equitype set, say that X is **weakly independent** if for any distinct $\bar{a},\bar{a}'$ in X one has $\mathrm{tp}(\bar{a}-\bar{a}')=p$ (cf. [Gar80a; Defn5], also [Mrt75; p331]). In particular direct-sum independence implies weak independence (by 4.51), but there is one more component to weak independence, and I examine this next. Perhaps I should point out that this notion is not comparable with the model-theoretic notion of independence (see §5.1); the "weak" refers to comparison with ds-independence.

If there is an upper bound, necessarily finite (exercise!), on the cardinality of weakly independent sets of realisations of p in models of T, then set $\mathrm{wkdim}(p,T)=k$. Otherwise set $\mathrm{wkdim}(p,T)=\infty$. For $N \in \mathcal{I}(T)$ define $\mathrm{wkdim}(p,N)$ (finite or an infinite cardinal) similarly, restricting the realisations to N. We examine this weak dimension now. When considering those types p with $\mathrm{wkdim}(p,T)$ finite, the next lemma allows us to restrict attention to limited types. The lemma is a direct consequence of the definition of unlimited and 4.51.

<u>Lemma 4.52</u> *If* $p \in S^T(0)$ *is unlimited then* $\mathrm{wkdim}(p, T)=\infty$. □

The converse, at least for the t.t. case will follow from **4.53**.

From now on in this section assume that T is totally transcendental.

Suppose that p is irreducible. Since $N(p)$ is the prime model of its own theory (**3.9**; so we are already using the assumption on T) it follows that if p_0 is that type over 0 in the theory of $N(p)$ with $p^+ = p_0{}^+$ then p_0 is isolated (prime models, where they exist, realise precisely the isolated types). There is a pp formula φ such that p_0 is equivalent modulo $\mathrm{Th}(N(p))$ to φ (**3.1**). Since p_0 is isolated there are pp formulas $\psi_1,\ldots,\psi_n$ with $\mathrm{Th}(N(p)) \vdash p_0 \leftrightarrow \varphi \wedge \bigwedge_1^n \neg\psi_i$. It is an easy exercise using **4.29** (see **9.19**) to show that one may take $n=1$: say p_0 is isolated by $\varphi\wedge\neg\psi$. Note that ψ is the unique maximal pp formula below φ in $N(p)$. Suppose now that p is T-limited so, by **4.46**, p_0 is $\mathrm{Th}(N(p))$-limited; hence, by **4.41**, $|\varphi(N(p))/\psi(N(p))|$ is finite. The next result shows that this value equals $\mathrm{wkdim}(p,N(p))$ and that this integer is actually an invariant of $N(p)$.

Proposition 4.53 [Pr81; 5.10] *Suppose that $N=N(p)$ is an indecomposable totally transcendental module. Let $\varphi\wedge\neg\psi$ isolate p (in $\mathrm{Th}(N)$) with φ,ψ pp and with φ equivalent to p^+ (in $\mathrm{Th}(N)$). Let D_N be the division ring $\mathrm{End}N/J(\mathrm{End}N)$. Then $\varphi(N)/\psi(N)$ has the structure of a 1-dimensional vectorspace over D_N and $\mathrm{wkdim}(p,N(p)) = |\varphi(N)/\psi(N)| = |D_N| = d_N$ (say).*

Proof If $\bar a$ and $\bar a'$ realise p in N then $\bar a-\bar a'$ realises p^+, so satisfies φ. Therefore, since ψ is the unique maximal pp-definable subgroup below φ, one has $\mathrm{pp}(\bar a-\bar a')\neq p^+$ iff $\psi(\bar a-\bar a')$ holds, and this is so iff $\bar a$ and $\bar a'$ lie in the same coset of $\psi(N)$. Therefore $\mathrm{wkdim}(p,N)=[\varphi(N):\psi(N)]$.

If $\bar a$ realises p in N then, by **2.8**, $p^+(N)=S\bar a$ where $S=\mathrm{End}N$. So define a left S-linear map $\theta: S\longrightarrow(\varphi(N)/\psi(N))$ by $\theta f=f\bar a+\psi(N)$. Then $f\in\ker\theta$ iff $f\bar a\in\psi(N)$, iff $\mathrm{pp}(f\bar a)>p$, and this happens (by **4.27**) iff $f\in JS$. Thus (by **2.8**, θ is onto), $\varphi(N)/\psi(N)$ is isomorphic to S/JS under the left action of S, so carries the induced structure of a 1-dimensional D_N-space. So the result follows. □

Corollary 4.54 *Suppose that $N\simeq N(p)\simeq N(q)$ is indecomposable and totally transcendental. Then $\mathrm{wkdim}(p,N)=\mathrm{wkdim}(q,N)=d_N$.* □

Corollary 4.55 *Suppose that N is an indecomposable totally transcendental module and let φ,ψ be pp formulas with $\varphi(N)>\psi(N)$ and φ/ψ an N-minimal pair. Then $[\varphi(N):\psi(N)]=d_N$.*

Proof Choose any $\bar a\in\varphi(N)\setminus\psi(N)$ and let φ' generate the pp-type of $\bar a$. By modularity of the lattice of pp-definable subgroups (**2.2**) it follows that $(\varphi\wedge\varphi'=\varphi')/\psi\wedge\varphi'$ is an N-minimal pair and so, by **4.29**, $\varphi\wedge\varphi'/\psi\wedge\varphi'$ isolates the type of $\bar a$. By **4.53** it follows that $[(\varphi\wedge\varphi')(N)/(\psi\wedge\varphi')(N)]=d_N$. Since, using the fact that φ/ψ is a minimal pair, one has the isomorphism of groups $\varphi/\psi\simeq(\varphi\wedge\varphi')/(\psi\wedge\varphi')$, the result follows. □

Corollary 4.56 *Suppose that N is an indecomposable totally transcendental module and let p be a (pp-)type with $N\simeq N(p)$. Then:*

$d_N=\min\{\mathrm{Inv}(N,\varphi,\psi) : \mathrm{Inv}(N,\varphi,\psi)>1\}$
$\quad=\min\{\mathrm{Inv}(N,\varphi,\psi) : \varphi/\psi\in p\}$.

Proof This is immediate by **4.53** and **4.55**. □

Corollary 4.57 *If N is an indecomposable totally transcendental module and if $\varphi(N)$ is any minimal pp-definable subgroup of N then $|\varphi(N)|=d_N$.* □

Although not obvious from the definition, it follows from **4.53** that the cardinal d_N is independent not just of the particular type but also of the number of free variables. In particular, in 4.56 one may take φ and ψ to have $n(\in\omega)$ free variables and still obtain the same minimal index of pp-definable subgroups.

These results will be generalised in 9.6.

Example 5 Consider: (i) $R=\mathbb{Z}$, $N=\mathbb{Z}_{p^\infty}$; (ii) $R=\mathbb{Z}$, $N=\mathbb{Z}_{p^n}$; (iii) $R=K[X,Y]/\langle X,Y\rangle^2$ (K a field), $N=R$. Then 4.56 shows that the uniformity, in each example, of minimal non-zero indices of pairs of pp-definable subgroups is no accident.

Consider also (iv) $R=\mathbb{Z}$, $N=\overline{\mathbb{Z}_{(p)}}$. Although this is not t.t., it is clear that the proof of 4.56 works for this example. In fact one has the following.

Exercise 4 Suppose that N is an indecomposable pure-injective such that every pp-type realised in N is finitely generated modulo $\mathrm{Th}(N)$ (Piron's example 7.2/6 shows that even a superstable indecomposable pure-injective may fail to satisfy this condition). Then the above analysis and (marginally modified) conclusions go through by the same arguments. Observe that if $T=\mathrm{Th}(\mathbb{Z}_{(p)})$ then each $N\in\mathcal{I}(T)$ satisfies this condition.

The next result is essentially [Gar80a; Lemma19] (see also various arguments towards the end of [Gar80]). It says that the action of each element of the division ring associated to an indecomposable t.t. module is (pp-)definable (also cf. §9.2).

<u>Theorem 4.58</u> *Let N be an indecomposable totally transcendental module. Let a be a non-zero element of N; let $\varphi>\psi$ be pp formulas such that $\varphi\wedge\neg\psi$ isolates the type, p, of a in the theory of N. Then, using a as a parameter, each element of the division ring D_N is interpretable in N.*

Proof Let $f\in\mathrm{Aut}(N)$ and set $b=fa$. By 4.11, there is a pp formula $\theta(v,w)$ with $\theta(a,b)\wedge\neg\theta(a,0)$: it may be assumed (replace $\theta(v,w)$ by $\theta(v,w)\wedge\varphi(v)\wedge\varphi(w)$ if necessary) that $\exists w\,\theta(v,w)$ is equivalent to $\varphi(v)$ and that $\exists v\,\theta(v,w)$ is equivalent to $\varphi(w)$.

I claim that the relation $\theta(c,d)$ induces, by $c+\psi(N)\mapsto d+\psi(N)$, the isomorphism $f+J(S)$ from the factor group $\varphi(N)/\psi(N)$ to itself. For, by choice of θ, if $\theta(c,d)$ holds then so does $\varphi(d)$. Also, if $\theta(c,d)$ holds and d is in $\psi(N)$ then so is c: for otherwise, since c realises p, there is d' realising p and with $\theta(c,d')$. Then we have $\theta(0,d'-d)$. Now $d'-d$ satisfies $\varphi\wedge\neg\psi$: that is, satisfies p. But then, since b and $d'-d$ have the same type, we obtain $\theta(0,b)$ - contradiction. This argument shows (by symmetry) both that we have a well-defined map from $\varphi(N)/\psi(N)$ to itself and that this map is monic. Since it has the same action, modulo $J(S)$, as does f, it must equal f modulo $J(S)$, as required. □

The essential content of the result above is set in a more general context in §9.2. We obtain the following corollary, which will be used in §16.1.

<u>Corollary 4.59</u> [Gar80; Lemma9] *Suppose that R is a commutative ring and that N is an indecomposable totally transcendental module. Then the minimal non-zero pp-definable subgroups of N are all isomorphic as R-modules.*

Proof Let $\varphi(N)$ and $\varphi'(N)$ be distinct minimal non-zero pp-definable subgroups of N. Take non-zero elements a in $\varphi(N)$ and b in $\varphi'(N)$. Let $\theta(v,w)$ be a pp formula linking a and b and chosen as in the proof above. Essentially by that argument (exercise or see 9.7), θ induces a group isomorphism from $\varphi(N)$ to $\varphi'(N)$. Since every element, r, of R is central, one has that $\theta(c,d)$ implies $\theta(cr,dr)$; so this map is an R-isomorphism. □

Now I want to show that if N is T-limited (T t.t.) and indecomposable then N occurs exactly $\mathrm{wkdim}(p,T)/d_N$ times in the direct sum decomposition of any model, where p is any type such that $N\simeq N(p)$. The only point which prohibits the conclusion being immediate from 4.52 and 4.53 is the possibility that there are non-isomorphic indecomposables $N,N'\in\mathcal{I}(T)$ and pp formulas φ,ψ such that φ/ψ is a minimal pair for T and such that $\varphi(N)>\psi(N)$ and

$\varphi(N')>\psi(N')$. We will see later (9.3) that indecomposable pure-injectives sharing a minimal pair must be isomorphic. In the t.t. case one may proceed directly.

For any pure-injective module M (t.t. or not) and any indecomposable pure-injective N, define the **multiplicity** of N in M, to be the number of copies of N which occur in a direct sum decomposition, in the sense of 4.A14, of M (by 4.A7, this is well-defined).

Proposition 4.60 [Pr81; 5.14] *Suppose that T is totally transcendental and let $p\in S^T(0)$ be irreducible and T-limited (so isolated). Then the multiplicity of $N(p)$ in any model of T is $n(p)=\mathrm{wkdim}(p,T)/d_N$. Moreover $n(p)=\mathrm{Inv}(T,\varphi,\psi)/\mathrm{Inv}(N(p),\varphi,\psi)$ where φ/ψ is a T-minimal pair such that $\varphi(N(p))>\psi(N(p))$.*

Proof Take $\varphi\wedge\neg\psi$ isolating p in T (4.44 and proof of 4.55). Note that by 4.53 $d_N=\mathrm{wkdim}(p,N(p))=\mathrm{Inv}(N(p),\varphi,\psi)$. Moreover, by 4.44, $\mathrm{Inv}(T,\varphi,\psi)$ is finite.

Let M be a model of T and set $M=N(p)^{(\kappa)}\oplus N'$ where N' contains no copy of $N(p)$. Then $\mathrm{Inv}(M,\varphi,\psi)=\mathrm{Inv}(N(p)^{(\kappa)},\varphi,\psi)\times\mathrm{Inv}(N',\varphi,\psi)$: this equals $\mathrm{Inv}(N(p),\varphi,\psi)^\kappa$ since $\varphi\wedge\neg\psi$ isolates p (and κ is finite); this in turn is $d_N{}^\kappa$. Thus, $\mathrm{Inv}(T,\varphi,\psi)=d_N{}^\kappa$ where $\kappa=n(p)$ is the number of appearances of $N(p)$ in the decomposition of any model (being independent of choice of M), as required. □

Now we have enough to give a complete description of all possible models of a t.t. theory in terms of their direct-sum decompositions. Let T be t.t. and, as before, let $\mathcal{I}(T)$ contain exactly one copy of each indecomposable direct summand of the monster model.

Let M be any model of T. By 3.14, M has a decomposition in an essentially unique way as a direct sum of indecomposable submodules: say $M\simeq\bigoplus\{N^{(\delta N)}:N\in\mathcal{I}(T)\}$ for suitable cardinals δN. Following a suggestion of Simmons, I present the restrictions on the number of times each indecomposable can occur as restrictions on the **decomposition function** of M: the map $\delta_M:\mathcal{I}(T)\longrightarrow\mathrm{Card}$ (the class of all cardinals) given by $\delta_M(N)=\delta N$ as above – the multiplicity of N in M.

Conversely, any function $\delta:\mathcal{I}(T)\longrightarrow\mathrm{Card}$ determines a module $M_\delta=\bigoplus\{N^{(\delta N)}:N\in\mathcal{I}(T)\}$. So the question of describing all the models of T may be phrased as: for which "decomposition functions", δ, is M_δ a model of T? By $\delta_M p$ I will mean $\delta_M(N(p))$.

Among those irreducible types whose hulls appear in the decomposition of some model of T we should first distinguish between those which are isolated (those whose hulls must appear in every model) and those which are non-isolated. Since T is t.t., it follows easily that if p is isolated and irreducible and if $q\sim p$ then q is isolated.

Within the isolated case there are essentially three possibilities, described in the next result.

Proposition 4.61 *Suppose that T is totally transcendental. Let $p\in S^T(0)$ be irreducible and isolated – say by $\varphi\wedge\neg\psi$ with φ,ψ pp (cf. before 4.53). Then exactly one of the following possibilities obtains.*

(a) p is T-unlimited: this occurs iff $\mathrm{Inv}(T,\varphi,\psi)=\infty$. In this case either:

(i) $N(p)\equiv N(p)^{\aleph_0}$ and then for any model, M, of T, $\delta_M p\geq 1$, and in the prime model M_0 of T, $\delta_M p=1$; or

(ii) $N(p)\not\equiv N(p)^{\aleph_0}$ and then for any model, M, of T, $\delta_M p\geq\aleph_0$, and in the prime model M_0 of T, $\delta_{M_0}p=\aleph_0$.

Case (i) occurs iff $\mathrm{Inv}(N(p),\varphi,\psi)$ is infinite.

(b) p is T-limited: this occurs if $\mathrm{Inv}(T,\varphi,\psi)$ is finite and then for any model, M, of T, $\delta_{M_0}p=n(p)$ (as in 4.60).

Proof This now follows directly from **4.42**, **4.60** and the fact that the module $M_0 = \oplus\{N^{(\kappa)} : N \simeq N(q)$ for some isolated irreducible $q \in S^T(0)$, and $\kappa = 1$ in case (a)(i), $\kappa = \aleph_0$ in case (a)(ii) and $\kappa = n(q)$ in case (b)$\}$ is the prime model of T.

To see this, let M be any model of T and let q be irreducible and isolated by $\varphi \wedge \neg\psi$ say. By **3.14** and **4.60** $M = N(q)^{(\kappa)} \oplus N'$ for some N' containing no realisation of q and hence with $\mathrm{Inv}(N', \varphi, \psi) = 1$. It should be clear that κ is restricted as described in the proposition (consider $\mathrm{Inv}(-, \varphi, \psi)$) and that if M is the prime model then κ has the minimal possible value. Since (**3.14**) every model is determined by its indecomposable direct summands the only point which remains to be checked is that M_0 really is a model of T (i.e., that realising all *irreducible* isolated types ensures that all isolated types are realised). But this is again a consequence of **3.14**: clearly M_0 is a direct summand of the prime model, it already has enough copies of each "isolated" indecomposable pure-injective, and the prime model realises no non-isolated (irreducible) type. □

<u>Corollary 4.62</u> [**Pr81**; 5.15], [**Zg84**; 9.1] *Let T be totally transcendental. Then the prime model M_0 of T is given by the decomposition function δ_0 where:*
$\delta_0 N = 1$ *if N is isolated (T-unlimited) and $N(p) \equiv N(p)^{\aleph_0}$* (IUI)
$= \aleph_0$ *if N is isolated T-unlimited and $N(p) \not\equiv N(p)^{\aleph_0}$* (IUF)
$= n(N)$ *if N is (isolated and) limited* (IL)
$= 0$ *if N is non-isolated* (N). □

Since to the prime model may be added any unlimited $N \in \mathcal{I}(T)$, the main theorem now follows.

<u>Theorem 4.63</u> [**Pr81**; 5.15], [**Zg84**; 9.2] *Suppose that T is totally transcendental, and let $\mathcal{I}(T)$ contain one copy of each indecomposable direct summand of models of T. Then the models of T are given by those decomposition functions $\delta : \mathcal{I}(T) \longrightarrow \mathrm{Card}$ (by $\delta \mapsto M_\delta = \oplus\{N^{(\delta N)} : N \in \mathcal{I}(T)\}$) which satisfy the following conditions (and, by uniqueness of decomposition, $\delta \neq \delta'$ implies $M_\delta \not\simeq M_{\delta'}$):*
$\delta N \geqslant 1$ *if N is isolated (T-unlimited) and $N \equiv N^{\aleph_0}$;*
$\geqslant \aleph_0$ *if N is isolated, T-unlimited and $N \not\equiv N^{\aleph_0}$;*
$= n(N)$ *(finite, determined as in* **4.60**) *if N is limited (isolated and) $N \not\equiv N^{\aleph_0}$;*
$\geqslant 0$ *if N is non-isolated.* □

For Σ-injective modules, **4.62** and **4.63** are [**Pr82**; Thm 18]: the consequences for commutative noetherian rings and for general noetherian rings are spelled out after that theorem.

<u>Corollary 4.64</u> [**Gar80**; Thm 6] (see §3.2) *Let T be an ω-stable theory of modules. Then the number of countable models of T is either* 1, $\aleph_0$ *or* $2^{\aleph_0}$. □

Thus Vaught's Conjecture for ω-stable modules follows directly from the structure theorem (it is known to hold for any ω-stable theory [**HMS84**], also see [**BoLa83**], but that is a much deeper result!). Vaught's Conjecture is that, for any countable theory T, the number $n(\aleph_0, T)$ of non-isomorphic models of T of cardinality $\aleph_0$ is either countable or $2^{\aleph_0}$. The conjecture is important for us (for its general significance, see [**Las85a**]) in that one expects that, in order to be able to settle it for a particular class of theories, one will likely have to develop a reasonable structure theory for the models. Here I have followed this course, but it was pointed out in §3.2 that a direct proof due to Garavaglia [**Gar80**; Thm **6**] may be given, using uniqueness of decomposition (**3.14**) prime and universal models, and some idea of unlimited type. In another case (§7.2) we will have a decent structure theory for the models which, nevertheless, is not fine enough for Vaught's Conjecture (in that case, Vaught's Conjecture was proved by Buechler working from a much more general standpoint).

Notice also that **4.64** gives that the number of countable models, if finite, is 1. In fact this is true for arbitrary theories of modules ((**6.32**) and beyond this see **[Pi84a]**).

Exercise 5 Describe the possible uncountable spectrum functions $(\lambda \mapsto n(\lambda,T)$ for $\lambda > \aleph_0)$ for ω-stable theories of modules. Give an example for each case and show that your list of possibilities is exhaustive (cf. §7.1). Note that even if T is uncountable and t.t., all that is needed to count the number of models in a given infinite cardinality is the function $N \mapsto |N|$ for $N \in \mathcal{I}(T)$.

Exercise 6 Theories of abelian groups illustrate various cases in **4.63**.

(i) $T = \mathrm{Th}(\mathbb{Q})$. Here only the case (IUI) occurs, and the models are $\mathbb{Q}^{(\kappa)}$ $\kappa \geqslant 1$.

(ii) $T = \mathrm{Th}(\mathbb{Z}_2^{(\aleph_0)})$. Here only the case (IUF) occurs and the models are $\mathbb{Z}_2^{(\kappa)}$ $\kappa \geqslant \aleph_0$.

(iii) $T = \mathrm{Th}(\mathbb{Z}_2^{\,n})$. The only case ocurring is (IL) and the only model is $\mathbb{Z}_2^{\,n}$.

(iv) $T = \mathrm{Th}(\mathbb{Z}_2^{(\aleph_0)} \oplus \mathbb{Q})$. The cases which occur are (IUI) and (IUF), and the models are $\mathbb{Z}_2^{(\kappa)} \oplus \mathbb{Q}^{(\lambda)}$ where $\kappa \geqslant \aleph_0$ and $\lambda \geqslant 1$.

Adding on, say, one copy of $\mathbb{Z}_3$, one also has the case (IL).

(v) $T = \mathrm{Th}(\mathbb{Z}_{2^\infty}^{(\aleph_0)})$. Here one has the cases (IUF) and (N), the models being $\mathbb{Z}_{2^\infty}^{(\kappa)} \oplus \mathbb{Q}^{(\lambda)}$ with $\kappa \geqslant \aleph_0$ and $\lambda \geqslant 0$.

A copy of $\mathbb{Z}_3$ added on also gives (IL).

(vi) $T = \mathrm{Th}(\mathbb{Z}_{2^\infty})$. Here one has (IL) and (N). The models are $\mathbb{Z}_{2^\infty} \oplus \mathbb{Q}^{(\kappa)}$ with $\kappa \geqslant 0$.

Observe that since there is only one indecomposable pure-injective $\mathbb{Z}$-module, N, with $N \equiv N^{\aleph_0}$ (namely $\mathbb{Q}$), and since $\mathbb{Q}$ is the only possibility for a non-isolated indecomposable in a t.t. theory of $\mathbb{Z}$-modules, one must look over other rings to combine (IUI) and (N).

(vii) $R = \mathbb{Z} \times \mathbb{Z}$. Set $e_1 = (1,0)$ and $e_2 = (0,1)$ in R and let $T = \mathrm{Th}(e_1\mathbb{Q} \oplus e_2\mathbb{Z}_{2^\infty})$. Then one has the cases (IUI), (IL) and (N), and the models are $(e_1\mathbb{Q})^{(\kappa)} \oplus (e_2\mathbb{Z}_{2^\infty}) \oplus (e_2\mathbb{Q})^{(\lambda)}$ where $\kappa \geqslant 1$ and $\lambda \geqslant 0$.

For (t.t.) examples with infinitely many non-isolated indecomposables take, for example, the theory of a large enough injective module over $K[X,Y]$ (K a field) - the non-maximal primes give non-isolated injectives. Alternatively, take the theory of the pre-injective modules over the path algebra of an extended Dynkin quiver over an infinite field (see **[Len83]**, **[Pr85a]**; cf. §13.3).

From the classification theorem **4.63** various consequences concerning categoricity, saturation, as well as effective listings of theories, are immediate. Note that the dimension of T, $\mu(T)$ (see §6.4), is just the cardinality of the subset of $\mathcal{I}(T)$: $\mathcal{I}_*(T) = \{N \in \mathcal{I}(T) : N$ falls into one of the classes (IUI), (IUF) or (N)$\}$. For any complete theory of modules T, we will define $\mathcal{I}_*(T) = \{N \in \mathcal{I}(T) : N$ is T-unlimited$\} = \mathcal{I}(T_U)$.

<u>Corollary 4.65</u> *Suppose that T is totally transcendental and let $\kappa \geqslant |T|$. Then the κ-saturated model of T of cardinality κ is given by $\delta N = \kappa$ in all cases except* (IL), *where $\delta N = n(N)$.* ◻

The saturated abelian groups (not just the t.t. ones) are described in [EF72; §3].

Many of the ideas and results of this section (which is based on **[Pr81**; §5**]**) are seen in more general contexts in later chapters (9 and 10) of these notes based on **[Zg84]** (which was independent of **[Pr81]**). A number of the key ideas may be found more or less explicitly in [Gar80a].

4.C Categoricity

Let T be a countable theory: T is said to be $\aleph_0$-categorical if there is, up to isomorphism, only one countable model of T; T is $\aleph_1$-categorical if there is, up to isomorphism, only one model of T of cardinality $\aleph_1$ equivalently ([Mor65]), if for any uncountable cardinal λ there is just one model of T of cardinality λ; T is totally categorical is T is both $\aleph_0$-categorical and $\aleph_1$-categorical.

A countable complete theory is $\aleph_0$-categorical iff, for each $n \in \omega$, there are only finitely many n-types over $\emptyset$ and this is so iff, for each $n \in \omega$, there are, up to logical equivalence, only finitely many formulas in the free variables $v_1, \ldots, v_n$ ("Ryll-Nardzewski Theorem" - see [Poi85; 10.11]).

If the complete theory T is $\aleph_1$-categorical then T is totally transcendental ([Mor65]): indeed, the notion of a t.t. theory and the associated (Morley) rank was introduced by Morley when he proved the equivalence mentioned in the definition of $\aleph_1$-categoricity.

Until further notice, R is a countable ring. What follows is rather immediate from 4.63 and is also spelled out by Ziegler [Zg84; 10.6].

$\aleph_0$-categoricity Suppose that T is $\aleph_0$-categorical. Since $S_1{}^T(0)$ is finite, it follows that T is totally transcendental (3.1) and of finite Morley rank (5.13 and 5.18). (It is an open question whether, in general, a stable $\aleph_0$-categorical theory is totally transcendental.) Also, since $S_1{}^T(0)$ is finite, $\mathcal{I}(T)$ is finite: say $\mathcal{I}(T) = \{N_1, \ldots, N_k\}$. Let M_0 be the prime model of T.

Referring back to the notation introduced in 4.62 in connection with the description of the models of an arbitrary t.t. theory, we may ask what kinds of indecomposable summand can occur.

(i) There can be no non-isolated point N of $\mathcal{I}(T)$ (type (N)). For otherwise, we have $M_0 \not\cong M_0 \oplus N$ both countable models, but non-isomorphic (by 4.A14 and 4.63), so contradicting $\aleph_0$-categoricity. So $\mathcal{I}(T)$ carries the discrete topology. (Alternatively, since the Morley rank is finite, each irreducible type has a minimal pair, so its hull is isolated in $\mathcal{I}(T)$ by 9.23.)

(ii) There is no indecomposable N which satisfies $N \equiv N^{\aleph_0}$ (type (IUI)). For again one would have two non-isomorphic countable models: M_0 and $M_0 \oplus N^{\aleph_0}$.

(iii) So $\mathcal{I}(T) = \{N_1, \ldots, N_t, N_{t+1}, \ldots, N_k\}$ where each N_i $(1 \leqslant i \leqslant t)$ occurs a fixed, finite number of times, κ_i, in every model and each of $N_{t+1}, \ldots, N_k$ occurs at least $\aleph_0$ times in the decomposition of every model.

(iv) Since each member N of $\mathcal{I}(T)$ satisfies $N \not\equiv N^{\aleph_0}$, by 4.55, each minimal index (>1) of one pp-definable subgroup in another is finite. But $\aleph_0$-categoricity implies that there are only finitely many inequivalent formulas (in particular, only finitely many pp-definable subgroups). Hence each $N \in \mathcal{I}(T)$ is a finite module.

(v) Therefore, each model of T has the form $M = N' \oplus N_{t+1}{}^{\kappa_{t+1}} \oplus \ldots \oplus N_k{}^{\kappa_k}$, where N' is a finite module, each κ_i is infinite and each N_j is a finite indecomposable (pure-injective) module.

(vi) Conversely, any module of the above form is $\aleph_0$-categorical (exercise).

(vii) It may as well be assumed that R is a finite ring since, if M is $\aleph_0$-categorical then $R/\mathrm{ann}M$ is finite. For $\mathrm{ann}M = \mathrm{ann}(N' \oplus N_{t+1} \oplus \ldots \oplus N_t)$ is the annihilator of a finite module, and a finite module has only finitely many endomorphisms!

The conclusions (v)-(vii) are due to Baur [Bau75; Lemma 2, Thm 2]. Also, see [EF72] for the abelian groups case.

Exercise 1 Show that an abelian group is $\aleph_0$-categorical iff it is of bounded exponent.

Exercise 2 If a module is $\aleph_0$-categorical, need its injective hull be $\aleph_0$-categorical?

Example 1 The following are non-$\aleph_0$-categorical, so not all types in finitely many variables are isolated but, in the first example, all 1-types over 0 are isolated and, in the second example, all irreducible 1-types are isolated even though $S_1(0)$ is infinite.
(i) $T = \mathrm{Th}(\mathbb{Q})$: one has to look at 2-types in order to see that this is not $\aleph_0$-categorical.
(ii) $T = \mathrm{Th}(R)$ where R is the ring $\mathbb{Q}[X,Y]/\langle X,Y\rangle^2$, considered as a module over itself. There is just one non-isolated 1-type, and it is reducible (cf. Ex 6.4/3 below).

Exercise 3 [Gar80; §2] In general, for $\aleph_0$-categoricity, one needs to check finiteness of $S_n{}^T(0)$ for all $n \in \omega$. For modules, $S_2{}^T(0)$ being finite is enough.

Total categoricity The complete theory T is totally categorical iff it satisfies the conditions above and, furthermore, $k = t+1$ – that is, if every model has the form $N' \oplus N^\kappa$ where N and N' are finite, N is indecomposable and $\kappa \geq \aleph_0$. For, if there exist two indecomposables which can occur infinitely often, then we get non-isomorphic models of cardinality $\aleph_1$ of the form $X \oplus N_{k-1}{}^{(\aleph_0)} \oplus N_k{}^{(\aleph_1)}$ and $X \oplus N_{k-1}{}^{(\aleph_1)} \oplus N_k{}^{(\aleph_0)}$.

$\aleph_1$-categoricity The characterisation of $\aleph_1$-categorical abelian groups is [Mac70; Thm 2]. As already mentioned T $\aleph_1$-categorical implies that T is totally transcendental. As in the argument above, there can be only one indecomposable with infinite multiplicity in models of T (and this condition suffices for a t.t theory to be $\aleph_1$-categorical).

There can be infinitely many indecomposables (but there can be only one cluster point in $\mathcal{I}(T)$). Consider, for example, the theory of $\bigoplus\{\mathbb{Z}_{p^\infty} : p \text{ prime}\}$: the typical model just adds $\kappa(\geq 0)$ copies of $\mathbb{Q}$.

Uncountable rings If R is uncountable then the situation regarding categoricity changes.

For example, let R be the ring of p-adic integers, $\overline{\mathbb{Z}(p)}$ and consider this a module over itself. This is not $|R| = 2^{\aleph_0}$-categorical, since R and $R \oplus Q$ are elementarily equivalent as R-modules (where Q is the field of quotients of $\overline{\mathbb{Z}(p)}$). But it is λ-categorical for all $\lambda > 2^{\aleph_0}$.

If R is arbitrary and T is λ-categorical for some $\lambda > |T|^{|T|}$ (note that T has a discrete pure-injective model of cardinality $\geq |R|^{|R|}$ (**6.31**)) then T is "unidimensional" in the sense that there is just one unlimited indecomposable pure-injective and there are no continuous pure-injectives. So, by **7.9**, T is superstable (the above example shows that T need not be t.t.).

It would be appropriate at this point to mention the paper [BCM79], which has given rise to a great deal of further work, since the authors do use (and, in an appendix, extend) Baur's results from [Bau75]. The main results of [BCM79] are: any $\aleph_0$-categorical stable group is nilpotent by finite; any $\aleph_0$-categorical t.t. group is abelian by finite and, this uses the work on modules, they give a complete description of the totally categorical groups [BCM79; Thm 64].

Exercise 4 In [Kre82], [Kre84] Kremer considers a notion (introduced by Belegradek) – "almost categoricity" – for modules. It is shown in [Kre82] (see [Kre84; Fact 4]) that a module M is almost categorical iff $\mathcal{I}(M)$ is finite and if every point in it is isolated, with an isolating neighbourhood of the form $(\varphi(v)/v=0)$ (i.e., every indecomposable realises a $\mathrm{Th}(M)$-critical type). It follows by **6.28** below that M is totally transcendental of finite Morley rank (this uses only the fact that $\mathcal{I}(M)$ contains no non-isolated points). Kremer shows the following [Kre84; Thm 1, Thm 2, Thm 3]:
(i) Every module is almost categorical iff R is semisimple artinian.
(ii) Every injective module is almost categorical iff R is right artinian.
(iii) Every projective module is almost categorical iff R is left artinian (actually, [Kre84; Thm 3] says "iff R is right perfect and left coherent", but finite Morley rank of the theory of projective modules forces R to be left artinian (cf. [Pr84; 3.18]): note, re. Kremer's proof, that $e_iRe_j \neq 0$ need not imply $e_iR \simeq e_jR$).

4.7 The space of indecomposables

In [Zg84] Ziegler provided a topology on the set, $\mathcal{I}(T)$, of isomorphism types of non-zero indecomposable pure-injective direct summands of models of T, under which the components of T "are" precisely the closed sets. (This ties in with [Pr80e], where the author had suggested that the "components" (in the sense of §2.6) of a theory T (complete and closed under products) should be regarded as localisations of that theory.) Moreover the order proposed in §2.6 (cf. also [Gar79; §1]) on theories closed under products is just inclusion of the corresponding closed sets (so this poset is the dual of a complete Heyting algebra). This indicates that the subject of this section may reasonably be said to be the "spectrum" of the theory of R-modules (I haven't explored the connection with the topos-theoretic notion).

So, from now, $\mathcal{I}(T)$ may be either of two things: simply the set of (isomorphism classes of) indecomposable pure-injective direct summands of models of T; or this set equipped with the topology ([Zg84]) generated by open sets of the form $(\varphi/\psi)=\{N\in\mathcal{I}(T) : \mathrm{Inv}(N,\varphi,\psi)>1\}$ where φ, ψ are pp formulas in one free variable (and we may suppose that ψ implies φ in all modules). This actually gives a basis for the topology (4.66).

Herzog has pointed out [Her87] that this topology is simply that induced from the relative topology on the irreducible 1-types. Let X be the subset of $S_1{}^T(0)$ consisting of all the irreducible types, equipped with the relative topology. Now identify two points of X if they have isomorphic hulls: the result, equipped with the quotient topology, is precisely $\mathcal{I}(T)$.

Throughout this section, unless specified otherwise, all theories considered are assumed to be closed under products (so for example, in referring to a point N of $\mathcal{I}(T)$, I have in mind $\mathrm{Th}(N^{\aleph_0})$ rather than $\mathrm{Th}(N)$, if these are different). This exclusion of theories T not satisfying $T=T^{\aleph_0}$ and the concentration on indecomposables is justified by 4.39, 4.40 and by 4.36 (and, *a posteriori*, by the analyses in Sections 4.6 and 10.4).

Let us first note the basic properties of this space.

Theorem 4.66 [Zg84; 4.9] *Let T be any complete theory of modules. The sets (φ/ψ) form a basis for a topology on $\mathcal{I}(T)$. With this topology, $\mathcal{I}(T)$ and in fact every basic open set, is a compact space (but not even T_0 in general). If $p\in S^T(0)$ is irreducible then a neighbourhood basis for $N(p)$ is given by $\{(\varphi/\psi) : \varphi/\psi\in p\}$.*

Proof The last statement is proved first: from this will follow the first assertion So let $p\in S^T(0)$ be irreducible, and suppose that $N=N(p)$ lies in the intersection of (φ_1/ψ_1) and (φ_2/ψ_2). It must be shown that there is $\varphi/\psi\in p$ with $(\varphi/\psi)\subseteq(\varphi_1/\psi_1)\cap(\varphi_2/\psi_2)$.

Let a realise p in $N=N(a)$. By assumption, there are a_1,a_2 in N with $\varphi_i(a_i)\wedge\neg\psi_i(a_i)$ $(i=1,2)$. Since $\mathrm{pp}^N(a_i/a)\vdash\mathrm{tp}^N(a_i/a)$ (by 4.17) there exists $\theta_i\in\mathrm{pp}^N(a_i/a)$ such that $\theta_i(w,a)$ proves $\varphi_i(w)\wedge\neg\psi_i(w)$ (modulo T) $(i=1,2)$. So the formulas $\exists w\,(\theta_i(w,v)\wedge\varphi_i(w))$ and $\neg\exists w\,(\theta_i(w,v)\wedge\psi_i(w))$ are formulas in $p(v)$.

Set $\varphi'(v)$ to be $\exists w\,(\theta_1(w,v)\wedge\varphi_1(w))\wedge\exists w\,(\theta_2(w,v)\wedge\varphi_2(w))$ - a formula in p. Also, since p is irreducible, 4.29 gives that there is $\varphi\in p$ which (replace it by $\varphi\wedge\varphi'$) may be taken with $\varphi\leqslant\varphi'$ such that the formula $\psi(v)$, being $[\varphi(v)\wedge\exists w\,(\theta_1(w,v)\wedge\psi_1(w))]+[\varphi(v)\wedge\exists w\,(\theta_2(w,v)\wedge\psi_2(w))]$, is in p^-.

Thus $(\varphi/\psi)\in p$ and hence $N\in(\varphi/\psi)$. It must be checked that (φ/ψ) is contained in $(\varphi_1/\psi_1)\cap(\varphi_2/\psi_2)$.

So suppose that $N'\in(\varphi/\psi)$ - say $b\in\varphi(N')\setminus\psi(N')$ - and let $i\in\{1,2\}$. Since $\varphi(b)$ holds, certainly $\exists w\,(\theta_i(w,b)\wedge\varphi_i(w))$ holds. This formula, being pp, is satisfied in N' which is pure in the monster model of T, so take $c\in N'$ with $\theta_i(c,b)\wedge\varphi_i(c)$. It will be enough to show that $\neg\psi_i(c)$ holds. For then one will have $\mathrm{Inv}(N',\varphi_i,\psi_i)>1$ - that is, $N'\in(\varphi_i/\psi_i)$.

Since one has $\neg\psi(b)$, one concludes $\neg(\varphi(b)\wedge\exists w\,(\theta_i(w,b)\wedge\psi_i(w)))$. Together with $\varphi(b)$ this gives $\neg\exists w\,(\theta_i(w,b)\wedge\psi_i(w))$. Since one already has $\theta_i(c,b)$ the conclusion $\neg\psi_i(c)$ follows - as required.

The proof of the result is completed by showing that each basic open set (φ/ψ) is compact (see §10.V for examples showing that this space is not even T_0). Notice that $\mathcal{I}(T)$ is the special case $(v=v\,/\,v=0)$.

Therefore let $\{(\varphi_i/\psi_i): i\in I\}$ be a cover of (φ/ψ) in $\mathcal{I}(T)$ (we're already using what has been shown above). Suppose that there is no finite sub-cover. Then the set $\{\exists v\,(\varphi(v)\wedge\neg\psi(v))\}\cup\{\forall v\,(\varphi_i(v)\to\psi_i(v)): i\in I\}\cup$ "$T_1\leqslant T$" is consistent (exercise) where "$T_1\leqslant T$" is the set of sentences $\mathrm{Inv}(-,\varphi',\psi')\leqslant\mathrm{Inv}(T,\varphi',\psi')$ for those φ',ψ' with $\mathrm{Inv}(T,\varphi',\psi')$ finite.

Let M be a saturated model of the above set of sentences. By **2.9** and **2.30**, $M\in\mathcal{P}(T)$ and hence $\mathcal{I}(\mathrm{Th}(M))\subseteq\mathcal{I}(T)$ as topological spaces (another exercise in the definition of the topology). By **4.35**, there is an irreducible type p (for $\mathrm{Th}(M)$) with $\varphi/\psi\in p$. By construction, one has (for $N(p)$ a direct summand of M) $\mathrm{Inv}(N(p),\varphi_i,\psi_i)=1$ for all $i\in I$. Since the topology on $\mathcal{I}(\mathrm{Th}(M))$ is just the relative one, we have $N(p)$ in (φ/ψ) but not in any (φ_i/ψ_i) - contradiction, as required. ◻

There is a very satisfying and useful description of the closed sets in this topology.

<u>Theorem 4.67</u> [Zg84; 4.10] *Suppose that $T=T^{\aleph_0}$. Then the map $T_1\mapsto\mathcal{I}(T_1)$ is an order-preserving bijection from the poset $\{T_1: T_1=T_1^{\aleph_0}$ and $T_1\leqslant T\}$ of component theories (closed under product) of T to the set of closed subsets of $\mathcal{I}(T)$, ordered by inclusion.*

Proof Observe first that $\mathcal{I}(T_1)$ is a closed subset of $\mathcal{I}(T)$. For N lies in $\mathcal{I}(T)\setminus\mathcal{I}(T_1)$ iff there are φ,ψ with $\mathrm{Inv}(N,\varphi,\psi)>\mathrm{Inv}(T_1,\varphi,\psi)$ (by **2.30**) and this is so iff there are φ,ψ with $\mathrm{Inv}(N,\varphi,\psi)>1$ and $\mathrm{Inv}(T_1,\varphi,\psi)=1$ (since $T_1=T_1^{\aleph_0}$), that is, iff there is a (basic) open neighbourhood of N not intersecting $\mathcal{I}(T_1)$.

Suppose, conversely, that S is a closed subset of $\mathcal{I}(T)$. Let $M=\bigoplus\{N^{(\aleph_0)}: N\in S\}$. Set T_1 to be the theory of M: so $T_1\leqslant T$ and $T_1=T_1^{\aleph_0}$. It will be shown that $\mathcal{I}(T_1)=S$: the inclusion $S\subseteq\mathcal{I}(T_1)$ is by construction. Therefore let $N'\in\mathcal{I}(T_1)$ and take (φ/ψ) to be any basic open neighbourhood of N' - so $\mathrm{Inv}(N',\varphi,\psi)>1$. Since $M\equiv M\oplus N'$ (clearly!), it must be that $\mathrm{Inv}(M,\varphi,\psi)>1$ and hence, for some $N\in S$, $\mathrm{Inv}(N,\varphi,\psi)>1$. Thus every open neighbourhood of N' intersects S. Since S is closed it follows that $N'\in S$, as required.

Finally note that if $T_2\leqslant T_1(\leqslant T)$ then $\mathcal{I}(T_2)\subseteq\mathcal{I}(T_1)$. Furthermore, $T_1=T_2$ iff $\mathcal{I}(T_1)=\mathcal{I}(T_2)$. One direction is trivial. In the other direction, if $\mathcal{I}(T_1)=\mathcal{I}(T_2)$ then, by **4.36**, one has $M'=\bigoplus\{N^{(\aleph_0)}: N\in\mathcal{I}(T_1)=\mathcal{I}(T_2)\}$ is a common model of T_1 and T_2; so $T_1=T_2$. ◻

It follows that the poset of complete theories closed under product (introduced in §2.6) is the dual of a complete Heyting algebra, since the closed sets of any topology form such a structure.

It will be seen from Ex 2 below that, when applied to primes over a commutative noetherian ring, the topology is almost the "opposite" of the Zariski topology (Zariski open sets are closed in $\mathcal{I}(T)$). On the other hand, this topology does exactly generalise that Pierce spectrum (Ex 3). The peculiar relation to the Zariski topology is actually seen in the algebraic theory of non-commutative localisation (see [Gol80]) and corresponds to a distinction which has to be made between primes and localisations. Also, comparison with this algebraic theory suggests that one should be prepared to forget the points of $\mathcal{I}(T)$ for some purposes and work instead on the level of frame morphisms (consider the relation from $\mathcal{I}_S$ to $\mathcal{I}_R$ induced by a ring morphism from R to S), see [Si81], [Si84].

Exercise 1
(i) Show that $\mathcal{I}(M_1 \oplus M_2) = \mathcal{I}(M_1) \cup \mathcal{I}(M_2)$.
(ii) A closed set is **irreducible** if it cannot be expressed as the union of two closed proper subsets. Deduce that $\mathcal{I}(M)$ is irreducible iff, whenever $M \equiv M_1 \oplus M_2$ one has either $M \equiv M_1$ or $M \equiv M_2$ (this is a weak version of Garavaglia's "T-indecomposability in §2.6).

Example 1 Take $R = \mathbb{Z}$. Then, as a set, $\mathcal{I}_{\mathbb{Z}}$ is the union of the sets (after **2.ℤ11**): $\{\mathbb{Z}_{p^n} : p$ a prime, $n \geqslant 1\}$; $\{\mathbb{Z}_{p^\infty} : p$ a prime$\}$; $\{\overline{\mathbb{Z}_{(p)}} : p$ a prime$\}$; $\{\mathbb{Q}\}$. The topology will be described by giving a neighbourhood basis at each point (and, for convenience, the closure of each singleton X, denoted X^-).

$\mathbb{Z}_{p^n}$: $\{\mathbb{Z}_{p^n}\}$; $\{\mathbb{Z}_{p^n}\}^- = \{\mathbb{Z}_{p^n}\}$; so $\mathbb{Z}_{p^n}$ is isolated.
$\mathbb{Z}_{p^\infty}$: $\{\mathbb{Z}_{p^\infty}\} \cup \{\mathbb{Z}_{p^m} : m \geqslant n\}$ for $n \geqslant 1$; $\{\mathbb{Z}_{p^\infty}\}^- = \{\mathbb{Z}_{p^\infty}, \mathbb{Q}\}$.
$\overline{\mathbb{Z}_{(p)}}$: $\{\overline{\mathbb{Z}_{(p)}}\} \cup \{\mathbb{Z}_{p^m} : m \geqslant n\}$ for $n \geqslant 1$; $\{\overline{\mathbb{Z}_{(p)}}\}^- = \{\overline{\mathbb{Z}_{(p)}}, \mathbb{Q}\}$.
$\mathbb{Q}$: $\{\mathbb{Q}\} \cup \{\mathbb{Z}_{p^\infty} : p$ a prime$\} \cup \{\overline{\mathbb{Z}_{(p)}} : p$ a prime$\} \cup \{\mathbb{Z}_{p^n} : n \geqslant 1,\ p$ a prime, $p^n \geqslant m\}$ for $m \geqslant 2$; $\{\mathbb{Q}\}^- = \{\mathbb{Q}\} \cup \{\mathbb{Z}_{p^\infty} : p$ a prime$\} \cup \{\overline{\mathbb{Z}_{(p)}} : p$ a prime$\}$.

This space is layered by the Cantor-Bendixson analysis (see §5.2): the $\mathbb{Z}_{p^n}$ are the isolated points; the Prüfers and p-adics are isolated once those are removed; and this leaves $\mathbb{Q}$ as the only point of CB-rank 2.

Exercise 2 Describe $\mathcal{I}_R$ for: (i) $R = K[X]$; (ii) $R = K[X, X^{-1}]$; (iii) $R = K[X, X^{-1}, (1-X)^{-1}]$, where K is a field, and note the similarity to the case of $\mathbb{Z}$.

Exercise 3 Suppose that T is totally transcendental with $\mathcal{I}(T)$ infinite. Then there is a non-isolated irreducible 1-type over 0 (cf. [Pr84; 3.10], §9.3, §11.4).

Example 2 Let R be a commutative noetherian ring and let Spec$R$ be the set of prime ideals of R, endowed with the **Zariski topology**. A basis of open sets for this topology is given by the $D(I) = \{Q : Q$ does not contain $I\}$ where I is any ideal of R (P and Q will be used to denote prime ideals in this example). Given any ideal I, let $U(I) = \{Q : Q$ contains $I\}$. On account of the ring being noetherian, this topology has dcc on closed sets and one has the following characterisation:
typical Zariski closed: $U(I) = U(P_1) \cup \ldots \cup U(P_n)$;
typical Zariski open: $D(I) = D(P_1) \cap \ldots \cap D(P_n)$; where the P_i are the minimal primes above I.

How should we try to encompass Spec$R$? The only natural solution seems to be that taken in localisation theory (see [LM73], [Gol75]), where the prime P is "identified" with $E_P = E(R/P)$ - the injective hull of an element, $1+P$, whose annihilator is exactly P. This approach does give a module-theoretic interpretation of Spec$R$ which makes sense also in the non-commutative case (although there is the "direction" of the topology to be explained).

In our terms therefore, it seems that the relevant theory is the largest theory of injective modules: $T_{\text{inj}} = \text{Th}(\oplus\{E_P^{(\aleph_0)} : P \in \text{Spec}R\})$ (which is also the model-completion of the theory of R-modules - see **15.36**). Then the set Spec$R$ has been captured in the form $\mathcal{I}(T_{\text{inj}}) = \{E_P : P \in \text{Spec}R\}$. Let us compare the Zariski topology with the topology on the set Spec$R$ which is induced by that on $\mathcal{I}(T_{\text{inj}})$.

Consider the open set $(vP = 0 / v = 0)$ (if P is generated as a (right) ideal by $r_1, \ldots, r_n$ then "$vP = 0$" may be taken to be the formula $\bigwedge_1^n vr_i = 0$). One has $E_Q \in (vP = 0 / v = 0)$ iff there is a non-zero element $a \in E_Q$ with $aP = 0$ - that is, with ann $a \geqslant P$. But $a \in E_Q$ implies that ann a is Q-primary ([Mat58]) and hence that $Q \geqslant P$.

Thus $U(P) = (vP = 0 / v = 0)$ is open in $\mathcal{I}(T_{\text{inj}})$. Hence (see above) each Zariski-closed (resp. open) set is open (resp. closed) in $\mathcal{I}(T_{\text{inj}})$.

In any but the most trivial case, infinite unions of Zariski-closed sets need not be Zariski-closed. So to get the topology on $\mathcal{I}(T_{inj})$ one must "reverse" the basic sets of the Zariski topology and then close under the infinite operations. I now justify this remark by showing that there are no more closed sets in $\mathcal{I}(T_{inj})$ beyond infinite intersections of Zariski-open sets.

A typical intersection of Zariski-opens has the form (see above) $S = \bigcap\{D(P_\lambda) : \lambda \in \Lambda\} = \{Q : Q$ contains no $P_\lambda\}$. Since R is noetherian every member of S is below a maximal member of S and so S has the form $\{Q : Q \leqslant Q_\mu$ for some $\mu\}$....(*) for a suitable set $(Q_\mu)_\mu$ of primes. Conversely and similarly, any set of this form is an intersection of Zariski-open sets.

It must be shown that every $I(T_{inj})$-closed set has the form (*). By 4.67 the $\mathcal{I}(T_{inj})$-closed sets are "really just" the component theories of T_{inj}. These have been classified already in [Pr82], where Example 2 shows that the components of T_{inj} are indeed just those given by sets of the form (*).

Observe (exercise) that these correspond exactly to all the possible localisations (given a set S to be inverted, form $\{Q : Q \cap S = \emptyset\}$ - a typical set of the form (*)). More on injectives may be found in §6.1.

Example 3 Let R be a commutative (von Neumann) regular ring. The Pierce spectrum ([Pie67]) on Spec$R$ has as basic open sets the $\mathcal{O}_e = \{P \in \text{Spec}R : e \notin P\}$ where $e^2 = e$ is an idempotent of R. All such sets are clopen, since $\mathcal{O}_e$ has as complement $\mathcal{O}_{1-e}$. It is easy to check that this coincides with the Zariski topology on Spec$R$. Since there is a basis of clopen sets it should be no surprise (cf. Ex 2 above) that this topology turns out to be that on $\mathcal{I}(T_{inj})$ ($= \mathcal{I}_R$ in this case).

Using the fact that T has complete elimination of quantifiers (see 16.16), one deduces (16.18) that the basic open subsets of $\mathcal{I}(T_{inj}) = \mathcal{I}_R$ have the form $(ve = 0 / v = 0)$ where e is an idempotent. This clearly defines the open set $\mathcal{O}_{1-e}$ in the Pierce topology, so the topologies do coincide.

Question The definition of the Pierce topology makes sense over any ring. Does this correspond to $\mathcal{I}(T)$ for suitable T? (also, cf. [BV83], [BSV84])

Example 4 It is quite possible to have an indecomposable pure-injective (even t.t.) module N such that $\mathcal{I}(N)$ is infinite.

Take $R = K[X,Y]_{(X,Y)}$ - the ring of polynomials over the field K in two indeterminates, localised at the maximal ideal $\langle X,Y\rangle$. This ring is local, with unique simple module $S = R/\langle X,Y\rangle$, so it follows (exercise) that every injective module is a direct summand of a power of the injective hull, $E(S)$, of S. Hence $\mathcal{I}(E(S))$ contains the set of all indecomposable injective modules. But, for each $\alpha \in K$, the injective hull of $R/\langle X + \alpha Y\rangle$ is indecomposable, and distinct values of α give non-isomorphic injectives (see [Kap70; Thm 35]).

Given $T = T^{\aleph_0}$ and $N \in \mathcal{P}(T)$, define the subset $\mathcal{U}(N) = \{N' \in \mathcal{I}(T) : N'$ is a direct summand of $N\}$ of $\mathcal{I}(T)$. Note that this may be stricly smaller than $\mathcal{I}(N)$, which, if N is discrete, is the closure of $\mathcal{U}(N)$ in $\mathcal{I}(T)$. It will be convenient in what follows to restrict attention to discrete pure-injectives (or at least to those elementarily equivalent to their discrete parts): let $\mathcal{P}_d(T)$ denote the collection of such pure-injectives.

Problem Are there other topologies on the set of indecomposable pure-injectives which are useful? (say, which are more similar to the Zariski topology, or which allow some sort of sheaf representation).

<u>Lemma 4.68</u> [Zg84; 4.11] *Let $T = T^{\aleph_0}$ and suppose that $N \in \mathcal{P}_d(T)$. Then $\mathcal{U}(N)$ is dense in $\mathcal{I}(T)$ iff $N^{\aleph_0}$ is a model of T.*

Proof $\Rightarrow$ Let φ, ψ be pp such that $\mathrm{Inv}(T,\varphi,\psi)>1$. By assumption, there is $N' \in \mathcal{U}(N)\cap(\varphi/\psi)$; hence $\mathrm{Inv}(N,\varphi,\psi)>1$ since $\mathrm{Inv}(N',\varphi,\psi)>1$. Thus $\mathrm{Inv}(N^{\aleph_0},\varphi,\psi)=\infty=\mathrm{Inv}(T,\varphi,\psi)$. Therefore $\mathrm{Th}(N^{\aleph_0})\geqslant T$. The other inequality is immediate since $N\in\mathcal{P}(T)$.

$\Leftarrow$ Let (φ/ψ) be a non-empty basic open subset of $\mathcal{I}(T)$. From $\mathrm{Inv}(T,\varphi,\psi)>1$ one has $\mathrm{Inv}(N^{\aleph_0},\varphi,\psi)>1$ and hence $\mathrm{Inv}(N,\varphi,\psi)>1$. Now we use that N is discrete to conclude from **2.23** and **2.27** that there is $N'\in\mathcal{U}(N)$ with $\mathrm{Inv}(N',\varphi,\psi)>1$ – that is, with $N'\in(\varphi/\psi)$, as required. ◻

Example 5 [Bou79; II.2.2] Let R be commutative regular and let T^* be the largest theory of R-modules. A module $M = \bigoplus\{(R/I)^{(\kappa I)} : I\in X\subseteq \mathrm{Spec}R\}$ which is a direct sum of simples is a model of T^* iff $M\equiv M^{\aleph_0}$ and X is dense in $\mathrm{Spec}R$.

<u>Corollary 4.69</u> *If* $T=T^{\aleph_0}$ *has continuous part zero (see after* 4.A14) *and if* $N\in\mathcal{P}(T)$ *then* $N^{\aleph_0}$ *is a model of* T *iff* $\mathcal{U}(N)$ *is dense in* $\mathcal{I}(T)$. ◻

I will now use the topological space $\mathcal{I}(T)$ to give a rather abstract generalisation of the results of §4.6. More concrete and detailed results will be presented in §10.4. The remainder of this section is taken from [Pr82a].

First define the equivalence relation "$\approx$" on $I(T)$ by: $N\approx N'$ if $N'\in\{N\}^-$ and $N\in\{N'\}^-$ – that is, if each point is in the closure of the other, so that they are topologically indistinguishable. Clearly $N\approx N'$ iff N and N' belong to precisely the same (basic) open sets, hence iff $\{N\}^-=\{N'\}^-$. We will work in the factor space $\mathcal{I}(T)/\approx$ where these topologically indistinguishable points have been identified. The next lemma says what this means in terms of pp formulas.

<u>Lemma 4.70</u> *Let* $N, N'\in\mathcal{I}(T)$.

(*a*) $N\in\{N'\}^-$ *iff for all* φ,ψ *one has that* $\mathrm{Inv}(N,\varphi,\psi)>1$ *implies* $\mathrm{Inv}(N',\varphi,\psi)>1$.

(*b*) $N\approx N'$ *iff for all* φ,ψ *one has* $\mathrm{Inv}(N,\varphi,\psi)>1$ *exactly when* $\mathrm{Inv}(N',\varphi,\psi)>1$, *and this occurs iff* $N^{\aleph_0}\equiv(N')^{\aleph_0}$. ◻

It follows that if $N\approx N'$ then, whenever copies of N appear in the decomposition of a model of $T=T^{\aleph_0}$, they may all be replaced by a suitable number of copies of N' to yield a model. For purposes of classifying models, therefore, there can be no real distinction between such indecomposables (the only point to be watched is that if d_N (see **9.6**) is not equal (mod ∞) to $d_{N'}$ (can this happen?) then there will be limitations on such replacements). So, for some purposes, we may as well identify such points and work in the corresponding quotient space $\mathcal{I}(T)/\approx$. By **4.70** the topology (on the level of open sets) is the same, and is now at least T_0 (namely, given any two points, there is an open set containing just one of them). For convenience I will identify each $N\in I(T)$ with its image $N/\approx$ in $\mathcal{I}(T)/\approx$ when no confusion should arise.

It was found in §4.6 that for a t.t. theory T satisfying $T=T^{\aleph_0}$ the indecomposables of $\mathcal{I}(T)$ fall into two classes: those which must appear in every model (i.e., those realising an isolated type) and those which need not. Here I attempt to make the same division insofaras this is possible. The connection with notions of isolation and the generalisation of the more precise results of §4.6, I leave until §9.1 and §10.4.

Say that $N\in\mathcal{P}(T)$ is **ommissible** if whenever $M\in\mathcal{P}_{\mathrm{d}}(T)$ is such that $\mathcal{U}(M\oplus N)$ is dense in $\mathcal{I}(T)$ then $\mathcal{U}(M)$ already is dense (cf. 4.68). Thus N is omissible iff whenever $M\oplus N^{(\kappa)}$ is a discrete model of T then so is M. Examples are $\mathbb{Q}$ when $T=\mathrm{Th}(\mathbb{Z}_{p^\infty}{}^{\aleph_0})$ and also when $T=\mathrm{Th}(\mathbb{Z}_{(p)}{}^{\aleph_0})$. The question of describing the omissible indecomposables is largely answered in §10.4. Here I just note the topological description.

Lemma 4.71 *Suppose that* $T=T^{\aleph_0}$ *and let* $N\in\mathcal{I}(T)$. *Then* N *is omissible iff* $\{N\}^-$ *is not a neighbourhood of* N *(i.e., does not contain an open set).*

Proof $\Leftarrow$ Suppose that $M\in\mathcal{P}_d(T)$ is such that $\mathcal{U}(M\oplus N)$ is dense in $\mathcal{I}(T)$. To establish that N is omissible it is enough to show that $N\in\mathcal{U}(M)^-$.

So suppose that $N\in(\varphi/\psi)$: by hypothesis there is $N'\in(\varphi/\psi)\setminus\{N\}^-$. Since N' is not in the closure of $\{N\}$ there is (φ'/ψ') containing N' but not N. Thus $N'\in(\varphi/\psi)\cap(\varphi'/\psi')$ - in particular this set is non-empty - and $N\notin(\varphi/\psi)\cap(\varphi'/\psi')$. Since $\mathcal{U}(M\oplus N)$ is supposed to be dense, there is N'' in the intersection of this set with $(\varphi/\psi)\cap(\varphi'/\psi')$. It cannot be that $N''\simeq N$. Hence $N''\in(\varphi/\psi)\cap\mathcal{U}(M)$ - which in non-empty, as required.

$\Rightarrow$ Suppose that $\{N\}^-$ is a neighbourhood of N: say $N\in(\varphi/\psi)\subseteq\{N\}^-$. Let $M'=\bigoplus\{N'^{(\aleph_0)} : N'\in\mathcal{I}(T),\ N'\notin(\varphi/\psi)\}$. Then, by construction, $\mathcal{U}(M')$ (a closed set) is not dense in $\mathcal{I}(T)$. On the other hand $U(M'\oplus N)$ clearly is dense in $\mathcal{I}(T)$. So N is not omissible. □

Lemma 4.72 *Suppose that* $T=T^{\aleph_0}$ *and let* $N\in\mathcal{I}(T)$. *If* $N/\approx$ *is isolated in the space* $\mathcal{I}(T)/\approx$ *then, in the decomposition of every discrete pure-injective model of* T, *there occurs some* N' *which is* $\approx$*-equivalent to* N.

Proof The hypothesis implies that there is some (φ/ψ) with $(\varphi/\psi)=\{N' : N\approx N\}$. So if N_1 is not $\approx$-equivalent to N then $\mathrm{Inv}(N_1,\varphi,\psi)=1$. Since $\mathrm{Inv}(T,\varphi,\psi)>1$ the result is clear. □

I finish this section with a characterisation of when prime and minimal models exist for the category of discrete pure-injective models. If $\mathcal{C}$ is a category of models with elementary embeddings then $M\in\mathcal{C}$ is **prime** for $\mathcal{C}$ if it embeds in every object of $\mathcal{C}$, and M' is **minimal** for $\mathcal{C}$ if it has no proper subobjects.

Theorem 4.73 [Pr82a; 1.26] *Suppose that* $T=T^{\aleph_0}$. *Then:*

(a) T *has a prime discrete pure-injective model iff*
  (i) *whenever* $N/\approx$ *is isolated and* $N'\approx N$ *then* $N'\simeq N$, *and*
  (ii) *the isolated points of* $\mathcal{I}(T)$ *are dense;*

(b) T *has a minimal discrete pure-injective model iff* $\mathcal{I}(T)/\approx$ *is the closure of a discrete (with the relative topology) subset* S *such that each point of* S *contains some* $N\in\mathcal{I}(T)$ *with* $N\equiv N^{\aleph_0}$;

(c) T *has a model which is both prime and minimal in the category of discrete pure-injective models iff*
  (i) *whenever* $N/\approx$ *is isolated and* $N'\approx N$ *then* $N'\simeq N$, *and*
  (ii) *the isolated points of* $\mathcal{I}(T)$ *are dense, and*
  (iii) *if* $N\in\mathcal{I}(T)$ *is isolated then* $N\equiv N^{\aleph_0}$.

Proof (a) If the conditions are satisfied then let M be the pure-injective hull of $\bigoplus\{N^{(\kappa(N))} : N\in\mathcal{I}(T)$ is isolated, $\kappa(N)=1$ if $N\equiv N^{\aleph_0}$, $\kappa(N)=\aleph_0$ if $N\not\equiv N^{\aleph_0}\}$. It is claimed that M is the prime discrete pure-injective model of T.

Certainly M is a model of T: for given $\varphi\leqslant\psi$ with $\mathrm{Inv}(T,\varphi,\psi)>1$ there is (by (ii)) some isolated $N\in(\varphi/\psi)$. So, by construction, $\mathrm{Inv}(M,\varphi,\psi)=\infty$.

Furthermore, by **4.72**, every discrete pure-injective model must contain at least one copy of each indecomposable direct summand of M. So to show that M embeds in every such model it remains to show that if $N\not\equiv N^{\aleph_0}$ is isolated then at least $\aleph_0$ copies of N occur in the decomposition of every model.

Since N is isolated there are φ, ψ such that $\mathrm{Inv}(N,\varphi,\psi)>1$ but, for any $N' \in \mathcal{I}(T)$, $N' \not\approx N$ implies $\mathrm{Inv}(N',\varphi,\psi)=1$. So clearly the result follows.

As for the converse: suppose condition (ii) fails. Then if M were a prime discrete pure-injective model one would have (**4.68**) $\mathcal{U}(M)$ dense in $\mathcal{I}(T)$. So there would be some non-isolated point, N, in $\mathcal{U}(M)$. Set $M_1 = \bigoplus\{N_1^{(\aleph_0)} : N_1 \in \mathcal{I}(T),\ N_1 \not\approx N\}$. Since N is not isolated it is in the closure of $\mathcal{U}(M_1)$ and hence $\mathcal{U}(M_1)$ is dense - so M_1 is a model of T. But M_1 does not contain a copy of N and so M cannot embed elementarily (hence as a direct summand) in M_1: therefore M is not prime - contradiction as required.

If, on the other hand, condition (ii) holds but condition (i) does not, and if M purports to be a prime model, then choose points N, N' in $\mathcal{I}(T)$ which are $\approx$-equivalent, non-isomorphic, with isolated image in $\mathcal{I}(T)/\approx$ and such that N occurs in M (since $\mathcal{U}(M)$ is dense in $\mathcal{I}(T)$ there are such points). Replace the copies of N in the decomposition of M by 1 or $\aleph_0$ copies (as appropriate) of N'. Again (by **4.68**) one has a model of T with no possible embedding of M into it - contradiction, as required.

(b) If the conditions are satisfied then choose, for each point of S, some $N \in \mathcal{I}(T)$ in its class with $N \equiv N^{\aleph_0}$. Define M to be the direct sum of all these (one copy of each).

Since S is dense in $\mathcal{I}(T)/\sim$ the corresponding set of indecomposables is dense in $\mathcal{I}(T)$ so (**4.68**) M is a model of T. Since S is discrete in the relative topology, no proper subset is dense. Thus, if any summand of M is deleted, one no longer has a model. Hence M is a minimal discrete pure-injective model.

Conversely, if the condition fails, and if M purports to be a minimal model then consider the dense (**4.68**) set $\mathcal{U}(M)$.

If $\mathcal{U}(M)$ contained a discrete dense (in $\mathcal{U}(M)$, equally in $\mathcal{I}(T)$) subset S, then all points, N, of some member of S would (by the assumed failure) satisfy $N \not\equiv N^{\aleph_0}$. Let (φ/ψ) be a neighbourhood of such an $N/\approx$ isolating it from the other points of S. Since $N \not\equiv N^{\aleph_0}$, there is some $\varphi > \psi$ with $\mathrm{Inv}(N,\varphi,\psi)$ finite. Since $\mathrm{Inv}(M,\varphi,\psi)$ is infinite, it follows (by isolation) that some infinite direct sum of members of $N/\approx$ purely embeds in M. But then any one of these points may be omitted, while retaining a model, so M is isomorphic to a proper direct summand of itself and hence is not minimal - contradiction.

Therefore $\mathcal{U}(M)$ does not contain a discrete subset which is dense. Let S denote the set of points of $\mathcal{U}(M)$ which are isolated in the relative topology. Then there is N in $\mathcal{U}(M)\setminus\overline{S}$ and this set is a relatively open neighbourhood of N in $\mathcal{U}(M)$ which does not intersect S. Hence N is a limit of points of $\mathcal{U}(M)$ which are themselves non-isolated in $\mathcal{U}(M)$.

Set $M = M_1 \oplus \mathrm{pi}(N^{(\kappa)})$ where M_1 splits off no copy of N (actually minimality of M implies $\kappa = 1$). I claim that M_1 is a model of T, thus contradicting minimality of M.

So let φ, ψ be such that $\mathrm{Inv}(N,\varphi,\psi)>1$. Then the relative neighbourhood $(\varphi/\psi)\cap\mathcal{U}(M)$ of N contains some $N' \in \mathcal{U}(M)$ non-isolated in $\mathcal{U}(M)$ and with $N' \not\approx N$. So $\mathrm{Inv}(M_1,\varphi,\psi)>1$. Since N' is a direct summand of M_1 one draws the conclusion $\mathrm{Inv}(M_1,\varphi,\psi)>1$: one needs that $\mathrm{Inv}(M_1,\varphi,\psi)$ is infinite.

Since M is minimal, $\mathcal{U}(M)$ is T_0 in the relative topology (it is an easy exercise - compare with the first part of the argument - to see that $\mathcal{U}(M)$ cannot intersect any $\approx$-equivalence class in more than one point).

The relatively open neighbourhood $(\varphi/\psi)\cap(\mathcal{U}(M)\setminus\overline{S})$ is therefore a T_0 space with no isolated points - hence is infinite. So M_1 has infinitely many indecomposable direct summands N', apart from N, with $\mathrm{Inv}(N',\varphi,\psi)>1$. Hence $\mathrm{Inv}(M_1,\varphi,\psi)$ is infinite, as required.

(c) This follows directly from (a) and (b). □

CHAPTER 5 FORKING AND RANKS

Linear independence in vectorspaces and transcendence in algebraically closed fields both are kinds of independence. In fact, they are rather simple examples of a very general notion of independence which has arisen in model theory and which is dignified by the name "non-forking". Stability theory is concerned with classifying and investigating structures using this and derived concepts.

We begin by characterising (non-)forking in modules. Thus we give meaning to the phrases: the type q is a non-forking (=free) extension of the type p; the element a is independent from the set B over the set C. The description is in terms of the groups $G(-)$ introduced in §2.2. Since some of the material in this and the next two chapters is used in algebraic applications, I do not assume that the reader has already encountered ideas from stability theory and so I give illustrative examples and state (and, I hope, explain) the main background theorems. All of that is in the first section.

In the examples which I mentioned above, there is a dichotomy - algebraic(= completely dependent)/independent - but most theories are more complicated than this, with elements exhibiting degrees of dependence on one another. In at least some cases there are ordinal ranks which measure degree of dependence: these are discussed in §2. It turns out that the rank of a type, p, is the foundation rank of the connected component of the associated group $G(p)$ in the lattice of such connected groups. It follows that, for modules, the various stability-theoretic ranks coincide in so far as they exist (this is not true of arbitrary stable theories).

The characterisation of independence given in §1 is in terms of formulas represented in a type: in particular, it is not very "algebraic". But it turns out that independence is, more or less, direct-sum independence (of hulls). More specifically, if two sets A, B are such that there is a pure-injective model with a decomposition $N\oplus N'\oplus N''$ with $A\leq N$ and $B\leq N'$, then A and B are independent over 0. In the case where the theory is closed under products, the converse is true also so, there, independence over 0 is just direct-sum independence (cf. §4). In the general case, the converse is a little more subtle. The section (§3) actually considers independence over an arbitrary set in place of 0.

The description of independence is more clear-cut when the class of models is closed under products, and the simplified theorems are stated at the beginning of §4. Then we go on to show that, under the hypothesis $T=T^{\aleph_0}$, independence is exactly pushout in an appropriate category. The relevant category is not just (a part of) $\mathcal{M}_R$ since (cf. above paragraph) the sets A and B need not be pure in a model: in fact the category is (part of) $\mathcal{M}_R$, enriched by adding information about pp-types.

5.1 Forking and independence

At the centre of stability theory is a notion of dependence/independence. The formalisation of this as forking/non-forking gives precise meaning to the statement "A and B are independent from each other over C" where A, B and C are sets of parameters.

Example 1

(i) Let T be the theory of algebraically closed fields of a given characteristic (this theory is complete, because any two uncountable algebraically closed fields of the same cardinality and characteristic are isomorphic - or see [Poi85; 6.04]). Then subsets A, B of the monster (or "universal") model are independent over the subset C iff $\mathrm{acl}(A)\cap \mathrm{acl}(B\cup C)=\mathrm{acl}(A)\cap \mathrm{acl}(C)$, where "acl" denotes algebraic closure.

(ii) Let T be the (complete) theory of infinite vectorspaces over some given division ring. Then subspaces A and B are independent over the subspace C iff $A\cap(B+C)=A\cap C$.

In each of these examples an element is either completely dependent on (i.e., algebraic over), or completely independent of, a given set. But most theories are much more complex than this, with elements exhibiting varying degrees of dependence. The stability ranks are introduced to give an ordinal measure of degree of independence, at least in the totally transcendental and superstable cases.

(iii) Let T be the theory of one equivalence relation E (i.e., structures are sets equipped with a distinguished equivalence relation) with infinitely many infinite equivalence classes. Then a is dependent on b over $\emptyset$ iff aEb holds, but a will not be algebraic over b unless $a=b$. So dependence is strictly weaker then algebraicity. If M is a model then a will be independent from M over $\emptyset$ iff a is in a new E-equivalence class (i.e., one which does not intersect M): existence of such classes is consistent since there are infinitely many E-classes.

Were this example to be modified by the requirement that every E-equivalence class be finite, then dependence and algebraicity would coincide. For if aEb and $b\in M$ then $a\in M$ also.

(iv) Let T be the theory of two equivalence relations, E_1 and E_2, each with infinitely many classes, such that each $E_1\cap E_2$-equivalence class is infinite. Let a and c be elements of a model. Then a is independent from c over $\emptyset$ iff a is in neither the same E_1-class nor the same E_2-class as c. If a is in the same E_1-class as an element b then a does depend on b over $\emptyset$. If also a does not lie in the same E_2-class as b then a still has two "degrees of freedom over b": one is its E_2-class, which is not yet specified; but even if one chooses some element b' in the same E_2-class as a then a still has one degree of freedom over $\{b,b'\}$ (since its $E_1\cap E_2$-class is infinite).

I will now quickly review some basic stability theory, leading up to the precise definition of the notion of (in)dependence which has been seen in simple form in the above examples. As usual we work within the monster model.

Let $p\in S(A)$ and suppose that $B\supseteq A$. We want to select from the various extensions of p to B (that is, from those types q over B with $q\restriction B=p$) those extensions which are non-forking or "free", in the sense that they contain no more information than they have to, given that they do extend p to B. The amount of information in the type p is measured by the **class** of p - the set of formulas **represented** in p: $\mathrm{cl}(p)=\{\varphi(\bar{v},\bar{y}) : \varphi(\bar{v},\bar{a})\in p(\bar{v})$ for some $\bar{a}$ (in A)$\}$.

It seems only reasonable in our case to consider the set of pp formulas represented in p: $\mathrm{cl}^+(p)=\{\varphi\in\mathrm{cl}(p) : \varphi$ is pp$\}$. Actually something very like $\mathrm{cl}^+(p)$ has already been seen in §2.2, where $\mathcal{G}(p)$ was defined to be the filter of subgroups/formulas obtained from $\mathrm{cl}^+(p)$ by replacing the parameter-variables $\bar{y}$ by zero-tuples $\bar{0}$. Since it is possible to have $\varphi(\bar{v},\bar{0})$ equivalent to $\psi(\bar{v},\bar{0})$ with φ,ψ quite different pp formulas, $\mathcal{G}(p)$ contains a little less information than $\mathrm{cl}^+(p)$, but it will be seen that this lost information is not important and, indeed, $\mathcal{G}(p)$ is actually the better invariant of p.

It was shown in **2.17** that $\mathrm{tp}(\bar{a})=\mathrm{tp}(\bar{a}')$ iff $\mathrm{pp}(\bar{a})=\mathrm{pp}(\bar{a}')$ (the presence of a complete theory modulo which we work is presumed throughout this chapter). That is, $p=q$ iff $p^+=q^+$, for any types p,q. It will be seen below that for types p,q over a model one has $\mathrm{cl}(p)=\mathrm{cl}(q)$ iff $\mathrm{cl}^+(p)=\mathrm{cl}^+(q)$ and $p\restriction 0=q\restriction 0$; in fact $\mathrm{cl}(p)\le\mathrm{cl}(q)$ iff $\mathrm{cl}^+(p)\le\mathrm{cl}^+(q)$ and $p\restriction 0=q\restriction 0$ (**5.8**).

Next, some account must be taken of the possibility that a certain formula is not represented in a type, yet must be represented in any extension of that type to a model.

Exercise 1 Let T be the (complete) theory of algebraically closed fields of characteristic zero and let p be the type over $\emptyset$ of a square root of 2. Find a formula which is not represented in p but which must be represented in every extension of p to a model of T.

In our case this phenomenon typically arises as follows: there is a pp formula $\varphi(\bar{v},\bar{a}) \in p$ and there is ψ pp with the index $[\varphi(\bar{v},\bar{0}):\psi(\bar{v})]$ finite. If M is any model containing $\bar{a}$ then (exercise) M must contain a representative of each coset of ψ in $\varphi(\bar{v},\bar{a})$. Thus any extension of p to a (complete!) type over M must represent the formula $\psi(\bar{v}-\bar{y})$. Here is a concrete example.

Example 2 Take the abelian group $\mathbb{Z}_4 \oplus \mathbb{Z}_2^{(\aleph_0)}$, let b be the element $(1,\bar{0})$ of $\mathbb{Z}_4 \oplus \mathbb{Z}_2^{(\aleph_0)}$ and let p be the type over 0 of an element of order 4. It is not difficult to see that if q is any extension of p to a model - say to $\mathbb{Z}_4 \oplus \mathbb{Z}_2^{(\kappa)}$ - then $q(v)$ must contain the formula $(v-b).2=0$. So the formula $(v-y)2=0$ must be represented in q, yet clearly it is not represented in p (that is, $(v-0)2=0$ is not in p).

If the numbers 4 and 2 were to be replaced by 9 and 3 then there would be a choice to be made between cosets (exercise).

In cases such as that just described, one considers that, in adding such formulas and if necessary in making such choices, no essentially new information has been added beyond the necessary minimum. It would be quite a different matter were $[\varphi(\bar{v},\bar{0}):\psi(\bar{v})]$ (as above) infinite. For then it would be consistent that a tuple be in $\varphi(\bar{v},\bar{a})$ yet not be in any "named" coset of $\psi(\bar{v})$ (even over a model).

Therefore, it is best to consider $\mathrm{cl}(p)$ only for types p over models, and the following theorem makes the situation clear. It says that there is a set of formulas which must be represented in every extension of p to a model; moreover, there is an extension of p to a model which represents just these formulas.

<u>Theorem 5.A</u> ([LP79], see [Pi83; 3.4]) (*T a complete stable theory*)
Let $A \subseteq \tilde{M}$ and suppose that p is a type over A. Then the set $\{\mathrm{cl}(q) : q$ is a type over a model containing A and q extends $p\}$ has a (unique!) minimum element (with respect to inclusion) which is denoted $\beta(p)$ and is called the **bound** *of p. If M is any model containing A then there is at least one type q over M extending p and with $\mathrm{cl}(q)=\beta(p)$.* □

(The set of all classes of (say 1-)types over models is termed the **fundamental order**; see [Poi85; Chpt 13]. Thus the bound of a type is the least point of the fundamental order which contains the class of that type. The description of the fundamental order for the largest theory of injectives over a right coherent ring is due to Bouscaren [Bou79] and, independently, to Kucera (see [Kuc87; 2.4]) for the noetherian (=t.t.) case. We will not have ocassion to refer to this order explicitly, but its translation-invariant version - the "stratified order" - shows up in §6.1.).

Now one can make a sensible definition of "free" extension of a type and hence, of independence.

Suppose that $p \in S(A)$, that $A \subseteq B$ and that $q \in S(B)$ is an extension of p. Say that q is a **non-forking extension** of p if it has the same bound as p (one always has the inclusion $\beta(q) \supseteq \beta(p)$). One also says that q **does not fork over** A in this situation (note that $q \restriction A = p$).

Given parameters $\bar{a},\bar{b},\bar{c}$ say that $\bar{a}$ and $\bar{b}$ are **independent over** $\bar{c}$, and write $\bar{a} \perp \bar{b}/\bar{c}$ if $\mathrm{tp}(\bar{a}/\bar{b}^\frown\bar{c})$ is a non-forking extension of $\mathrm{tp}(\bar{a}/\bar{c})$. The notation and terminology is extended in the obvious way to sets (in place of tuples). It is a theorem (see [Pi83; 3.9]) that, as is implicit in the terminology, this relation is symmetric. The terms, **forking** and **dependent**, have the expected meanings.

Forking is a finitary property: $\bar{a} \not\perp \bar{b}/\bar{c}$ iff there are finite sub-sequences $\bar{a}' \le \bar{a}$ and $\bar{b}' \le \bar{b}$ with $\bar{a}' \perp \bar{b}'/\bar{c}$ - for forking is witnessed by a single formula.

A point which I will use repeatedly is the existence of non-forking extensions (for stable theories): given a type p over A and a set B containing A, there is at least one $q \in S(B)$ which is a non-forking extension of p (this follows quickly from 5.A).

Since modules are stable (3.1) it follows that, given any "situation" inside a module and given any new data - say a submodule B - there is a copy of the original situation, in some elementary extension of the over-module, which is free with respect to B. From the algebraic point of view it is not clear that any precise meaning could be given to this statement. Stability theory does provide a formal rendering of the statement. The existence of free extensions is guaranteed; and the resulting notion of "free" surely does generalise those cases - for example vectorspaces - where there is a clear meaning for the term.

One should note exactly what 5.A says. Although any type p has a unique bound it need not have a unique non-forking extension to a given model. In fact, the number of distinct non-forking extensions to any model is an invariant of p (see [Pi83; 3.27]), called its **multiplicity** and denoted mult(p). One has (see [Pi83; 3.24]) $\mathrm{mult}(p) \leq 2^{|T|}$ for a type, p, in finitely many variables. If $\mathrm{mult}(p)=1$ then p is said to be **stationary**. More will be said below concerning this, but one may note the following examples.

Examples 3

(i) Let T be the theory of the abelian group $\mathbb{Z}_9 \oplus \mathbb{Z}_3^{(\aleph_0)}$. Let $p \in S_1(0)$ be the type of an element of order 9. Note that $[v=v: v3=0]=3$ so, by the argument for Ex 2 above, if q is a type extending p to a model M then $q(v)$ contains the formula $(v-m).3=0$ for some $m \in M$ (that is, q has to decide in which coset of $M.3$ does "v" lie). Note that $m \notin M.3$, since q extends p and the latter contains the formula $v.3 \neq 0$. Apart from this, there is no restriction on m, so there are essentially two possibilities (conjugate over 0) for "m". Therefore $\mathrm{mult}(p)=2$.

One may note that in Ex 2 above, the type p was stationary.

(ii) Let T be the theory of the abelian group $\mathbb{Z}_{(p)}$ and let $p \in S_1(0)$ be the type of any one of the non-zero elements of $\mathbb{Z}_{(p)}$ (there are $\aleph_0$ types - see Ex 2.1/6(ii)). It is left as an exercise to show that $\mathrm{mult}(p)=2^{\aleph_0}$ (the proof of 3.1(i)⇒(iii)′ shows the sort of argument to use).

Exercise 2 Prove that if $T=\mathrm{Th}(\mathbb{Z}_4 \oplus \mathbb{Z}_2^{(\aleph_0)})$ then every 1-type over 0 is stationary. Is the same is true for 2-types? Rothmaler [Rot83c; Thm1] shows that, for modules, every n-type over a set A is stationary iff every non-algebraic 2-type over A is stationary. In [BR84; 3.7] it is shown that this is true for any stable theory.

Exercise 3 (for those who know what is meant by a definable type) Since modules are stable (3.1), all types are definable. Show this directly.

In this section, these basic notions of stability theory are interpreted in modules.

There is another way of looking at non-forking extensions which will be quite useful to us: this is usually presented by making use of "ideal types" (types over the monster model) but, for purposes of presentation, I use a slightly different approach. One uses the following fact (1.5): given a set A and a cardinal κ, there exists a model M which contains A, is $(2^{|T|})^+$-saturated and is such that, whenever $\bar{b}, \bar{c}$ are finite tuples from M and have the same type over A, there is an A-automorphism of M taking $\bar{b}$ to $\bar{c}$. The group of A-automorphisms of M is denoted $\mathrm{Aut}_A M$.

Let p be a type over A. Observe that $\mathrm{Aut}_A M$ has an action on the set, S, of types over M extending p, as follows. If $q \in S(M)$ extends p and if $f \in \mathrm{Aut}_A M$, then set $fq(\bar{v}) = \{ \varphi(\bar{v}, f\bar{m}) : \varphi(\bar{v}, \bar{m}) \in q(\bar{v})\}$ (If $\bar{c}$ realises q in the monster model and if M' is a sufficiently saturated elementary extension of M containing $\bar{c}$, then there is at least one extension, f', of f to M'; then $f'\bar{c}$ realises $fq(\bar{v})$. Since f' fixes A and is an automorphism, one has $\mathrm{tp}(f'\bar{c}/A) = p$.) An approach to non-forking may be based on the fact that there is a unique "small" orbit in S under this action, where "small" means of cardinality no greater than $2^{|T|}$. This orbit consists precisely of the non-forking extensions of p to M (and the cardinality of the orbit is the multiplicity of p).

So: given a type p and an extension, q, of p to the set B, one may "test" whether or not q is a non-forking extension of p by taking such a large "containing" model M for B, then using the fact that q is a non-forking extension of p iff q has an extension to a type in the small orbit of extensions of p to M.

<u>Lemma 5.1</u> *Let p be a type over the set A and let q be an extension of p to a model M. Then $\mathcal{G}(q) \geq \mathcal{G}_0(p)$.*

Proof Let $\psi(\bar{v},\bar{y})$ be pp with $\psi(\bar{v},\bar{0}) \in \mathcal{G}_0(p)$, say $\varphi(\bar{v},\bar{a}) \in p^+$ is such that $[\varphi(\bar{v},\bar{0}):\varphi(\bar{v},\bar{0}) \wedge \psi(\bar{v},\bar{0})]$ is finite. Let $\bar{m}_1,\ldots,\bar{m}_k$ be a complete set of coset representatives of $\varphi(\bar{v},\bar{0}) \wedge \psi(\bar{v},\bar{0})$ in $\varphi(\bar{v},\bar{a})$. Then $M \models \forall \bar{v}\,(\varphi(\bar{v},\bar{a}) \to \bigvee_1^k \psi(\bar{v}-\bar{m}_i,\bar{0}))$. So, since q is a complete type over M, one has, say, $\psi(\bar{v}-\bar{m}_1,\bar{0}) \in q$. Thus $\psi(\bar{v},\bar{0}) \in \mathcal{G}(q)$. □

Thus one may say "$\beta(p) \geq \mathcal{G}_0(p)$". More precisely, if $\psi(\bar{v},\bar{y})$ is a pp formula such that $\psi(\bar{v},\bar{0}) \in \mathcal{G}_0(p)$ and $\exists \bar{y}\,\psi(\bar{v},\bar{y}) \in p(\bar{v})$, then $\psi(\bar{v},\bar{y}) \in \beta(p)$. The converse will be proved.

<u>Lemma 5.2</u> [PP83; 2.5], [Zg84; 11.1] *Let p be a type over the set A. Then there exists an extension of p to a type q over a model with $G(q) = G_0(p)$. Any such type q is a non-forking extension of p.*

Proof Let M be a large containing model for A. Consider the following set of formulas: $p \cup X$, where $X = \{\neg\psi(\bar{v},\bar{m}) : \psi \notin \mathcal{G}_0(p)$ and $\bar{m}$ is in $M\}$. This set if consistent. For otherwise there is $\varphi(\bar{v},\bar{a}) \in p^+$, $\theta_1(\bar{v},a_1),\ldots,\theta_n(\bar{v},\bar{a}_n) \in p^-$ and $\neg\psi_1(\bar{v},\bar{m}_1),\ldots,\neg\psi_k(\bar{v},\bar{m}_k) \in X$ such that $\varphi(\bar{v},\bar{a}) \leq \bigvee_1^n \theta_i(\bar{v},\bar{a}_i) \vee \bigvee_1^k \psi_j(\bar{v},\bar{m}_j)$. Since, by consistency of p, the $\theta_i(\bar{v},\bar{a}_i)$ do not suffice to cover $\varphi(\bar{v},\bar{a})$, it must be, by Neumann's Lemma (2.12), that at least one of the subgroups $\psi_j(\bar{v},\bar{0})$ has finite index in $\varphi(\bar{v},\bar{0})$ – in contradiction to the defining property of X. Thus the set is indeed consistent, so has an extension to a type q over M. Clearly $G(q) = G_0(p)$.

If $f \in \mathrm{Aut}_A M$ then, for the type fq, one has (clearly) $G(fq) = G_0(p)$. But an extension of p satisfying this equation is completely determined by specifying, for each subgroup lying between $G(p)$ and $G_0(p)$, which coset of that subgroup is "in" the type. There are, therefore, at most $2^{|T|}$ possibilities for such a type. Hence (any such) q belongs to the orbit of non-forking extensions of p, as required. □

Some more information is extracted from this proof in 5.11 and 6.44 (where the non-forking extensions are counted). The previous two results combine to give us the following characterisation of non-forking.

<u>Theorem 5.3</u> [PP83; 2.6], [Zg84; 11.1] *Let p be a type and suppose that q is any extension of p. Then q is a non-forking extension of p iff $G_0(q) = G_0(p)$.* □

<u>Corollary 5.4</u> *Let p be any type. If $G_0(p) = G(p)$ then p is stationary.*

Proof It is sufficient to observe that, in the proof of 5.2, if $G_0(p) = G(p)$ then the set of formulas shown to be consistent is actually a complete type. Hence p has just one non-forking extension to a model, as required. □

The converse fails (consider Ex 2 above).

<u>Corollary 5.5</u> [PP83; 2.6] *Let A, B, C be sets of parameters. Then $B \perp C/A$ iff for any pp formula ψ such that $\psi(\bar{b},\bar{c},\bar{a})$ holds (where $\bar{a}$ is in A, $\bar{b}$ in B and*

$\bar{c}$ in C) there holds a pp formula $\varphi(\bar{b},\bar{a}')$ with $\bar{a}'$ in A and the index $[\varphi(\bar{v},\bar{0}):\varphi(\bar{v},\bar{0})\wedge\psi(\bar{v},\bar{0},\bar{0})]$ finite. □

Corollary 5.6 cf. [Gar81; Lemma3] *Let B and C be sets of parameters. Then $B \not\perp C/0$ iff there is a pp formula ψ with $\psi(\bar{b},\bar{c})$ for some $\bar{b}$ in B and $\bar{c}$ in C and such that for every pp formula φ with $\varphi(\bar{b})$, the index $[\varphi(\bar{v}):\varphi(\bar{v})\wedge\psi(\bar{v},\bar{0})]$ is infinite (in particular, $\neg\psi(\bar{b},\bar{0})$ holds).* □

As a consequence, the bound of a type may be described. Garavaglia [Gar81] has this in the case where $T=T^{\aleph_0}$ and $A=0$.

Corollary 5.7 [PP83; 2.5] *Let p be a type over A. Then the pp formula, $\varphi(\bar{v},\bar{y})$, is in the bound of p iff $\varphi(\bar{v},\bar{0})\in\mathcal{G}_0(p)$ and $\exists\bar{y}\,\varphi(\bar{v},\bar{y})\in p(\bar{v})$.* □

One may also specify when the class of one type is contained in the class of another. This was proved for modules in [PP83] and for the more general normal theories (see just below) in [Sr84] (also see [Rot83b]).

Corollary 5.8 [PP83; 2.3], [Sr84; §5.B] *Let p,q be types over the model M. Then:*
$\mathrm{cl}(p)\subseteq\mathrm{cl}(q)$ iff $\mathrm{cl}^+(p)\subseteq\mathrm{cl}^+(q)$ and $p\restriction 0=q\restriction 0$.

Proof Of course the right-hand conditions are necessary and, by **5.7** and **5.1**, they are also sufficient, for $\mathrm{cl}(p)$ to be contained in $\mathrm{cl}(q)$. □

Another approach to **5.3** is taken in [PP83] - one which involves first characterising the bound of a type by proving **5.8**. The direct proof of **5.8** (given by Garavaglia [Gar81] in the $T=T^{\aleph_0}$ case and generalised in [PP83]) does have one advantage: it does not depend on Neumann's Lemma but, as pointed out by Hodges, only on the following combinatorial property of pp formulas:

$$\varphi(\bar{a},\bar{b})\wedge\varphi(\bar{a}',\bar{b}')\wedge\varphi(\bar{a}',\bar{b})\rightarrow\varphi(\bar{a},\bar{b}') \quad \ldots.\ (**).$$

This property ties in with Pillay's definition [Pi84a] of a "normal" theory, and Srour's definition of an "equational" theory [Sr81α], [Sr81].

In [Sr81α] and [Sr81] (see also [Sr84; §5.B]) Srour considered what he termed "equations" and "equational theories". A formula $\varphi(\bar{v},\bar{y})$ is an "**equation** " in this sense if one has the dcc on intersections of sets of the form $\varphi(\tilde{M},\bar{a})$. A theory is "**equational**" (of sort "=" - see below) if every formula is equivalent to a boolean combination of "equations" (since these are not equations in the usual sense, I retain quotes around them, so as to avoid any misunderstanding). This notion includes that of a normal theory (see below) and so, in particular, modules. Srour independently developed a good deal of the theory of the "pure-injective" models of an "equational" theory (generalising 3.1(a), every equational theory is stable). He also generalised a number of results on the stability theory of modules to this context. He payed particular attention to the case where the "equations" are closed under existential quantification, since the theory may be developed further under that hypothesis.

Independently, Pillay introduced the notion of a normal theory. Say that a formula $\varphi(\bar{v},\bar{y})$ is **normal** (more precisely, $\bar{v}$-**normal**) if, for all $\bar{a},\bar{a}'$, one has that either $\varphi(\tilde{M},\bar{a})$ and $\varphi(\tilde{M},\bar{a}')$ are equal or have empty intersection (this generalises the coset property). Clearly normality of a formula is equivalent to the property (**) above (note that each presupposes a certain fixed partition of the free variables). A theory is **normal** if every formula is equivalent to a boolean combination of normal formulas.

Some normal theories are: modules; any theory of equivalence relations which has elimination of quantifiers; any ω-stable theory of Morley rank $\leqslant 2$ (for sort "=") ([PS84;

6.5]); also see [Pal85]. Examples of "equational" theories include: algebraically closed fields; differentially closed fields of characteristic zero (see [Sr81]).

Pillay and Srour together developed these ideas further, using a framework suggested by algebraic geometry (see [PS84]). They also introduced the notion of a weakly normal theory: a formula $\varphi(\overline{v},\overline{y})$ is **weakly normal** if every intersection of infinitely many distinct sets of the form $\varphi(\widetilde{M},\overline{a}_i)$ is empty, and a theory is **weakly normal** if every formula is equivalent to a boolean combination of weakly normal formulas (such a theory is "equational"). An example of a weakly normal, non-normal, theory is that of un-ordered pairs (two distinct conjugates of a pair may intersect, but no more than two). This notion of weak normality has turned out to connect with ideas arising from other investigations in stability theory but, before I say a little about that, let me present the generalisation of **5.8** for normal theories.

Let C be a class of $\overline{v}$-normal formulas. Say that the theory T is **($\overline{v}$-)normal with respect to** C if every formula $\chi(\overline{v})$ is equivalent modulo T to a boolean combination of formulas in C. Thus **2.16** says that any theory of modules is normal with respect to the class of pp formulas. Define $\mathrm{cl}_C(p)$ from C just as $\mathrm{cl}^+(p)$ was defined from the class of pp formulas - namely as $C \cap \mathrm{cl}(p)$.

The following result, for modules, is just **5.8** above. I sketch its proof (for details, presented in the modules case, see [PP83; 2.3]).

<u>Proposition 5.9</u> *Let T be complete and let C be a class of normal formulas. Suppose that T is normal with respect to C. Let p and q be types over a model M.*

Suppose that $p\restriction\emptyset = q\restriction\emptyset$ and $\mathrm{cl}_C(p)\subseteq\mathrm{cl}_C(q)$. Then $\mathrm{cl}(p)\subseteq\mathrm{cl}(q)$.

Proof (The proof here is that used in [PP83]: another proof may be given using the fact that the non-forking extensions of a type form the unique small orbit (see earlier in this section)). It is immediate that we may suppose that C is closed under conjunction; moreover disjunctions give no problem. It suffices, therefore, to check the conclusion for formulas which are the conjunction of a formula in C with negations of formulas in C. Let $\overline{a}$ realise p and $\overline{b}$ realise q in $\widetilde{M}$. Suppose that $\varphi,\psi_1,\dots,\psi_n$ are in C and are such that the formula $(\varphi \wedge \bigwedge_1^n \neg\psi_i)(\overline{v},\overline{y})$ is not in $\mathrm{cl}(q(\overline{v}))$. It may be supposed that each set $\psi_i(\overline{b},M)$ is non-empty - say $\overline{d}_i$ is in M with $\psi_i(\overline{b},\overline{d}_i)$; for otherwise, since $\mathrm{cl}_C(p)\subseteq\mathrm{cl}_C(q)$, $\neg\psi_i$ may be removed .

Suppose, for a contradiction, that we had $\varphi \wedge \bigwedge_1^n \neg\psi_i$ in $\mathrm{cl}(p)$. So in particular $\varphi(\overline{v},\overline{y})$ would be represented in $p(\overline{v})$ - hence, by assumption, represented in $q(\overline{v})$ - say $\varphi(\overline{b},\overline{c})$ holds with $\overline{c}$ in M.

Now, since there is $\overline{m}$ in M with $\varphi(\overline{v},\overline{m}) \wedge \bigwedge_1^n \neg\psi_i(\overline{v},\overline{m}) \in p(\overline{v})$, one has that $\varphi(\overline{a},M)$ is not contained in the union of the $\psi_i(\overline{a},M)$. So (since $\varphi(\overline{a},M)=M\cap\varphi(\overline{a},\widetilde{M})$ etc.), the same is true with $\widetilde{M}$ in place of M. Thus $\widetilde{M}\models\exists\overline{y}\,(\varphi(\overline{a},\overline{y})\wedge\bigwedge_1^n\neg\psi_i(\overline{a},\overline{y}))$. Therefore the formula $\exists\overline{y}\,(\varphi(\overline{v},\overline{y})\wedge\bigwedge_1^n\neg\psi_i(\overline{v},\overline{y}))$ is in the restriction of p to 0 and hence, by asumption, in the restriction of q to 0. Thus $\varphi(\overline{b},\widetilde{M})$ is not contained in the union of the $\psi_i(\overline{b},\widetilde{M})$.

Therefore one has $\widetilde{M}\models\varphi(\overline{b},\overline{c})\wedge\bigwedge_1^n\psi_i(\overline{b},\overline{d}_i)\wedge\exists\overline{y}\,(\varphi(\overline{b},\overline{y})\wedge\bigwedge_1^n\neg\psi_i(\overline{b},\overline{y}))$. Since the $\overline{c},\overline{d}_i$ are in $M(\prec\widetilde{M})$ there exists $\overline{b}'$ in M such that M satisfies the formula above but with $\overline{b}'$ in place of $\overline{b}$. Choose $\overline{m}$ in M to witness $\overline{y}$ in that formula.

Thus we have $\overline{c}$ in $\varphi(\overline{b},M)\cap\varphi(\overline{b}',M)$, and also $\varphi(\overline{b}',\overline{m})$ holds. So, by the combinatorial property (**) above, $\varphi(\overline{b},\overline{m})$ holds. Similarly, from the fact that $\overline{d}_i$ lies in $\psi_i(\overline{b},M)\cap\psi_i(\overline{b}',M)$, from $\neg\psi_i(\overline{b}',\overline{m})$ one deduces $\neg\psi_i(\overline{b},\overline{m})$. Therefore one has $\varphi(\overline{b},\overline{m})\wedge\bigwedge_1^n\neg\psi_i(\overline{b},\overline{m})$ - and $\overline{m}$ in M - contradicting that the formula $\varphi\wedge\bigwedge_1^n\neg\psi_i$ is not represented in q. Thus the result is proved. ◻

Note that one of the points of the proof is that if $\bar{b}$ is arbitrary and if φ is normal then $\varphi(\bar{b},M)$ is definable with parameters in M *via* the same formula: that is, there is some $\bar{b}'$ in M with $\varphi(\bar{b},M)=\varphi(\bar{b}',M)$ (in general, one has rather less: see [PS84] for more on this).

Suppose that $\varphi(v,\bar{y})$ and $\psi(v,\bar{y})$ are normal formulas: write $\varphi\geqslant\psi$ if each $\varphi(v,\bar{a})$ is a disjoint union of $\psi(v,\bar{y})$-classes. One might try to define the index of ψ in φ, *however*, this "index" may depend on $\bar{a}$.

Exercise 4 Show this dependence.
[Hint: take E_1, E_2 to be equivalence relations such that each E_1-class is the union of a certain number of E_2-classes.]
Srour has some results on this (personal communication).

The definitions of "equational" and (weakly) normal were adequate for what we have discussed so far. However, the work that I describe next is placed in the context of T^{eq} (see §10.T), so now the definitions of these notions should be strengthened so that every formula (with variables of whatever sort) is a boolean combination of "equations", or (weakly) normal formulas, (with variables of the appropriate sorts). So, for the discussion below, understand the terms in this stronger sense.

It has turned out that many stable theories are weakly normal. By results on "normalisation", one has that every $\aleph_0$-categorical ω-stable theory is weakly normal (this is implicit in [CHL85] and is explicitly pointed out in [PS84; 6.8]). Pillay ([Pi84b]) showed that any countable stable theory is either weakly normal or "type-interprets" a pseudo-plane (for the definition of this, see [Pi84b]). This was strengthened in [HP87], where it is shown that the following conditions on a complete theory are equivalent: T is weakly normal; T is stable and 1-based; T is stable and does not type-interpret a pseudoplane. The condition of being 1-based arose in the work of Buechler [Bue86b] and Pillay [Pi84b]: T is 1-based if whenever p is a type over a saturated model M of T, there is a subset A of M over which p does not fork and there is $\bar{a}$ in M realising the restriction of p to A, such that p is a non-forking extension of its restriction to $\bar{a}$ (cf. 5.11 below).

Hrushovski and Pillay prove the following about weakly normal groups. If the language is that of groups then a group is weakly normal iff it is abelian-by-finite: beyond this, if G is a definable set (in some language) with a definable group operation and G is weakly normal (for definable subsets: definable subsets of G^2 are needed) then G is abelian-by-finite [HP87; 3.2] (this should be interpreted in T^{eq}, so G could be a definable quotient for example). They also show that if G is a weakly normal group (in the second, stronger sense) then every definable subset is a boolean combination of cosets of subgroups definable with parameters from the algebraic closure of the empty set (and conversely) [HP87; 4.1]. This may be phrased in terms of abelian structures: if G is a weakly normal group then G is bi-interpretable (perhaps using parameters) with an abelian structure (see [HP87; §4]).

A number of results in this chapter and the next have generalisations (see [Sr84]) or analogues in other contexts (see [SPS85]).

5.2 Ranks

It is very useful to be able, not only to detect forking, but also to measure degree of forking. The various stability ranks provide a measure of this. I will concentrate on the U-rank of Lascar [Las76] which is defined directly in terms of forking: I include the relevant definitions, specialised to modules. Then it is shown that the other ranks - in particular Shelah degree and Morley rank - coincide with U-rank when they are defined (U-rank and Shelah degree are defined for all types exactly if the theory is superstable; Morley rank is globally defined iff the theory is t.t.). Thus for modules the only differences between the ranks are their domains of definition (for general theories one may have $\mathrm{UR}(p)<\mathrm{D}(p)<\mathrm{MR}(p)<\infty$). The common description of these ranks in modules is in terms of a more algebraic one which will have uses elsewhere and which is given in terms of pp-definable subgroups.

Let T be any stable theory and let p be a type; the **U-rank** of p, $\mathrm{UR}(p)$, is defined as follows (see **[Pi83**; 5.11**]**, **[Poi85**; §17.a**]**). For α an ordinal set $\mathrm{UR}(p)>\alpha$ iff p has a forking extension q with $\mathrm{UR}(q)\geqslant\alpha$ - one begins the inductive definition by setting $\mathrm{UR}(p)\geqslant 0$ for all (consistent) types p. Thus $\mathrm{UR}(p)=0$ iff p is algebraic; $\mathrm{UR}(p)=1$ iff p is not algebraic but every forking extension of p is algebraic. Set $\mathrm{UR}(p)=\infty$ if $\mathrm{UR}(p)>\alpha$ for all ordinals α.

I will use the fact (see **[Poi85**; p445**]**) that if $\mathrm{UR}(p)=\alpha<\infty$ and if $\beta<\alpha$ then p has an extension q, necessarily forking (since $\mathrm{UR}(q)<\mathrm{UR}(p)$), with $\mathrm{UR}(q)=\beta$. Since forking is witnessed by formulas, it follows that $\mathrm{UR}(p)=\infty$ iff $\mathrm{UR}(p)>|T|$ (for p a type in finitely many free variables). One sets $\mathrm{UR}(T)=\sup\{\mathrm{UR}(p) : p\in S_1{}^T(0)\}$.

Example 1

(i) Let T be the theory of equality. There are only two kinds of types: realised types and the single unrealised type (over any given set). The former have U-rank 0; the latter must therefore have U-rank 1 since any forking extension is a realised type.

(ii) Let T be the theory of an equivalence relation E with infinitely many infinite equivalence classes. Then, over any given set, there are: the realised types - of U-rank 0; the unrealised types which nevertheless specify an E-class - these are of U-rank 1 since the only way to extend them by (non-trivial) information is to specify an element; the type which specifies no E-class - so is of U-rank 2. Therefore $\mathrm{UR}(T)=2$.
In this example we see how U-rank measures "degrees of freedom" or extent of forking.
One may note that the Cantor-Bendixson rank (see below) of $S_1{}^T(\emptyset)$ is just 0 (since it contains just one point - all elements look the same over $\emptyset$) - parameters are needed to display the full complexity of the type structure. This is somewhat less the case for modules.

(iii) Refer back to Ex 2.2/5 ($T=\mathrm{Th}(\mathbb{Z}_4{}^{\aleph_0})$). With notation as there, one sees that the only forking extensions of p and q are realised types; so p and q have U-rank 1. There is a type over M_0 of U-rank 2 - the type of an element of order 4 whose difference from any element of M_0 is still of order 4. There are no more kinds of types in this example, and so $\mathrm{UR}(T)=2$.

Exercise 1

(i) Show that $\mathrm{UR}(\mathrm{Th}(\mathbb{Z}_8{}^{\aleph_0}))=3$.

(ii) Find $\mathrm{UR}(\mathrm{Th}(\mathbb{Z}_2\oplus\mathbb{Z}_4{}^{\aleph_0}))$.

(iii) Show that $\mathrm{UR}(\mathrm{Th}(\mathbb{Z}_{(p)}))=1$ (use the results of §1).

<u>Theorem 5.B</u> (see **[Pi83**; 5.15**]**) *A stable theory T is superstable iff every (1-)type has* U-*rank* (*i.e., has* U-*rank* $<\infty$). ▫

The U-rank is actually the smallest rank among those which satisfy certain reasonable conditions which one would want to impose on an independence rank **[Las76]**. An implication of **5.B** is therefore that these stability ranks will be useful mainly in the case of superstable or even nicer theories. It will be seen in Chapter 10, however, that for modules there do exist certain useful and more widely defined ranks.

Define $\mathrm{PP}_0{}^{(n)}$ to be the set of those $\mathbb{M}$-pp-definable subgroups of $\widetilde{M}^{(n)}$ which are connected - that is, (**5.10** below) those of the form $G_0(p)$ for some $p\in S_n(0)$. Order PP_0 by inclusion: by **2.3** this is a modular lattice.

Exercise 2 Show that a ($\mathfrak{M}$-)definable connected group is ($\mathfrak{M}$-)pp-definable. Is this still true for non-connected groups? (I don't know).

Now, it is immediate from **5.3** that if p is a type and if the interval $[G_0(p):0]=\{H\in PP_0 : H\leqslant G_0(p)\}$ is well-founded (i.e., has the dcc), then $UR(p)$ exists and is bounded by the foundation rank of $[G_0(p):0]$. The **foundation rank** of a point, x, in a poset is defined to be 0 if it is minimal and is defined to be $\sup\{1+\text{foundation rank of } y : y<x\}$ otherwise. So a point acquires foundation rank (less than ∞) iff the poset of points below it has the dcc (exercise).

To prove that $UR(p)$ is exactly this foundation rank, it must be shown that if $H\in PP_0$ lies below $G_0(p)$ then there is some extension q of p with $G_0(q)=H$.

Lemma 5.10 *Let $H\in PP_0{}^n$. Then there is $p\in S_n(0)$ with $G(p)=H$.*

Proof Use Neumann's Lemma (**2.12**) to show that the following set is consistent and generates the required type: $\{\varphi(\bar{v}) : \varphi$ is pp and $\varphi(\tilde{M})\geqslant H\}\cup\{\neg\psi(\bar{v}) : \psi$ is pp and $\psi(\tilde{M})$ does not contain $H\}$. ▫

Proposition 5.11 [PP87; 2.8] (also cf. [Sr84; V.C.9-12]) *Let $p\in S(A)$ and suppose that $H\in PP_0$ is such that $H\leqslant G_0(p)$. Let $\bar{c}$ realise p.*
Then there is $q\in S(\bar{c})$ with $G(q)=G_0(q)=H$ such that the unique non-forking extension of q to $A^\frown\bar{c}$ extends p.
In particular there is an extension q' of p with $G(q')=H$.

Proof By 5.10 there is $q_0\in S(0)$ with $G(q_0)=H$. Let $\bar{b}$ realise the non-forking extension (5.4) of q_0 to $\bar{c}$ and set $q=tp(\bar{c}+\bar{b}/\bar{c})$. Therefore $q^+(\bar{v})=\langle\varphi(\bar{v}-\bar{c}) : \varphi(\bar{v})\in q_0{}^+\rangle$ (by **5.3**) and q is stationary by **5.4** since $G_0(q)=G_0(q_0)=H$.

Let q' be the (unique) non-forking extension of q to $A^\frown\bar{c}$. It must be shown that q' extends p; it will be enough to establish that $(q'\upharpoonright A)^+=p^+$.

Therefore let $\varphi(\bar{v},\bar{a})$ be in p^+: so $\varphi(\bar{c},\bar{a})$ holds and also $\varphi(\bar{v},\bar{0})$ is in $G(p)$, hence $\varphi(\tilde{M},\bar{0})\geqslant H$. So by construction, $\varphi(\bar{v},\bar{0})$ is in $q_0{}^+$; hence $\varphi(\bar{v}-\bar{c},\bar{0})\in q\subseteq q'$. Together with $\varphi(\bar{c},\bar{a})$, this yields that $\varphi(\bar{v},\bar{a})$ is in q' and so $\varphi(\bar{v},\bar{a})\in q'\upharpoonright A$.

Conversely, if $\varphi(\bar{v},\bar{a})$ is a pp formula in the restriction of q' to A then, since $\varphi(\bar{v},\bar{0})\in G(q')=G(q)=G(q_0)$, one has $\varphi(\bar{v},\bar{0})$ in q_0. So, by construction, one has $\varphi(\bar{v}-\bar{c},\bar{0})\in q\subseteq q'$. Combining these formulas in q' yields $\varphi(\bar{c},\bar{a})$. Since $\bar{c}$ realises p the conclusion is that $\varphi(\bar{v},\bar{a})$ is in p, as required. ▫

Let pp_0-rk denote the foundation rank on PP_0.

Theorem 5.12 [PP87; 2.10] *Let p be a type. Then* $UR(p)=pp_0\text{-rk}\,G_0(p)$. *In particular* $UR(p)<\infty$ *iff the interval* $[G_0(p):0]$ *in* PP_0 *has the dcc.*

Proof The proof is by induction on $\alpha=pp_0\text{-rk}\,G_0(p)$. If $\alpha=0$ then $G_0(p)=0$, so (exercise) every extension of p to a model is algebraic and hence p itself is algebraic - that is $UR(p)=0$.

Now suppose the truth of the result for ordinals $\beta<\alpha$. Let q be a forking extension of p; so by **5.3** we have $G_0(q)<G_0(p)$. Thus $pp_0\text{-rk}\,G_0(q)<pp_0\text{-rk}\,G_0(p)=\alpha$ and so, by the induction hypothesis, $UR(q)=pp_0\text{-rk}\,G_0(q)$. Then it is immediate from the definition of the two ranks, UR and pp_0-rk, that $UR(p)=\alpha=pp_0\text{-rk}\,G_0(p)$. ▫

As an immediate corollary of **5.8**, **5.12** and **3.1**, one has the following.

Corollary 5.13 *The theory T is superstable iff* PP_0 *has the dcc, in which case* $UR(T)=pp_0\text{-rk}\,PP_0$ *- the foundation rank of (the top element in)* PP_0. ▫

Now I digress briefly to present some examples which show that the statements of **5.11** and **5.13** cannot be weakened in certain ways, and which are perhaps of independent interest.

Example 2 PP_0 is not a sublattice of the lattice of all $\bigwedge\!\!\!\bigwedge$-pp-definable subgroups.

Let $R=\mathbb{Z}_{(2)}[X,Y]$ - so an R-module is a $\mathbb{Z}_{(2)}$-module equipped with two commuting endomorphisms (given by the actions of X and Y). Let N be the module whose underlying $\mathbb{Z}_{(2)}$-structure is $\mathbb{Z}_{(2)}^{(\aleph_0)} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{(2)}^{(\aleph_0)}$, with the actions of X, Y given as follows.

On the first copy of $\mathbb{Z}_{(2)}^{(\aleph_0)}$ the action of X is multiplication by 0, and that of Y is multiplication by 2. On the second copy of $\mathbb{Z}_{(2)}^{(\aleph_0)}$ the roles of X and Y are reversed. On $\mathbb{Z}_2$ both X and Y act as 0.

Then $N_1=\mathbb{Z}_{(2)}^{(\aleph_0)} \oplus \mathbb{Z}_2 \oplus 0$ is defined by the pp formula $vX=0$. Similarly $N_2=0 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{(2)}^{(\aleph_0)}$ is defined by $vY=0$. Both N_1 and N_2 may be seen to be connected (*). But their intersection, $0 \oplus \mathbb{Z}_2 \oplus 0$, is not. So in PP_0, the meet operation is "$(-\cap-)_0$" not "$\cap$". [(*) For example, on N_1, the action is just that of $K[Y]$ so, clearly, there is a non-trivial morphism from the first component to the second. So by **2.2**(i), any pp-definable subgroup which contains the first component must contain the second; thus the only possibility for a pp-definable subgroup of finite index in N_1 is eliminated.]

Example 3 In **5.11** we need H to be connected.

Take T to be the theory of the abelian group $\mathbb{Z}_4 \oplus \mathbb{Z}_2^{(\aleph_0)}$. Let $p \in S_1(0)$ be given (isolated) by $v2 \neq 0$. Set $G_3=M\models T$, $G_2=\{m\in M : m2=0\}$, $G_1=\{m\in M : \exists m'\in M\,(m=m'2)\}$, $G_0=0$. Then $G(p)=G_3$. Now $G(p)\geqslant G_1$ but note that $[G_1:G_0]=2$. I will show that there is no extension, q, of p with $G(q)=G_1$.

If q is a non-forking extension of p then $G(q)\geqslant (G_3)_0=G_2$ (by **5.3**).

If q is a forking extension then it is fairly clear (by **2.Z1**) that q must represent $2|(v-y)$ (representation of $(v-y).2=0$ does not give forking) - say $(2|v-a)\in q$. Then either $v=a$, in which case $G(q)=0$, or $v\neq a$, in which case $v-a$ is a non-zero element divisible by 2. But there is only one such element, b say. Thus $v-a=b$. Since "v" has order 4 and "$v-a$" has order 2, necessarily a has order 4. Therefore $a2=b$ and so $v=a3$ - hence $G(q)=G_0$.

Thus there is no extension q of p with $G(q)=G_1$.

Example 4 In order that T be superstable it is not enough to have dcc on connected pp-definable (as opposed to connected $\bigwedge\!\!\!\bigwedge$-pp-definable) subgroups.

Let K be a finite field and let $R=$"$K\{T\}^{\nu}$" - the K-span of $\{T^{\alpha} : \alpha<\nu\}$ equipped with multiplication given by $T^{\alpha}T^{\beta}=T^{\alpha+\beta}$ (so we choose ν so that $\alpha,\beta<\nu$ implies $\alpha+\beta<\nu$). If $\nu>\omega$ then this is a non-commutative ring.

Define the module M_R by: as a K-vectorspace $M=\bigoplus\{x_{\alpha}K : \alpha<\nu\}$; the action of R is given by $x_{\alpha}T^{\beta}=x_{\alpha+\beta}$. Let $T=\mathrm{Th}(M)$.

The pp-definable subgroups of M are (exercise) these: $M_{\beta}=\bigoplus\{x_{\alpha}K : \beta\leqslant\alpha<\nu\}$ (for $\beta<\nu$), which is defined by $\exists w\,(v=wT^{\beta})$; and no others, apart from 0. Clearly the index $[M_{\beta}:M_{\beta+1}]=|K|$ is finite. So the only connected pp-definable subgroup is 0.

But if $\nu>\omega^2$ then T is not superstable: $\{T^{\beta}|v : \beta<\lambda\}$ for limit λ defines a connected $\bigwedge\!\!\!\bigwedge$-pp-definable subgroup G_{λ} and, if $\lambda<\lambda'$ are limits, then $G_{\lambda}>G_{\lambda'}$.

Next I consider the other stability ranks (see [Pi83; 6.13ff], [Poi85; Chpt17]). These all have the form $R(-,\lambda)$ for some λ (an ordinal or ∞) - and there are two main ones, corresponding to $\lambda=\aleph_0$ (Morley rank) and $\lambda=\infty$ (Shelah degree). Unlike U-rank, these ranks are defined in the first instance on formulas, as follows.

$R(p,\lambda)=\min\{R(\varphi,\lambda) : \varphi\in p\}$ where p is any type in finitely many free variables.

$R(\varphi,\lambda)\geqslant 0$ iff φ is consistent.

$R(\varphi,\lambda)\geqslant\alpha$ if for each $\beta<\alpha$ and each $\mu<\lambda$ there is a set A and there are pairwise contradictory $q_i\in S(A)$ for $i\leqslant\mu$ such that $\varphi\in q_i$ for each i and $R(q_i,\lambda)\geqslant\beta$ for each i.

Note that $R(p,\lambda)=\alpha$ iff there is some $\varphi\in p$ with $R(\varphi,\lambda)=\alpha$ and for all $\varphi'\in p$ one has $R(\varphi',\lambda)\geqslant\alpha$ and then one has that any type q with $\varphi\in q$ satisfies $R(q,\lambda)\leqslant R(p,\lambda)$ (in general, though not in modules, one cannot define U-rank of types in terms of some rank on formulas). Thus the ranks $R(-,\lambda)$ are continuous: they are continuous maps from the spaces of types, $S(A)$, to the class of ordinals equipped with the order topology.

The rank $R(-,\aleph_0)$ is called **Morley rank** and is denoted "MR". The rank $R(-,\infty)$ is **Shelah degree** "D".

There is an alternative way of looking at Morley rank. Let X be a set endowed with a topology. The **Cantor-Bendixson derivative** of X is the (closed) set $X'=X\setminus\{p\in X : p$ is an isolated point$\}$ with the induced topology. The higher derivatives are defined by induction: $X^{(\alpha+1)}=(X^{(\alpha)})'$; $X^{(\lambda)}=\bigcap\{X^{(\alpha)} : \alpha<\lambda\}$ if λ is a limit ordinal. There is a least ordinal α such that $X^{(\alpha+1)}=X^{(\alpha)}$ (that is, such that $X^{(\alpha)}$ has no isolated points). If this $X^{(\alpha)}$ is empty and if X is a compact space then (exercise) α cannot be a limit ordinal, so one can define the **Cantor-Bendixson rank** of X, CB-rk(X), to be $\alpha-1$. If $p\in X^{(\beta)}\setminus X^{(\beta+1)}$ then set the **Cantor-Bendixson rank** of the point p to be β and write CB-rk$(p)=\beta$; if $p\in\bigcap\{X^{(\beta)} : \beta$ an ordinal$\}$ then set CB-rk$(p)=\infty$ (and similarly with X).

Exercise 3

(i) Suppose that X is a compact space.

(a) CB-rk$(X)=0$ implies that X is finite: the converse holds if X is also Hausdorff or at least if X does not contain topologically indistinguishable points.

(b) CB-rk$(X)=\max\{$CB-rk$(p) : p\in X\}$.

(ii) If X is the unit rational interval then $X'=X$.

(iii) Let $\alpha=\{\beta : \beta<\alpha\}$ be an ordinal equipped with basic open sets of the sort $\{\gamma : \beta<\gamma<\alpha\}$. When is this space compact?. What is its Cantor-Bendixson rank?.

To connect this with Morley rank, let p be any type in n free variables. Let M be any $\aleph_0$-saturated model containing the set over which p is defined and let q be any non-forking extension of p to M. Consider the CB-rank of q in the space $S_n(M)$. Then MR$(p)=$CB-rk(q) (exercise: one should first decode the definition of $R(-,\aleph_0)$ into terms of definable sets; see [Poi85; 17.17]). Note that the Morley rank of a type need not be equal to its CB-rank: consider Ex 1 above or $T=\mathrm{Th}(\mathbb{Z}_4{}^{\aleph_0})$, where one has CB-rk$(S_1{}^T(0))=0$ but CB-rk$(S_1{}^T(\mathbb{Z}_4{}^{(\aleph_1)}))=$CB-rk$(S_1{}^T(\mathbb{Z}_4{}^{(\aleph_0)}))=2$, so that MR$(T)=2$ (one should perhaps work through this example if the definitions are new).

The remainder of this section is taken rather directly from [PP87]. It is all the time assumed that we are working in a stable theory.

The following facts about the ranks $R=R(-,\lambda)$ will be used, but not proved here (see [Pi83; Chpt6]).

(i) $R(\bigvee_1^n\varphi_i)=\max\{R(\varphi_i) : 1\leqslant i\leqslant n\}$.

(ii) If $R(\varphi)=\alpha>\beta$ then there is ψ which implies φ (so defines a smaller set) with $R(\psi)=\beta$.

(iii) If T has a group operation "+" then $R(\bar{a}+\varphi)=R(\varphi)$ ($\bar{c}$ lies in $\bar{a}+\varphi$ iff $\varphi(\bar{c}-\bar{a})$ holds) (exercise).

(iv) T is totally transcendental iff MR$(v=v)$ is defined, and this will be so iff MR(p) is defined for all types (in any finite number of free variables and over any set).

(v) T is superstable iff $D(v=v)$ is defined, and this is so iff $D(p)$ is defined for all $p\in S_n{}^T(A)$.

The first step in showing that all these ranks coincide (in so far as they are defined) is to prove that they depend only on pp formulas. From now on we are back in the context of modules.

<u>Proposition 5.14</u> [PP87; 2.5] *Let* $\chi(\bar{v})=\varphi(\bar{v},\bar{a})\wedge\bigwedge_1^n\neg\psi_i(\bar{v},\bar{a}_i)$ *be a consistent formula with* $\varphi,\psi_1,\ldots,\psi_n$ *pp. Then* $R(\chi)=R(\varphi(\bar{v},\bar{a}))=R(\varphi(\bar{v},\bar{0}))$.

Proof (S. Thomas) The proof goes by induction on n, the case $n=0$ being trivial (the second equality follows directly from (iii) above). So assume the truth of the result for $n-1$. One may suppose, therefore, that for each j, χ is inequivalent to $\varphi\wedge\bigwedge\{\neg\psi_i : 1\leqslant i\leqslant n,\ i\neq j\}$. It may also be supposed, as usual, that $\varphi(\bar{v},\bar{0})\geqslant\psi_i(\bar{v},\bar{0})$ for each i.

Choose $i\leqslant n$. Since χ is consistent there is some coset $\psi_i(\bar{v}-\bar{c}_i,\bar{0})$ of $\psi_i(\bar{v},\bar{0})$ such that $\psi_i(\bar{v}-\bar{c}_i,\bar{0})\leq\varphi(\bar{v},\ \bar{a})$ and such that the formula $\theta(\bar{v})$, being $\psi_i(\bar{v}-\bar{c}_i,\bar{0})\wedge\bigwedge_1^n\neg\psi_j(\bar{v},\bar{a}_j)$, is consistent. In particular, $\psi_i(\bar{v}-\bar{c}_i,\bar{0})\cap\psi_i(\bar{v},\bar{a}_i)=\emptyset$ (cosets are equal or disjoint). So $\theta(\bar{v})$ is equivalent to the formula $\psi_i(\bar{v}-\bar{c}_i,\bar{0})\wedge\bigwedge\{\neg\psi_j(\bar{v},\bar{a}_j) : 1\leqslant j\leqslant n,\ \ j\neq i\}$.

Now $\theta(\bar{v})\to\chi(\bar{v})$ (clearly) so $R(\theta)\leqslant R(\chi)=\alpha$ say (the inequality is immediate from the definition of the ranks). By the induction hypothesis, $R(\theta(\bar{v}))=R(\psi_i(\bar{v}-\bar{c}_i,\bar{0}))$. So, by (iii) above, $R(\psi_i(\bar{v},\bar{a}_i))\leqslant\alpha$; and this holds for each $i\leqslant n$.

Note that $\varphi(\bar{v},\bar{a})\leftrightarrow\chi(\bar{v})\vee\bigvee_1^n\psi_i(\bar{v},\bar{a}_i)$. So, by (i) above, and what has just been shown, $R(\varphi(\bar{v},\bar{a}))=\alpha$, as required. □

<u>Corollary 5.15</u> [PP87; 2.6] *Let* p *be a type. Then there is some* $\varphi\in p^+$ *with* $R(p)=R(\varphi)$.

Proof By definition of R there is $\chi\in p$ with $R(\chi)=R(p)$. Then use 5.14. □

<u>Corollary 5.16</u> [PP87; 2.7] *Let* $\varphi(\bar{v},\bar{a})$ *be pp with* $R(\varphi(\bar{v},\bar{a}))=\alpha<\infty$. *Let* $\beta<\alpha$. *Then there is some pp formula* $\psi(\bar{v},\bar{b})$ *with* $\psi(\bar{v},\bar{b})\to\varphi(\bar{v},\bar{a})$ *and* $R(\psi(\bar{v},\bar{b}))=\beta$.

Proof By (ii) above there is some formula χ with $\chi\to\varphi(\bar{v},\bar{a})$ and $R(\chi)=\beta$. By property (i) above, χ may be supposed to be the conjunct of a pp formula and negations of pp formulas. On replacing χ by the equivalent formula $\chi\wedge\varphi(\bar{v},\bar{a})$ it follows that the positive conjunct, $\psi(\bar{v},\bar{b})$ say, of χ may be taken to imply $\varphi(\bar{v},\ \bar{a})$. Then apply **5.14**. □

<u>Lemma 5.17</u> *Suppose that* φ,ψ *are pp formulas with the index* $[\varphi(\bar{v},\bar{0}):\psi(\bar{v},\bar{0})]$ *finite. Then for any* $\bar{a},\bar{b}$ *one has* $R(\varphi(\bar{v},\bar{a}))=R(\psi(\bar{v},\bar{b}))$, *provided both* $\varphi(\bar{v},\bar{a})$ *and* $\psi(\bar{v},\bar{b})$ *are non-empty. Conversely, if* $\varphi(\bar{v},\bar{a})$ *and* $\psi(\bar{v},\bar{b})$ *are consistent with rank* $<\infty$ *and the index* $[\varphi(\bar{v},\bar{0}):\psi(\bar{v},\bar{0})]$ *is infinite, then* $R(\varphi(\bar{v},\bar{a}))>R(\psi(\bar{v},\bar{b}))$.

Proof This is immediate from (i) and (iii) above (and properties of cosets). The second part follows easily from the definition of the ranks R; also it is a consequence of **2.4** and 5.20 below. □

<u>Theorem 5.18</u> [PP87; 2.11] *Let* p *be a type with* $R(p)<\infty$ *where* $R(-)$ *is any of the ranks* $R(-,\lambda)$. *Then* $R(p)=UR(p)$.

Proof It is known (see [Pi83; 6.29, 6.36]) that in general $UR(p)\leqslant R(p)$. The proof that $UR(p)=R(p)$ goes by induction on $R(p)$.

Suppose that $R(p)=\alpha$ and let $\beta<\alpha$. By **5.15** there is some $\varphi(\bar{v},\bar{a})\in p^+$ with $R(p)=\alpha$. By **5.16** there is a pp formula $\psi(\bar{v},\bar{b})$ with $\psi(\bar{v},\bar{b})\to\varphi(\bar{v},\bar{a})$ and $R(\psi)=\beta$.

Let H be the connected component of the group $\psi(\bar{v},\bar{0})$; thus $H=\bigcap\{\varphi'(\bar{v}) : \varphi'$ is pp and $[\psi(\bar{v},\bar{0}):\psi(\bar{v},\bar{0})\wedge\varphi'(\bar{v})]$ is finite$\}$. Noting that, by **5.17**, the index $[\varphi(\bar{v},\bar{0}):\psi(\bar{v},\bar{0})]$ is infinite, one has $H<G_0(p)$ (by **2.4**).

So by **5.11** there is a, necessarily forking, extension q of p with $G(q)=H$. For $\psi'\in G(q)$ one has $R(\psi')\geqslant R(\psi(\bar{v},\bar{0}))=\beta$ (**5.17**) and so $R(q)\geqslant\beta$ (**5.14**). Since also $\psi(\bar{v},\bar{0})\in G(q)$ it follows that $R(q)=\beta$. So, by the induction hypothesis, $UR(q)=\beta$.

Thus for every $\beta<\alpha$ there is a forking extension of p with U-rank β. So, since $R(p)=\alpha$ and in general $UR\leqslant R$, it follows that $UR(p)=\alpha$. □

The coincidence of Morley rank and U-rank was shown by Bouscaren [Bou79; Prop 4] for the largest theory of injective (rather, absolutely pure) modules over a right coherent ring.

Corollary 5.19 [PP87; 2.13] *If* T *is a superstable theory of modules then the* U-*rank is continuous. In particular, if* p *is a type then there is a formula in its pp-part which has the same* U-*rank as* p. □

The result is immediate by **5.18**, (v) above and **5.15**.

It follows that in the context of modules it does make sense to refer to the U-rank of a formula φ: this should be defined as the foundation rank of $G_0(\varphi)=\bigcap\{\psi : \psi$ is pp and of finite index in $\varphi\}$ in PP_0. One then has as an immediate corollary of **5.18** and **5.12** the following local or relative version of **5.18**. It was obtained in the special case where T is the largest complete theory of injective modules over a right noetherian ring in [Gar80a; Thm6]: also see [Bou79; Prop4].

Corollary 5.20 [PP87; 2.12] *Let* $\varphi(\bar{v},\bar{a})$ *be pp with* $\mathrm{R}(\varphi(\bar{v},\bar{a}))<\infty$. *Then* $\mathrm{R}(\varphi)$ *is the foundation rank of* $G_0(\varphi(\bar{v},\bar{0}))$ *in* PP_0. *If* p *is such that* $\varphi(\bar{v},\bar{a})\in p$ *and* $\mathrm{R}(p)=\mathrm{R}(\varphi(\bar{v},\bar{a}))$, *then this common value is also* $\mathrm{UR}(p)$. □

Corollary 5.21 [PP87; 2.14] *If* M, M' *are modules with* M *pure in* M' *and if* $\chi(\bar{v},\bar{a})$ *is any formula with* $\bar{a}$ *in* M, *then* $\mathrm{R}^M(\chi)\leq \mathrm{R}^{M'}(\chi)$, *where superscripts indicate the theory with respect to which the rank is being measured.*

Proof This is immediate from **5.14**, properties of pp formulas and **5.12**. □

The next result is effectively [BR84a; Cor1] and it follows from **3.1**, **2.23** and **5.12** above; the result after that is then an immediate corollary.

Corollary 5.22 *Suppose that* $N \succ M$ *are superstable. Let* $a\in N\setminus M$. *Then the* U-*rank of* a *over* M *is equal to the Morley rank of its image* $a+M/M$, *regarded as an element of the monster model of* $\mathrm{Th}(N)$. □

Corollary 5.23 [Zg84; 2.3], [BR84a; Lemma3.2] *Let* T *be a complete theory of modules. Then* T *is superstable iff* T_{U}, *its unlimited part, is totally transcendental.* □

Example 5 For any integer n there is an abelian group with Morley rank n: take $\mathbb{Z}_{p^n}{}^{\aleph_0}$ where p is any prime (so PP_0 is a single chain of length n).

Example 6 Let p be any prime. Then $\mathrm{MR}(\mathbb{Z}_{p^\infty}{}^{\aleph_0}))=\omega$. One may check that $\mathrm{MR}(\bigoplus\{\mathbb{Z}_{p^\infty}{}^{\aleph_0} : p$ is prime$\})$ also is ω. Since (Exercise 3.1/6) every t.t abelian group has the form $\bigoplus_p \mathbb{Z}_{p^\infty}{}^{(\kappa_p)}\oplus \mathbb{Q}^{(\kappa)}\oplus B$ where B has bounded exponent (so finite Morley rank) and κ_p,κ are cardinals, it follows now, without much difficulty, that for any abelian group M, $\mathrm{MR}(M)<\omega+\omega$ provided $\mathrm{MR}(M)<\infty$, and this bound is optimal [Rot78]. Hence (consider the unlimited part) if M is a superstable abelian group, then $\mathrm{UR}(M)<\omega+\omega$ (and this bound is best possible) [BR84a; Thm 3].

Example 7 $R=\mathbb{Z}$, $M=\mathbb{Z}_{(p)}$. The only type in $S_1(0)$ with Morley rank defined is the trivial algebraic type $\langle v=0\rangle$. The type $p\in S_1(0)$ given by $p^+(v)=\langle p^n | v : n\in\omega\rangle$ has U-rank 1 (since $|\mathrm{PP}_0|=2$) and is stationary (by **5.4**, since $G(p)$ is connected). Since PP_0 is just a two-point chain, every other type q in $S_1(0)$ has $G_0(q)=G_0(p)$; hence $\mathrm{UR}(q)=1$.

Therefore $\mathrm{UR}(T)=1$.

Example 8 $M=(\mathbb{Z}_4\oplus\mathbb{Z}_2{}^{\aleph_0})_{\mathbb{Z}}$. There are exactly four types in $S_1(0)$ (there are two types of elements of order 2) and it is easy to see that $|\mathrm{PP}_0|=2$. Since M is t.t. one therefore has (using property (iv)) $\mathrm{UR}(M)=\mathrm{MR}(M)=1$.

On the other hand, it is easy to see that, in $M^{\aleph_0}$, PP_0 is a single chain of length 3; hence $MR(M^{\aleph_0})=3$.

Exercise 4

(i) Show that $MR(\mathbb{Z}_4{}^{\aleph_0}\oplus\mathbb{Z}_2)=2$.

(ii) Show that $MR(\mathbb{Z}_4{}^{\aleph_0}\oplus\mathbb{Z}_8{}^{\aleph_0})=5$ (cf. 11.39).

(ii) Suppose that M is an abelian group of bounded exponent. Give a formula for $MR(M)$ in terms of the canonical decomposition of M.

Exercise 5

(i) Let R be a left artinian ring. Find an algebraic estimate for $MR(R)$. The estimate should be exact if R is, for example, an algebra over an infinite central subfield.

(ii) Find the Morley rank of each of the examples in Ex 2.1/6.

Example 9 [Gar80; Thm 6], [Bou79; §I.4] Suppose that R is right noetherian and let T be the largest theory of injective modules (so T is t.t.). Then the Morley rank of T is the depth of the zero ideal (i.e., its foundation rank in the opposite of the lattice of right ideals of R).

The ranks discussed above take no account of intervals where the quotient group is finite: algebraically this may be inappropriate. So we may define the **pp-rank**, pp-rk(φ), of a pp formula φ to be its foundation rank in the lattice of *all* pp-definable subgroups. Thus pp-rk$(v=v)<\infty$ iff T is totally transcendental, and then pp-rk$(\varphi)=MR(\varphi)$ for all pp φ iff $T=T^{\aleph_0}$.

5.3 An algebraic characterisation of independence

In this section, independence will be given an algebraic characterisation - one in terms of direct-sum decompositions of models. Roughly, it says the following: B and C are independent over 0 iff they are contained in complementary direct summands of some pure-injective model. But an example like $T=\mathrm{Th}(\mathbb{Z}_3\oplus\mathbb{Z}_2{}^{\aleph_0})$ and $B=C=\mathbb{Z}_3$ shows that the direction "$\Rightarrow$" of the "iff" above is literally false (for $\mathbb{Z}_3$ is algebraic so, trivially, is independent from any set). On the other hand, the implication "$\Leftarrow$" is an easy consequence of **5.5**, and in fact the equivalence between these model-theoretic and algebraic expressions of independence does hold if $T=T^{\aleph_0}$ (5.36).

What we need is an analysis which allows us to take account of "trivial" independence such as in the example just given. We will also want to be able to deal with the case "B and C are independent over A".

Let us prove the easy half immediately: this will also give us a goal to work towards for a converse.

Proposition 5.24 [Gar81; Lemma 4], [PP87; 1.4] *Given sets A,B,C of parameters suppose that there is a direct summand $N=N_1\oplus N_0\oplus N_2$ of $\tilde{M}$ with $B\subseteq N_1\oplus N_0$, $C\subseteq N_0\oplus N_2$, $N_0=N(A)$ and such that $B\perp N_0/A$. Then $B\perp C/A$.*

Proof The criterion of 5.5 is used. So suppose that one has $\varphi(\bar{b},\bar{c},\bar{a})$ with φ pp, with $\bar{b}$ in B, $\bar{c}$ in C and $\bar{a}$ in A. Set $\bar{b}=(\bar{b}_1,\bar{b}_0,\bar{0})$ and $\bar{c}=(\bar{0},\bar{c}_0,\bar{c}_1)$ according to the given decomposition of N. One has, in particular, $\varphi(\bar{b},\bar{c}_0,\bar{a})$.

Since N_0 is assumed independent from B over A, 5.5 implies that there is $\psi(\bar{b},\bar{a}')$ true with ψ pp, $\bar{a}'$ in A and $\psi(\bar{v},\bar{0})\wedge\varphi(\bar{v},\bar{0},\bar{0})$ of finite index in $\psi(\bar{v},\bar{0})$.

But then one has the criterion of 5.5 satisfied for B,C and A. Hence $B\perp C/A$. □

Corollary 5.25 *Suppose that* A *is a direct summand of* $\tilde{M}$. *Suppose also that* $B, C \subseteq \tilde{M}$ *are such that there is* $N \in \mathcal{P}(T)$ *of the form* $N = N_1 \oplus A \oplus N_2$ *with* $B \subseteq N_1 \oplus A$ *and* $C \subseteq A \oplus N_2$. *Then* $B \downarrow C / A$. □

It is necessary in 5.24 to take care over the choice of copy of $N(A)$ - to ensure that it is independent over A from at least one of B and C - before direct-sum decomposing. This applies even if A is pure in $\tilde{M}$, indeed, even if $A \prec \tilde{M}$ - as the next example shows.

Example 1 Let A be such that $N(A)$ is not algebraic over A. For example $A = \mathbb{Z}_{(p)}$ in $T = \mathrm{Th}(A_{\mathbb{Z}})$. Let $C = B = N(A)$ - a particular choice of copy of the hull of A. Then certainly $B \not\downarrow C / A$. On the other hand, taking $N = N_0 = N(A)$ one has the situation in 5.24, excepting the requirement $B \downarrow N_0 / A$.

In the example described at the beginning of this section the problem arises because $\mathbb{Z}_3$ is limited to one occurence in each (decomposition of a) model. Recall that we have encountered this situation already (§4.5) and introduced the terms "limited" and "unlimited" for types and members of $\mathcal{P}(T)$.

The following may now be proved.

Proposition 5.26 (see [Pr85; 0.13]) *Let* p *be an unlimited type over* 0. *Then* p *is stationary. Also* $\varphi(\bar{v}, \bar{y})$ *is a pp formula in the bound of* p *iff* $\varphi(\bar{v}, \bar{0}) \in p$.

Proof By 4.42, $G(p) = G_0(p)$, so p is stationary (5.4), and also the description of the bound of p follows (by 5.7). □

Corollary 5.27 (see [Pr85; 0.14]) *Let* p *be an unlimited type over* 0 *and let* M *be any model. Consider* $M \oplus N(p)$. *Let* $\bar{a}$ *in* $N(p)$ *be any realisation of* p. *Then the type of* $\bar{a}$ *over* M *is* p^M *- the unique non-forking extension of* p *to* M.

Proof Suppose that $\varphi(\bar{v}, \bar{y})$ is represented in $\mathrm{tp}(\bar{a}/M)$: say $\varphi(\bar{a}, \bar{m})$ holds, where $\bar{m}$ is in M. Projecting to $N(p)$ gives $\varphi(\bar{a}, \bar{0})$ - so $\varphi(\bar{v}, \bar{0}) \in p(\bar{v})$. Thus, by 5.26 (and 5.A), $\mathrm{tp}(\bar{a}/M)$ is the non-forking extension of p to M. □

Next, I extend the notion of (un-)limited so as to be able to deal efficiently with independence over sets other than 0. Let A and B be sets of parameters. Say that B is unlimited over A if, when one sets $N(A \cup B) = N(A) \oplus N'$ for some N', one has N' unlimited. Note that this is well-defined and that it is really a property of the type of B over A (modulo the over-theory).

Generalising 5.26 one obtains the following.

Proposition 5.28 *Suppose that* B *is unlimited over* $A \in \mathcal{P}(T)$. *Then* $\mathrm{tp}(B/A)$ *is stationary and a pp formula is in the bound of* $\mathrm{tp}(B/A)$ *iff it is represented in* $\mathrm{tp}(B/A)$.

Proof Set $N(A \cup B) = A \oplus N'$. Let M be a model containing A, and consider $M \oplus N''$ where $N'' \simeq N'$ - this is a model since, by hypothesis, N' is unlimited. Applying an automorphism of the monster model if necessary, it may be supposed that $N'' = N'$.

Suppose $\varphi(\bar{v}, \bar{m}) \in \mathrm{pp}(B/M)$. As in the proof of 5.27, projecting to the factor $A \oplus N'$ of $M \oplus N'$ shows that $\varphi(\bar{v}, \bar{y})$ already is represented in $\mathrm{tp}(B/A)$. Then the result follows as in the case $A = 0$. □

Now one obtains the converse for 5.25.

Theorem 5.29 [PP87; 1.5] *Let* A *be a direct summand of* $\tilde{M}$ *and suppose that* B, C *are such that at least one of* B, C *is unlimited over* A. *Then* $B \downarrow C / A$ *iff there is* $N = N_1 \oplus A \oplus N_2$ *a direct summand of* $\tilde{M}$ *with* $B \subseteq N_1 \oplus A$ *and* $C \subseteq A \oplus N_2$.

Proof The direction "$\Leftarrow$" is 5.25.

For "$\Rightarrow$", one may suppose that B, C are modules which contain A. So one has $B = B_1 \oplus A$ and $C = A \oplus C_2$ for suitable modules B_1, C_2. Set $N_1 = N(B_1)$, $N_2 = N(C_2)$ and let us suppose that B, hence B_1, is unlimited over A. Choose a model M containing $N(C) = A \oplus N_2$ as a direct summand. Note that there is a copy N_1' of N_1 with $N_1' \oplus M$ a model. Let B_1' be the corresponding copy of B_1.

Thus we have $N_1' \oplus A \oplus N_2$ a direct summand of $\tilde{M}$. By hypothesis, $B_1 \perp C_2/A$ and, by construction and 5.28, $B_1' \perp C_2/A$. By 5.28 the type of B_1 over A is stationary; so $\mathrm{tp}(B_1/C_2{}^\frown A) = \mathrm{tp}(B_1'/C_2{}^\frown A)$. Hence there is an automorphism f (of $\tilde{M}$) which fixes $C_2{}^\frown A$ and takes B_1' to B_1.

Then $N = fN_1' \oplus A \oplus N_2$ is the required direct summand of $\tilde{M}$, with appropriate decomposition. ◻

In particular, one has the following corollaries (the second of which follows from the first by Zorn's Lemma) and then the converse for 5.24.

Corollary 5.30 (cf. [Gar81; Thm 1]) *Suppose that at least one of B, C is unlimited over 0. Then $B \perp C/0$ iff there is a direct summand $N_1 \oplus N_2$ of $\tilde{M}$, with $B \le N_1$ and $C \le N_2$.* ◻

Corollary 5.31 *Suppose that B_i is unlimited over 0 for each $i \in I$ and also suppose that $B_i \perp C_i/0$ for each $i \in I$, where $C_i = \bigcup\{B_j : j \in I, j \neq i\}$. Then $\bigoplus_i N(B_i)$ purely embeds in the monster model.* ◻

Corollary 5.32 *Suppose that at least one of B, C is unlimited over A. Then $B \perp C/A$ iff, given $N(A)$ chosen independent from $B \cup C$ over A, there is a direct summand $N = N_1 \oplus N(A) \oplus N_2$ of $\tilde{M}$ with $B \le N_1 \oplus N(A)$ and $C \le N(A) \oplus N_2$.*

Proof The choice of $N(A)$ means that $B \perp C/A$ implies $B \perp C/N(A)$: then 5.29 applies (if B is unlimited over A then, since $B \perp N(A)/A$, B certainly is unlimited over $N(A)$).

The direction "$\Leftarrow$" follows from 5.24. ◻

5.4 Independence when $T = T^{\aleph_0}$

The descriptions of independence and non-forking are simplified in theories closed under product; indeed the main results of section 3 were first discovered by Garavaglia [Gar81] in this case (though he only considered independence over 0).

Here I quickly state these simplified forms as corollaries of results of the last section, derive some results which hold in this special case, then give a category-theoretic characterisation of independence in this context. This section is largely based on [PP83].

It is notable that in this ($T = T^{\aleph_0}$) situation the main algebraic and model-theoretic notions often coincide.

The first result is a consequence of 5.3 and 5.4.

Theorem 5.33 [PP83; 3.3] *Suppose that $T = T^{\aleph_0}$ and let p, q be types with q extending p. Then q is a non-forking extension of p iff p^+ proves q^+; this is the case iff $G(q) = G(p)$.*
Furthermore, every type is stationary. ◻

Actually, 5.33 is true in wider circumstances. This seems to have been discovered independently by a number of people: indeed a number of such generalisations seem to be "folklore" (the folk include Makkai, Srour, Saffe+Palyutin+Starchenko). I have taken the proof below from [Kuc87], where it is attributed to Makkai.

Theorem 5.34 *Suppose that T is a complete stable theory, the class of models of which is closed under formation of products. Suppose also that T has pp-elimination of quantifiers. Let $p \leq q$ be types. Then q is a non-forking extension of p iff $p^+ \vdash q^+$. Moreover, every type is stationary.*

Proof The first point to note is that, even in this generality, pp formulas commute with direct product and direct summand (exercise). It will be sufficient to show that if $p \in S(A)$ and if M is a large containing model in the sense of §1 (before 5.1), then there is a unique extension q of p to M with $p^+ \vdash q^+$ (for then the orbit of q under $\mathrm{Aut}_A M$ is just $\{q\}$, and so q is the unique non-forking extension of p to M).

The proof begins as does that of 5.2: it is sufficient to show that the following set of formulas is consistent: $p^+ \cup \{\neg\psi(\overline{v},\overline{m}) : \psi$ pp, $\overline{m}$ in M and p^+ does not prove $\psi(\overline{v},\overline{m})\}$. The difference is that we don't have Neumann's Lemma available.

If this set were inconsistent then there would be pp formulas $\psi_i(\overline{v},\overline{m}_i)$ such that for each i one had $p^+ \vdash \neg\psi_i(\overline{v},\overline{m}_i)$ yet $p^+ \vdash \bigvee_1^n \psi_i(\overline{v},\overline{m}_i)$. For each i take $\overline{c}_i$ in M with $p^+(\overline{c}_i) \wedge \neg\psi_i(\overline{c}_i,\overline{m}_i)$. Consider the element $\overline{c} = (\overline{c}_i)_i \in M^n$; clearly the (pp-)type of this is $p^{(+)}$. Yet, on considering the components, we see that $\neg\psi_i(\overline{c},\overline{m}_i)$ holds for each i. This contradicts $p^+ \vdash \bigvee_1^n \psi_i(\overline{v},\overline{m}_i)$, as required. □

If T is stable and $\mathrm{Mod}(T)$ is closed under products, then pp formulas are normal (under any partition into free/parameter variables): indeed, T is normal. More generally, if T is normal and closed under products then every formula is equivalent to a boolean combination of those formulas which "commute" with direct products (*i.e.*, the formula holds of a tuple from the product iff it holds at each component), and these formulas are normal. These formulas are the Horn formulas and certain others which may be explicitly described. The above theorem is true under these weaker hypotheses. This is considered in [SPS85] (also see [BaLa73]).

The next result follows by 5.29, since all types are unlimited when $T=T^{\aleph_0}$. The result after that then follows by 5.33 and 5.35.

Theorem 5.35 [PP83; 5.3] *Suppose that $T=T^{\aleph_0}$. Let A be a direct summand of $\tilde{M}$ and let B,C be arbitrary subsets of $\tilde{M}$. Then $B \downarrow C/A$ iff there is $N=N_1 \oplus A \oplus N_2$ - a direct summand of $\tilde{M}$ - with $B \leq N_1 \oplus A$ and $C \leq A \oplus N_2$.* □

Corollary 5.36 [Gar81; Lemma3, Thm 1] *Suppose that $T=T^{\aleph_0}$. Given $B,C \leq \tilde{M}$ the following conditions are equivalent:*

(*i*) $B \downarrow C/0$;

(*ii*) *whenever $\varphi(\overline{b},\overline{c})$ holds with $\overline{b}$ in B, $\overline{c}$ in C and φ pp, then $\varphi(\overline{b},\overline{0})$ holds*;

(*iii*) *there is a direct summand $N_1 \oplus N_2$ of $\tilde{M}$ with $B \leq N_1$ and $C \leq N_2$*;

(*iv*) $N(B \cup C) \simeq N(B) \oplus N(C)$. □

If purity assumptions are added for some of A,B,C then the description of independence simplifies further.

Proposition 5.37 [PP83; 5.2] *Suppose that $T=T^{\aleph_0}$. Let A,B,C be submodules of $\tilde{M}$ and suppose that A is pure in each of B and C.*

(*a*) *If B is independent from C over A then $B \cap C = A$.*

(*b*) *If A is pure in $\tilde{M}$ then $B \downarrow C/A$ iff, whenever $\overline{b}$ is in B, $\overline{c}$ is in C and the pp formula $\varphi(\overline{b},\overline{c})$ holds, then $\varphi(\overline{b},\overline{a})$ holds for some $\overline{a}$ in A.*

(*c*) *If A is pure in $\tilde{M}$ and if $B+C$ also is pure in $\tilde{M}$ then B is independent from C over A iff $B \cap C = A$.*

Proof (a) Observe first that (quite generally) if $B \downarrow C/A$ then $B \cap C$ is algebraic over A. For let $b = c \in B \cap C$ and let $\varphi(v,w)$ be the formula "$v = w$" - so $\varphi(b,c)$ holds. By 5.5 there is ψ pp and $\bar{a}$ in A with $\psi(b,\bar{a})$ true and $\varphi(v,0)$ (=0!) of finite index in $\psi(v,\bar{0})$. Hence $\psi(v,\bar{0})$ is finite, so $\psi(v,\bar{a})$ is finite. Therefore, since $\psi(b,\bar{a})$ holds, b is indeed algebraic over A.

In the $T = T^{\aleph_0}$ case $\psi(v,\bar{0})$, being finite, must actually be 0; so there is just one solution of $\psi(v,\bar{a})$ in $\tilde{M}$, so (in particular) in B. But A pure in B implies that $\psi(A,\bar{a})$ is non-empty. Hence $b \in A$, as required.

(b) The direction "$\Leftarrow$" is immediate from 5.5.

For "$\Rightarrow$", 5.5 plus $T = T^{\aleph_0}$ implies that there is some pp ψ and $\bar{a}$ in A with $\psi(\bar{b},\bar{a})$ and $\psi(\bar{v},\bar{0}) \to \varphi(\bar{v},\bar{0})$. From $\varphi(\bar{b},\bar{c}) \wedge \psi(\bar{b},\bar{a})$ one has $\tilde{M} \models \exists \bar{v},\bar{w}\,(\varphi(\bar{v},\bar{w}) \wedge \psi(\bar{v},\bar{a}))$. So, since A is pure in $\tilde{M}$, A also satisfies this sentence. Thus there is $\bar{a}'$ in A with $(A \models)\ \exists \bar{v}\,(\varphi(\bar{v},\bar{a}') \wedge \psi(\bar{v},\bar{a}))$. Thus $\varphi(\bar{v},\bar{a}) \cap \psi(\bar{v},\bar{a}) \neq \emptyset$. Then, since $\psi(\bar{v},\bar{0}) \to \varphi(\bar{v},\bar{0})$, one concludes $\psi(\bar{v},\bar{a}) \to \varphi(\bar{v},\bar{a}')$. Hence $\varphi(\bar{b},\bar{a}')$ holds, as required.

(c) One direction follows from (a). For the other we apply the criterion for independence in part (b). So suppose that $\varphi(\bar{b},\bar{c})$ as in part (b) is $\exists \bar{v} \bigwedge_j t_j(\bar{b}) + t_j'(\bar{c}) + t_j''(\bar{v}) = 0$ where the $t_j^{(-)}$ are terms (linear combinations). Since $B + C$ is pure in $\tilde{M}$ there is $\bar{d} = \bar{b}' + \bar{c}'$ (say) with $\bar{b}'$ in B and $\bar{c}'$ in C, such that $\bigwedge_j t_j(\bar{b}) + t_j'(\bar{c}) + t_j''(\bar{b}' + \bar{c}') = 0$. By linearity this yields $\bigwedge_j t_j(\bar{b}) + t_j''(\bar{b}') = -t_j'(\bar{c}) - t_j''(\bar{c}')$. Then, since $B \cap C = A$, one has, for each j, both sides equal to $a_j \in A$ (say).

In particular this gives $\bigwedge_j t_j'(\bar{c}) + t_j''(\bar{c}') = -a_j$. Since A is pure in U, there are $\bar{a}', \bar{a}''$ in A such that $\bigwedge_j t_j'(\bar{a}') + t_j''(\bar{a}'') = -a_j$. Substitute this back, to obtain $\bigwedge_j t_j(\bar{b}) + t_j''(\bar{b}') = -t_j'(\bar{a}') - t_j''(\bar{a}'')$. Re-arrange to get $\bigwedge_j t_j(\bar{b}) + t_j'(\bar{a}') + t_j''(\bar{b}' + \bar{a}'') = 0$. Hence $\varphi(\bar{b},\bar{a}')$ holds, as required. □

When $T = T^{\aleph_0}$ there is a characterisation of independence as pushout. This is set in the category $\mathcal{C}_T$ defined below.

Consider the category $\mathcal{K}_T$ which has as its objects submodules of models of T (i.e., models of $T_\forall$) and which is a full subcategory of $\mathcal{M}_R$. In passing from T to $\mathcal{K}_T$ some information, namely that whose expression requires quantifiers, is lost (consider $\mathrm{Th}(\mathbb{Z}_4^{(\aleph_0)})$ and $\mathrm{Th}(\mathbb{Z}_2^{(\aleph_0)} \oplus \mathbb{Z}_4^{(\aleph_0)})$). The category $\mathcal{C}_T$ is designed to retain such information. I define $\mathcal{C}_T$ abstractly at first and then show how "small" parts of it may be realised within $\tilde{M}$. Actually the definition of $\mathcal{C}_T$ makes sense for any complete theory (of modules or otherwise) and so the definition is made in this generality.

Let T be any complete theory. The objects of $\mathcal{C}_T$ are pairs $(A, p = p_A(\bar{v}))$ where A is a set of symbols for constants, $\bar{v}$ is a sequence of variables indexed by A (as $\bar{a}$) and $p(\bar{v})$ is a set of pp formulas which is closed under conjunction and pp-implication modulo T (p is to be regarded as the pp-type of A), such that $T \cup p(\bar{v}) \cup \neg p^-(\bar{v})$ is consistent, $\neg p^-(\bar{v})$ being $\{\neg\varphi(\bar{v}) : \varphi$ is pp and not in $p(\bar{v})\}$.

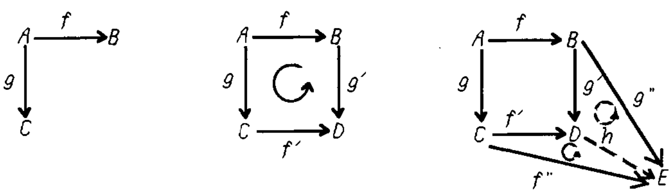
The morphisms of $\mathcal{C}_T$ are of the form $f : (A, p_A) \longrightarrow (B, p_B)$ where $A \xrightarrow{f} B$ is any map such that if $\varphi(\bar{a}) \in p_A(\bar{a})$ then $\varphi(f\bar{a}) \in p_B(\bar{b})$. That is, the morphisms of $\mathcal{C}_T$ are the "pp-type"-preserving maps (note that this implies, for example, linearity in the module case).

If $\tilde{M}$ is the very saturated model, then one may interpret "small" parts of $\mathcal{C}_T$ within $\tilde{M}$ by identifying (A, p_A) with a subset, A, of $\tilde{M}$ whose pp-type in $\tilde{M}$ is exactly p_A. The morphisms of $\mathcal{C}_T$ then become the pp-preserving maps between subsets of $\tilde{M}$. In the modules case, these are the restrictions of endomorphisms of $\tilde{M}$ and isomorphisms of $\mathcal{C}_T$ are the restrictions of automorphisms of $\tilde{M}$.

The main theorem, **5.40** below, seems to be the result of a happy marriage between the fact that it is precisely the pp formulas which are preserved by (algebraic) morphisms and the fact that it is just those formulas which determine complete types in modules.

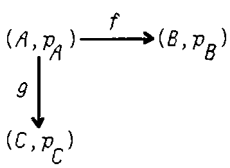
Before specialising to modules, a general result will be proved. Write A for (A, p_A) when no ambiguity is thereby produced.

Recall that, given a diagram as shown in the category $\mathcal{C}$, a commutative diagram as shown in the middle is a (or "the" since it is essentially unique) pushout of the first diagram if it is the least such commutative completion of the first diagram. That is, whenever one has the outer diamond commutative in the third diagram below, there is a unique morphism h, as shown, with $hg' = g''$ and $hf' = f''$.



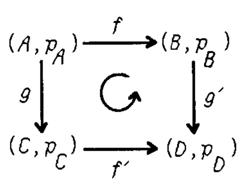
Pushouts in $\mathcal{C}_T$ will be related to a generalised notion of definition by generators and relations.

Consider a diagram of $\mathcal{C}_T$:

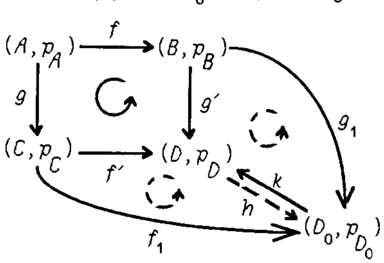


Expand the language L of T by constants symbols for the elements of B and C in such a way that if $a \in A$ then "a" will name both the element fa of B and the element ga of C (since no confusion should arise from my doing so, I do not distinguish between an element and the corresponding constant symbol). So one may regard $p_B(\overline{b}) \cup p_C(\overline{c})$ as the "pp-diagram" of this diagram of $\mathcal{C}_T$.

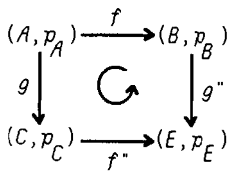
<u>Proposition 5.38</u> [PP83; 4.1] *Suppose that the diagram below is a diagram of* $\mathcal{C}_T$. *Then this is a pushout diagram iff:*
(i) D *is the union of the images of* f' *and* g'; *and*
(ii) $\varphi(g'\overline{b}, f'\overline{c}) \in p_D(\overline{d})$ *iff* $T \cup p_B(\overline{v}) \cup p_C(\overline{w})$ *proves* $\varphi(\overline{v}, \overline{w})$.
Furthermore, if this is the case and if f *is strictly pp-type-preserving then so is* f'.



Proof $\Rightarrow$ (i) Set $D_0 = \mathrm{im} f' \cup \mathrm{im} g'$ and $p_{D_0} = "p_D \upharpoonright D_0"$ and consider the diagram shown. Here g_1, f_1 are the obvious co-restrictions and k is the inclusion (note that these are all in $\mathcal{C}_T$). By hypothesis, there exists h as shown. By the uniqueness clause in the definition of pushout, $kh = 1_D$. Hence k must be surjective, and $D = D_0$ as required.

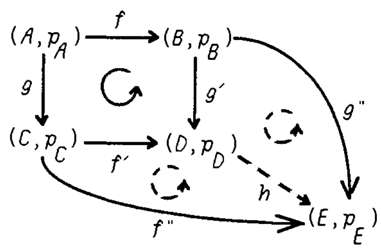


(ii) Certainly if $T \cup p_B(\overline{v}) \cup p_C(\overline{w}) \vdash \varphi(\overline{v}, \overline{w})$ then, since $p_D(\overline{d})$ contains both $p_B(g'\overline{b})$ and $p_C(f'\overline{c})$ (by definition of morphism), it must be that $\varphi(g'\overline{b}, f'\overline{c})$ is in $p_D(\overline{d})$.



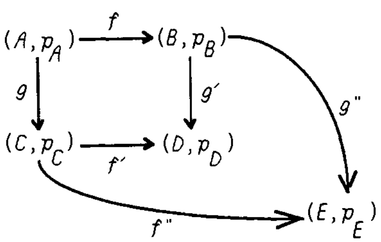
For the converse, suppose that $T \cup p_B(\bar{v}) \cup p_C(\bar{w})$ does not prove $\varphi(\bar{v},\bar{w})$. So there is a model for $T \cup p_B(\bar{v}) \cup p_C(\bar{w}) \cup \{\neg\varphi(\bar{v},\bar{w})\}$. Restrict this model to the set E consisting of the parameters corresponding to $\bar{v}\frown\bar{w}$, so that one has the diagram opposite in $\mathcal{C}_T$ with $\varphi(g''\bar{b}, f''\bar{c}) \notin p_E(\bar{e})$.

Then consider the diagram:



where h exists by hypothesis, making $g'' = hg'$ and $f'' = hf'$. But then, if one had $\varphi(g'\bar{b}, f'\bar{c}) \in p_D(\bar{d})$, applying h would yield $\varphi(hg'\bar{b}, hf'\bar{c}) \in p_E(\bar{e})$. That is $\varphi(g''\bar{b}, h''\bar{c}) \in p_E(\bar{e})$ - contradiction, as required.

$\Leftarrow$ Suppose that we are given a diagram as shown with the outer diamond commutative.



Since, by hypothesis, $D = \mathrm{im}\, g' \cup \mathrm{im}\, f'$ one may define a mapping of sets $h: D \longrightarrow E$ by the conditions $g'' = hg'$ and $f'' = hf'$. It must be shown that h is pp-type-preserving.
So suppose that $\varphi(g'\bar{b}, f'\bar{c})$ is in $p_D(\bar{d})$ (by (i) this is a typical formula of $p_D(\bar{d})$). Then by (ii) one has $T \cup p_B(\bar{v}) \cup p_C(\bar{w}) \vdash \varphi(\bar{v},\bar{w})$. Since g'' and f'' are pp-type-preserving one has $p_B(g''\bar{b}) \cup p_C(f''\bar{c})$ included in $p_E(\bar{e})$.
Hence $\varphi(g''\bar{b}, f''\bar{c}) \in p_E(\bar{e})$; that is, $\varphi(hg'\bar{b}, hf'\bar{c}) \in p_E(\bar{e})$ and h is indeed a $\mathcal{C}_T$-morphism.

Notice that h is unique by hypothesis (i).

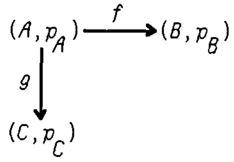
Finally suppose that f is strictly pp-type-preserving. Let $\bar{c}$ be in C and let $\varphi(\bar{z})$ be a pp formula in $\bar{z} \subseteq \bar{w}$ such that $\varphi(f'\bar{c})$ holds. So $\varphi(f'\bar{c}) \in p_D(\bar{d})$: hence by (ii) it must be that $T \cup p_B(\bar{v}) \cup p_C(\bar{w})$ proves $\varphi(\bar{z})$. It follows that if $\bar{u} = \bar{v} \setminus \bar{z}$ then $T \cup \exists\bar{u}\, p_B(\bar{z},\bar{u}) \cup p_C(\bar{w})$ proves $\varphi(\bar{z})$. But by assumption on f, one has $\exists\bar{u}\, p_B(\bar{z},\bar{u})$ contained in the intersection of the pp-type of "$\bar{z}$" with p_A (this will be empty if no variable in $\bar{z}$ corresponds to an element of A) and hence $\exists\bar{u}\, p_B(\bar{z},\bar{u})$ is contained in $p_C(\bar{w})$. Thus $\varphi(\bar{c})$ is already in the pp-type of C. So f' is strictly pp-type-preserving, as required. □

The above result appears in [PP83], but with "monic" replacing "strictly pp-type-preserving" in the last sentence of the statement. It was pointed out to me by Rothmaler that the "proof" of that given there was certainly incomplete (indeed it is simply not true that if f is a 1-1 map then so is f': for a counterexample take T to be the theory of $\mathbb{Z}_{(p)}{}^{\aleph_0} \oplus \mathbb{Z}_p{}^{\aleph_0}$; take $\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}, \mathbb{Z}_p$ for A, B, C respectively; take f to be multiplication by p and take g to be reduction mod p - the details are left as an exercise). The correct statement is as given above: that is, "monic" has to be given what is, after all, its natural meaning in the category $\mathcal{C}_T$. The last part of 5.38 is used implicitly in the way that 5.40 is stated.

Henceforth, T is a complete theory of modules.

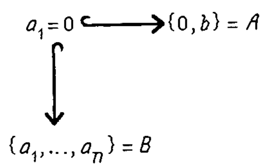
Proposition 5.39 [PP83; 4.2] *$\mathcal{C}_T$ has pushouts iff $T = T^{\aleph_0}$.*

Proof $\Leftarrow$ Given the diagram shown



consider the sets of formulas: $\Delta^+(\bar{v},\bar{w})=\{\varphi(\bar{v},\bar{w}) : \varphi$ is pp and $T\cup p_B(\bar{v})\cup p_C(\bar{w})\vdash\varphi(\bar{v},\bar{w})\}$; $\Delta^-=\{\neg\varphi(\bar{v},\bar{w}) : \varphi$ is pp and $\varphi(\bar{v},\bar{w})\notin\Delta^+(\bar{v},\bar{w})\}$. Set $\Delta=\Delta^+\cup\Delta^-$. By **5.38**, any model of $T\cup\Delta(\bar{v},\bar{w})$, reduced to the witnesses for $\bar{v},\bar{w}$, will provide us with a pushout for the diagram. So it need only be shown that $T\cup\Delta$ is consistent. But this is immediate since $T=T^{\aleph_0}$.

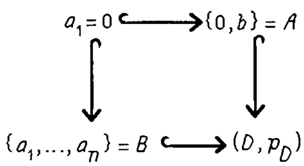
$\Rightarrow$ Suppose that $T\neq T^{\aleph_0}$. Then there are pp formulas $\varphi(v)$, $\psi(v)$ such that $\mathrm{Inv}(T,\varphi,\psi)=n$ and $1<n\in\omega$.



Pick, in $\tilde{M}$, elements $a_1=0,a_2,\dots,a_n$ to form a complete set of coset representatives of ψ in φ. Let b be any element of $\varphi(\tilde{M})\setminus\psi(\tilde{M})$. Set $A=\{a_1,\dots,a_n\}$ and $B=\{0,b\}$.

It will be shown that the diagram opposite has no pushout in $\mathcal{C}_T$.

Suppose that there were such a pushout: say

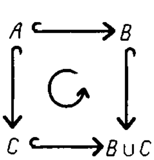


Then 5.38 would imply: $D=\{fa_1,\dots,fa_n,gb\}$ and, moreover, $\theta(f\bar{a},gb)\in p_D(\bar{d})$ iff $T\cup p_A(\bar{v})\cup p_B(\bar{w})\vdash\theta(\bar{v},\bar{w})$. It is then easy to see that, for all i, $\psi(fa_i-gb)\notin p_D(\bar{d})$. For $i=1$ this is so since $\neg\psi(b)$ holds; for $i>1$ note that $p_B(0,0)$ holds but $\psi(a_i)$ does not, so $\psi(fa_i-gb)$ cannot be in $p_D(\bar{d})$.

However, by definition of $\mathcal{C}_T$, $T\cup p_D(\bar{d})$ is consistent, yet $T\cup p_D(\bar{d})$ entails the existence of $n+1$ elements satisfying $\varphi(v)$ but lying in distinct cosets of $\psi(v)$ - contradicting $\mathrm{Inv}(T,\varphi,\psi)=n$.

Hence $\mathcal{C}_T$ does not have pushouts (actually it has been shown that $\mathcal{C}_T$ does not have coproducts - 0 is the initial object of $\mathcal{C}_T$ (if we exclude $\emptyset$)). ▫

Next, the main result of this section.

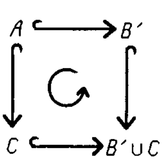


<u>Theorem 5.40</u> [PP83; 4.3] *Let $T=T^{\aleph_0}$ be a complete theory of modules and let A,B,C be parameters with $A\subseteq B,C$. Then B and C are independent over A iff the diagram shown, with canonical inclusions, is a pushout diagram in $\mathcal{C}_T$.*

Proof $\Rightarrow$ The conditions of **5.38** are checked. Certainly the first condition is satisfied, so consider the second.

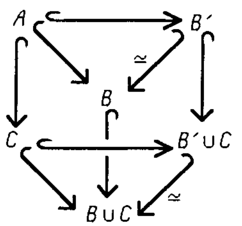
One direction is trivial. So suppose for the other that $\varphi(\bar{b},\bar{c})$ holds for some $\bar{b}$ in B, $\bar{c}$ in C and pp formula φ. By assumption and **5.33**, there is a pp formula $\psi(\bar{v},\bar{z})$ and $\bar{a}$ in A with: $\forall\bar{v}(\psi(\bar{v},\bar{0})\to\varphi(\bar{v},\bar{0}))$ (a sentence of T); $\exists\bar{v}(\psi(\bar{v},\bar{a})\wedge\varphi(\bar{v},\bar{w}))$ (in $p_C(\bar{w})$); $\psi(\bar{v},\bar{a})$ (in $p_B(\bar{v})$). Taken together, these formulas imply $\varphi(\bar{v},\bar{w})$, as required.

$\Leftarrow$ Let $p(\bar{v})$ be the type of B over A; let $p'(\bar{v})$ be its non-forking extension (note $T=T^{\aleph_0}$) to C and let B' realise p'.



Then $B'\mathrel{\unicode{x2AEB}}C/A$ so, by what has been proved already, the diagram opposite is a pushout in $\mathcal{C}_T$.

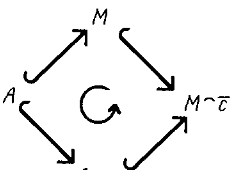
Since $\mathrm{tp}(B/A)=\mathrm{tp}(B'/A)$ there is a $\mathcal{C}_T$-isomorphism over A between B and B' as shown below.



So, by uniqueness of pushout, there is a $\mathcal{C}_T$-homomorphism between $B'\cup C$ and $B\cup C$ as shown.

Hence $B'\cup C$ and $B\cup C$ are isomorphic in $\mathcal{C}_T$ over A, so have the same type over A. Since B' and C are independent over A it follows that B and C are independent over A. □

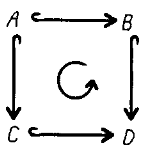
Exercise 1 Here is another proof of the direction "$\Leftarrow$" above. Note first (cf. §1, before **5.1**) that it is enough to show that if M is a sufficiently saturated containing model for A with $|M|>|A|+2^{\aleph_0}$ and if the diagram shown is a pushout in $\mathcal{C}_T$, then $\mathrm{tp}(\bar{c}/M)$ does not fork over A.



Suppose that $M, A, \bar{c}$ satisfy these conditions. Since the diagram is a pushout it follows (**5.38**) that, if $\overline{m},\overline{m}'$ are in M, if $\varphi(\bar{v},\bar{w})$ is pp with parameters in A and if $\mathrm{tp}(\overline{m}/A)=\mathrm{tp}(\overline{m}'/A)$, then $\tilde{M}\models\varphi(\bar{c},\overline{m})$ iff $\tilde{M}\models\varphi(\bar{c},\overline{m}')$. Let $p=\mathrm{tp}(\bar{c}/M)$: it follows by saturation of M and pp-elimination of quantifiers that if f is an A-automorphism of M then $fp=p$. Thus, the $\mathrm{Aut}_A M$-orbit of p is (very) small; so p does not fork over A.

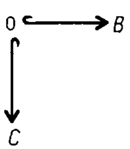
It has been remarked that there is no essential change produced if sets of parameters are replaced by the modules they generate. It might be that one wishes to deal only with modules of parameters. So it is useful to have at hand the statement of **5.40**, modified for the full subcategory $\mathcal{C}_T{}^*$ which has as its objects only the modules of parameters.

If $i:\mathcal{C}_T{}^*\longrightarrow\mathcal{C}_T$ is the inclusion functor and if $\gamma:\mathcal{C}_T\longrightarrow\mathcal{C}_T{}^*$ is the functor taking (B,p_B) to $(\gamma B,\gamma p_B)$, where γB is the module generated by B and $\gamma p_B=p_{\gamma B}$ is the corresponding pp-type, then clearly γ is left adjoint to i. Hence $\mathcal{C}_T{}^*$ is a reflective subcategory (see. e.g., [St75; §X.1]) of $\mathcal{C}_T$. This, together with the observation that B and C are independent over A iff γB and γC are independent over γA quickly yields the next result.



<u>Theorem 5.41</u> [PP83; 4.3*] *Let $T=T^{\aleph_0}$ be a complete theory of modules and let A,B,C be modules of parameters with $A\subseteq B,C$. Then B and C are independent over A iff the diagram shown, with canonical inclusions, is a pushout diagram in $\mathcal{C}_T{}^*$.* □

Example 1 In order to illustrate the difference between $\mathcal{K}_T$ and $\mathcal{C}_T{}^*$ consider the example $T=\mathrm{Th}(\mathbb{Z}_{(p)}{}^{\aleph_0})$. Within the indecomposable direct summand $\overline{\mathbb{Z}_{(p)}}$ of $\tilde{M}$ it is (Ex 4.1/1, also Exercise 4.2/3) possible to find non-zero submodules B,C with intersection 0. Thus $B+C$ is the pushout of the diagram shown in $\mathcal{K}_T$.



However, by **5.36**, B and C are not independent over 0 (this may be seen directly: there is $\overline{\mathbb{Z}_{(p)}}\oplus\overline{\mathbb{Z}_{(p)}}$ in $\mathcal{P}(T)$ and a $\mathcal{C}_T{}^*$-embedding of B, resp. C into the first, resp. the second, copy of $\overline{\mathbb{Z}_{(p)}}$). Hence $B+C$ is not the pushout of the diagram in $\mathcal{C}_T{}^*$ shown.

CHAPTER 6 STABILITY-THEORETIC PROPERTIES OF TYPES

The notion of dependence (forking) considered in the last chapter is the raw stuff of stability theory: for the analyses of models, we need some tools shaped from it. For instance, vectorspaces over a field are "unidimensional" in the sense that only one parameter (the dimension over the field) is needed to classify them. But, for classifying abelian groups of exponent 4, we need two "dimensions" (the number of copies of $\mathbb{Z}_2$ and the number of copies of $\mathbb{Z}_4$). These "dimensions" (unfortunately, the term is overworked) are exposed by using notions such as orthogonality and weight. The chapter is concerned with this "structural" level of stability theory.

Again, I have tried to include enough explanation for non-specialists, since some of the results are used elsewhere, and since an understanding of the main points adds another dimension to one's view of later results.

The first task is to extract that part of a type, its "free part", which controls the stability-theoretic properties of the type. This is done in section 1.

An element is said to dominate another if, whenever a third element is independent of the first, then it is independent of the second also. For instance, an element dominates all elements in any copy of its hull (at least, that is so for unlimited types). The exact connection between domination and hulls is clarified in §2. A type is RK-minimal if, whenever a realisation of it dominates another element, the latter also dominates the first. Since domination corresponds to realisation in prime pure-injective extensions, such RK-minimal types correspond to the simplest building blocks of models. Indecomposable pure-injectives are among these simplest blocks, but there can be others.

The idea in section 2 is that a pure-injective model can be built up around a skeleton of elements which correspond to minimal blocks. One may come from the other direction, and split a model into "orthogonal" pieces. Two types are orthogonal if (roughly) any realisation of the one is necessarily independent of any realisation of the other. It turns out (in general) that the minimal orthogonality classes are just the RK-minimal classes. In the third section, this is shown directly for modules.

Particularly important examples of RK-minimal types are the regular types. These are the types which are orthogonal to all their forking extensions, so they tend to enjoy properties similar to those of prime ideals (since they are "critical"). Outside of the superstable case, our main interest in them, as opposed to RK-minimal types, is that we may often show the existence of regular types with prescribed properties. In the fourth section regular types are characterised and a number of existence theorems are established for use in other parts of the notes.

There is a brief supplementary section which interprets what has been done, in the context of injective modules.

In the fifth section, saturation in modules is resolved into vertical and horizontal components and various notions of saturation and homogeneity are compared.

The sixth section is devoted to strong types. The multiplicity of a type p is described in terms of the index $[G(p):G_0(p)]$, and we finish the section by looking at the number of independent realisations of strong types in models.

Throughout this chapter, T is some fixed complete theory of modules, and M, M', M_1, ... will always be models of T.

6.1 Free parts of types and the stratified order

It has already been remarked that, outside the totally transcendental case, classification of all models should not be expected. In superstable theories which satisfy some further properties (see [Poi85; Chpt20], [Ho87] and references therein) which are satisfied in modules, there are good classification theorems for the $F^a_{\aleph_0}$-saturated models; it will be seen (6.37) that for superstable modules these are precisely the weakly saturated pure-injective models. The results for modules slightly improve on the general situation, in that one normally obtains theorems for pure-injective models (weakly saturated or not). The difference is not, however, very great since any pure-injective model may be expanded rather trivially, by adding on suitable direct summands, to obtain a weakly saturated model. Thus, again (cf. for example §§5.4, 12.3), the category in which we work is that of the pure-injective models (with pure = elementary = split embeddings). It will be seen that most of what is said concerns discrete pure-injectives; continuous pure-injectives have not been seriously investigated as yet. We see first that this category has "prime extensions". (In the general stable case, one would have only prime $|T|^+$-saturated extensions - see **[Poi85**; §18.d].)

Given $M \models T$ and $\bar{a}$ (in $\tilde{M}$) set $M(\bar{a}) = N(M \cup \{\bar{a}\})$. By **2.25**, $M(\bar{a})$ is a model of T. In fact, since $\bar{M}$ is a direct summand of $M(\bar{a})$, one has that $M(\bar{a})$ is an elementary extension of M. Moreover $M(\bar{a}) = \bar{M}(\bar{a})$ (or rather, there is an $M^\frown\bar{a}$-isomorphism between any choices of copies of the modules on either side of this equation).

<u>Lemma 6.1</u> *Suppose that p is a type over the model, M, of T. Let $\bar{a}$ and $\bar{a}'$ be realisations of p. Then there is an isomorphism $f: M(\bar{a}) \longrightarrow M(\bar{a}')$ which fixes M and takes $\bar{a}$ to $\bar{a}'$.*

Proof Since $\mathrm{tp}(\bar{a}/M) = \mathrm{tp}(\bar{a}'/M)$ one has $\mathrm{tp}(M^\frown\bar{a}) = \mathrm{tp}(M^\frown\bar{a}')$. So, by **4.15**, there is an isomorphism with the stated properties between any chosen hulls, $M(\bar{a})$ and $M(\bar{a}')$. ▫

Therefore if p is a type over a model M, one may, without ambiguity, set $M(p) = M(\bar{a})$ where $\bar{a}$ is any realisation of p. This is, in the following sense, a prime model extension of M by p in the category of pure-injective models.

<u>Lemma 6.2</u> *Let p be a type over the model M and suppose that M' is a pure-injective elementary extension of M which contains a realisation $\bar{a}$ of p. Then there is a copy of $M(p)$ which contains M purely, contains $\bar{a}$ and is a direct summand of M'.* ▫

This lemma is an immediate consequence of the existence of hulls (§4.1).

Now I define the stratified (pre-)order of Poizat **[Poi81]**. This is a coarsening of the fundamental order (§5.1) and, like the various stability ranks, allows one to compare types which do not have the same restriction to the empty set. I give the definition and just state some basic properties (refer to **[Poi81]** for details); then I compare this with ideas we have met already. Poizat defines the stratified order in any context with a definable group operation. Since our groups here are all abelian, I will use additive notation.

Let $p(\bar{v}) \in S(A)$ and suppose that $\bar{a}$ is in A with $l(\bar{a}) = l(\bar{v})$. Set $\bar{a} + p = \mathrm{tp}(\bar{a} + \bar{c}/A)$ where $\bar{c}$ is any realisation of p. Note that $\bar{a} + p$ is a well-defined element of $S(A)$ since $\chi(\bar{v}, \bar{b}) \in \bar{a} + p(\bar{v})$ iff $\chi(\bar{v} + \bar{a}, \bar{b}) \in p(\bar{v})$. If q is a non-forking extension of p then $\bar{a} + q$ is a non-forking extension of $\bar{a} + p$ (exercise).

Let p and q be types over models. Set $s(p) \geqslant s(q)$ if, for every formula $\chi(\bar{v}, \bar{w})$, if $\chi(\bar{u} + \bar{v}, \bar{w})$ is represented in $p(\bar{v})$ then it is represented also in $q(\bar{v})$. The resulting **stratified order** has the following properties (see **[Poi81]** for these, and more on the stratified order. Also see **[BR84a]** for discussion of the stratified order for modules).

(s0) If p and q are equivalent in the fundamental order (that is, since they are over models, if $\mathrm{cl}(p) = \mathrm{cl}(q)$) then $s(p) = s(q)$.

(s1) If p is a type over M and if $\bar{a}$ is in M, then $s(\bar{a} + p) = s(p)$ (strata are "translation-invariant").

(s2) If q extends the type p then $s(q) = s(p)$ iff q is a non-forking extension of p.

It will be seen that the stratified order for types over models is identical with that induced by inclusion of the corresponding groups $G_0(-)$. So, for types over models, the stratified order is just PP_0 (cf. §5.2).

Example 1 Take T to be the theory of the abelian group $M=\mathbb{Z}_2{}^{(\aleph_0)}\oplus\mathbb{Z}_4{}^{(\aleph_0)}$. Let M' be the model $M\oplus\mathbb{Z}_2\oplus\mathbb{Z}_4$ and let $m\in M$ be any element of order 4. With the obvious notation, let $a=(0,1,0)$, $b=(0,0,2)$, $c=(m,0,2)$ and let p,q,q' be their respective types over M.

Now, c is an element of order 4 and b is of order 2, so their types are not even comparable in the fundamental order. Yet q' is a translate, over M, of q: $q'=m+q$ and hence (s1) $\mathrm{s}(q')=\mathrm{s}(q)$. In fact, for most purposes of this chapter, q' is "equivalent" to q (q is actually the free part q'_* of q' - see below).

On the other hand, p is not equivalent in the stratified order to q.

Now we come to a central idea: that of the free part of a type. Consider a type $p\in S(M)$ where $M\models T$. Let $\bar{a}$ be any realisation of p in some model N containing M. By **2.23**, the factor N/M purely embeds in the monster model $\tilde{M}$ and by **4.41** it is unlimited. Let $\bar{c}$ be the image of $\bar{a}$ in $N/M(\prec^*_\cdot \tilde{M})$. Define the **free part**, p_*, of p to be the type of $\bar{c}$ over 0. Observe that the free part of a type is unlimited (and is zero only if p is algebraic). The free part of an arbitrary type, not necessarily one over a model, will be defined after **6.5** below. First I show that the free part is well-defined.

<u>Lemma 6.3</u> *Let $p\in S(M)$ with $M\models T$. Let $\bar{a}$ in N and $\bar{a}'$ in N' be realisations of p in elementary extensions of M. Regard N/M and N'/M as being purely embedded in the monster model of T and let $\bar{c}$, $\bar{c}'$ be respectively the images of $\bar{a}$ in N/M and $\bar{a}'$ in N'/M. Then* $\mathrm{tp}(\bar{c}/0)=\mathrm{tp}(\bar{c}'/0)$.

Proof Let M' be a containing model for $N\cup N'$. Since $\bar{a}$ and $\bar{a}'$ have the same type over M, there is $f\in\mathrm{Aut}_M M'$ with $f\bar{a}=\bar{a}'$. There is induced an endomorphism $\bar{f}$ of M'/M, given by $\bar{f}(m+M/M)=(fm+M/M)$. This is well-defined and an automorphism of M'/M since f is an automorphism of M' fixing M. Regard M'/M as purely embedded in $\tilde{M}$. Since $\bar{f}\bar{c}=\bar{c}'$ it follows that $\bar{c}$ and $\bar{c}'$ have the same type in $\tilde{M}$ over 0, as required. □

The free part of a type (defined in **[Pr85]**, following discussion with Bouscaren, see also [Zg84; 11.3]) was originally defined only for types over pure-injective models. Indeed, that is often the context in which I use them since, over a pure-injective model, the free part splits off.

<u>Theorem 6.4</u> [Pr85; 1.1] *Let p be a type over the pure-injective model M and let p_* be its free part. Then:*

(i) $M(p)=M\oplus N(p_*)$.

(ii) If $\bar{a}$ realises p then, decomposing $M(\bar{a})$ as $M\oplus N$ and accordingly setting $\bar{a}=(\bar{a}_0,\bar{a}_1)$, the type of $\bar{a}_1$ over M is p_*. In particular $p=\bar{a}_0+p_*{}^M$, where $p_*{}^M$ is the (unique) non-forking extension of p_* to M.
Furthermore, p_* is the unique unlimited type over 0 which is such that there exists $\bar{m}$ in M with $p=\bar{m}+p_*{}^M$.

Proof I use the notation set up in the statement of (ii). Observe first that N is the hull of $\bar{a}_1$. For certainly $N(\bar{a}_1)$ is a summand of $M(\bar{a})$. Conversely $M^\frown\bar{a}$ is contained in $M+\bar{a}_1R$ so, by minimality of $M(\bar{a})$, we do have $M\oplus N(\bar{a}_1)=M(\bar{a})$. Thus (i) is proved.

Since $(M\oplus N)/M\simeq N$, the type of $\bar{a}_1$ over 0 is, by definition, p_*.

By **5.27**, $p_*{}^M$ is the type of $\bar{a}_1$ over M, so $p=\bar{a}_0+p_*{}^M$. Finally, the last statement follows by well-definedness (**6.3**) of p_* and the prescription for realising $p_*{}^M$ (**5.27**). □

<u>Corollary 6.5</u> [Pr85; 1.1] *Let $p\in S(M)$. Then* $\mathrm{s}(p)=\mathrm{s}(p_*{}^M)$ *and* $G(p)=G(p_*)$.

Proof If M is pure-injective then property (s1) of the stratified order, together with **6.4**(ii), implies $\mathrm{s}(p)=\mathrm{s}(p_*{}^M)$. The general case follows by property (s2) of the stratified order.

Suppose that $\varphi(\bar{v},\bar{m}) \in p^+$. Then, immediately from the definition of p_* (M is factored out), one deduces $\varphi(\bar{v},\bar{0}) \in p_*$. Thus $G(p) \geq G(p_*)$. This argument may be reversed: alternatively, recall (**5.3**) that $G(p) = G(q)$ if q is a non-forking extension of p, so it may be supposed that we are in the situation of **6.4**(ii). Then $\varphi(\bar{v}) \in p_*$ yields $\varphi(\bar{a}-\bar{a}_0)$ and hence $\varphi(\bar{v}-\bar{0})$, i.e. $\varphi(\bar{v})$, is in $\mathcal{G}(p)$. ◻

Exercise 1 Show that p_* is the unique unlimited type in $S(0)$ satisfying $s(p_*{}^M) = s(p)$.

Exercise 2 Show that the splitting off of p_* may not be possible if M is not pure-injective. [Hint: consider $M = \mathbb{Z}_{(p)}$.]

It follows that we may define the **free part**, p_*, of an arbitrary type $p \in S(A)$ to be the free part of any non-forking extension to a model or, more simply, as that type, q, over 0 with $G(q) = G_0(p)$. The next corollary is immediate from what has been shown, on noting that, for unlimited types p', q', the equality $G(p') = G(q')$, implies $p' = q'$, as does the equality $s(p') = s(q')$ (for types over models).

<u>Corollary 6.6</u> [Pr85; 1.2] *Suppose that* p *and* q *are types. Then the following are equivalent*:
(*i*) $p_* = q_*$;
(*ii*) $G_0(p) = G_0(q)$.
If p *and* q *are over models then another equivalent is*:
(*iii*) $s(p) = s(q)$.
If p *and* q *are over a pure-injective model* M *then a further equivalent is*:
(*iv*) $p = \bar{m} + q$ *for some* $\bar{m}$ *in* M. ◻

It is the invariant described (in a number of ways) in **6.6** which is important in determining the stability-theoretic properties of a type p. For, after going to a non-forking extension over a pure-injective model, one has $p = \bar{m} + p_*{}^M$ for some (realised) $\bar{m}$ in M; so it should not appear unreasonable that it is this free part $p_*{}^M$ which controls those properties of p.

Example 2 Continue with Ex 1 above. Let p_1, p_2 be the types over 0 of an element of order 2 which is divisible, respectively not divisible, by 2. Then $p_* = p_2$ and $q_* = q'_* = p_1$.

I remark at this point that in various proofs I make a convenient choice of copy of an object when the object is defined only up to isomorphism.

Before going on to the next section, I should say something about attributions. Many of the results of this chapter are natural generalisations of what one sees in the special case of injective modules over (commutative noetherian) rings. That "algebraic" case was studied independently by Bouscaren, Kucera and the author: all three saw that the results could be generalised.

Kucera concentrated on the totally transcendental case, using injectives over (commutative) noetherian rings as his model (and also seeing to what extent that "algebraic" example is a good model for t.t. structures other than modules). He independently obtained, for t.t. theories, many of the results below (see [Kuc84], [Kuc87]).

In their general form, many of the results come from [Pr85]. A number of people had some influence on that paper.

While still at Leeds and concerned with injective modules, I prepared a paper [Pr82] on elementary equivalence of Σ-injective (= t.t. injective) modules: in particular, that paper included the classification of all the models of a t.t. theory of injective modules (cf. §4.6). After hearing me give a talk on this late in 1979, Poizat pointed out to me that I was dealing with special cases of concepts well-known in stability theory (regularity and orthogonality in particular). Also around that time, I received Bouscaren's thesis [Bou79] and Kucera's preprints [Kuc80]: both these contained related ideas and went some way into stability theory (a subject with which I was just starting to get to grips). Pillay gave a course on stability theory at Bedford College the next spring (from which sprung [Pi83]) and we collaborated to produce [PP83]. Certainly the seeds of [Pr85] were sown.

In the spring of 1981, I visited Paris and, while there, had some discussions with Bouscaren which resulted in the characterisations of regular and strongly regular types, as well as some notion of the free part of a type. I went on from there, and produced [Pr85].

Finally, by the time I came to prepare [Pr85] for publication, I had seen the first version of Ziegler's paper [Zg84], but the results were in place by then and, although there is considerable overlap, there was no real influence.

6.2 Domination and the RK-order

The central question of this section is: given types p and q over a model, when does realising p force a realisation of q? If one restricts attention only to pure-injective (or in the general case, sufficiently saturated) models, then a well-behaved order on (classes of) types is obtained - the (generalised) RK-order (RK = Rudin-Kiesler). The idea of orthogonality of types also arises: when does realising p have no effect on realisations of q? These somewhat different questions actually lead to the same structure - the RK-order. In this section I consider the first, rather more algebraic, approach *via* the idea of domination (of elements and types).

If M is a pure-injective model and p, q are types over M, set $q \triangleleft^* p$ if q is realised in $M(p)$. There are various ways of regarding this relation.

<u>Theorem 6.7</u> [Pr85; 1.3] *Suppose that M is a pure-injective model and let p and q be types over M. Then the following are equivalent:*

(i) $q \triangleleft^* p$;

(ii) $M(q)$ *is a direct summand, over* M, *of* $M(p)$;

(iii) q_* *is realised in* $N(p_*)$;

(iv) $N(q_*)$ *is a direct summand of* $N(p_*)$;

(v) ${q_*}^M \triangleleft^* {p_*}^M$.

Proof (i)⇒(ii), (iii)⇒(iv) These are immediate from the definition and properties of hulls.

(ii)⇒(iii) Set $M(q) = M(\bar{b})$ where $\bar{b}$ is a realisation of q. By assumption, there is a split embedding $M(q) \xrightarrow{f} M(p) = M \oplus N(p_*)$ with f fixing M. Decompose $f\bar{b}$ accordingly as $(\bar{b}_0, \bar{b}_1) \in M \oplus N(p_*)$. Since f fixes M the type of $f\bar{b}$ over M also is q. So, by uniqueness of q_* (6.4), the type of $\bar{b}_1$ over M is ${q_*}^M$; in particular its type over 0 is q_*, as required.

(iv)⇒(v) By 5.27 one has $M({q_*}^M) \simeq M \oplus N(q_*)$ and $M({p_*}^M) \simeq M \oplus N(p_*)$. By assumption, the former is (isomorphic over M to) a direct summand of the latter, which therefore realises ${q_*}^M$, as required.

(v)⇒(i) We know that $M(q) \simeq M \oplus N(q_*)$ by 6.4, and this in turn is isomorphic to $M({q_*}^M)$ by 5.27. Similarly $M(p) \simeq M({p_*}^M)$. Thus (i) follows from (v). □

<u>Corollary 6.8</u> [Pr85; 1.4] *Let p, q be types over the pure-injective model M. Then $q \triangleleft^* p$ iff every pure-injective elementary extension of M which realises p also realises q.* □

Now this algebraic-looking notion will be compared with a more model-theoretic one.

Given parameters $\bar{a}, \bar{b}, \bar{c}$, say that $\bar{b}$ **dominates** $\bar{c}$ **over** $\bar{a}$ (see, e.g., [Poi85; Chpt19]) and write $\bar{b} \triangleright \bar{c}/\bar{a}$, if every set of parameters which depends on $\bar{c}$ over $\bar{a}$ also depends on $\bar{b}$ over $\bar{a}$: $\bar{d} \not\downarrow \bar{c}/\bar{a}$ implies $\bar{d} \not\downarrow \bar{b}/\bar{a}$. Given types p, q over a model M, we say that p **dominates** q (over M), writing $p \triangleright q$, if there is some realisation $\bar{b}$ of p and a realisation $\bar{c}$ of q such that $\bar{b}$ dominates $\bar{c}$ over M. It is equivalent (exercise) that for every $\bar{c}$ realising q there should be a realisation $\bar{b}$ of p dominating $\bar{c}$ over M.

<u>Proposition 6.9</u> [Pr85; 1.5], [Sr81; Lemma10] also see [Zg84; 6.4] *Suppose that the type of* $\bar{a}$ *over* 0 *is unlimited and let* $\bar{b}$ *be in the hull of* $\bar{a}$. *Then* $\bar{a}$ *dominates* $\bar{b}$ *over* 0. *In particular,* $\bar{a} \not\perp \bar{b}/0$.

Proof Let $p(\bar{v},\bar{a})$ be the pp-type of $\bar{b}$ over $\bar{a}$. Recall (4.10) that p is maximal in $S^+(\bar{a})$. Suppose that $\bar{c}$ depends on $\bar{b}$ over 0: it must be shown that $\bar{c}$ depends on $\bar{a}$ over 0.

By 5.6, the dependence of $\bar{c}$ on $\bar{b}$ is witnessed by a pp formula: $\psi(\bar{c},\bar{b}) \wedge \neg\psi(\bar{0},\bar{b})$ (we note that since $\bar{a}$ has unlimited type over 0 so has $\bar{b}$).

Let $\varphi(\bar{v},\bar{a})$ be in $p(\bar{v},\bar{a})$; then $\psi(\bar{c},\bar{b}) \wedge \varphi(\bar{b},\bar{a})$ holds. Hence $\exists \bar{v}\,(\psi(\bar{c},\bar{v}) \wedge \varphi(\bar{v},\bar{a}))$ holds. So, were $\bar{c}$ and $\bar{a}$ independent over 0, 5.6 would imply that $\exists \bar{v}\,(\psi(\bar{0},\bar{v}) \wedge \varphi(\bar{v},\bar{a}))$ held. In particular one would have $\psi(\bar{0},\bar{v}) \wedge p(\bar{v},\bar{a})$ consistent. Then maximality of p would yield $\psi(\bar{0},\bar{b})$ - contradiction. Hence $\bar{c} \not\perp \bar{a}/0$ as required. ◻

A similar result, with a similar proof, holds over models (6.11), but an extra lemma (6.10) is required. An analogous result is true for arbitrary stable theories provided one works over a $|T|^+$-saturated module (see, e.g., [Bal8?]). This illustrates how the class of pure-injective models behaves with respect to pp formulas (and so, for modules, often with respect to all formulas), as does the smaller class of $|T|^+$-saturated models behave with respect to all formulas in the general case.

<u>Lemma 6.10</u> [Pr85; 1.6] *Let* M *be a pure-injective model and let* $\bar{a}$ *be any set of parameters. Let* p *be a pp-type over* $M^\frown\bar{a}$ *and suppose that* p *is finitely satisfied by tuples of* M. *Then* p *is realised in* M.

Proof Set $M(\bar{a}) = M \oplus N$ and write $\bar{a} = (\bar{a}_0, \bar{a}_1)$ accordingly. Suppose that $\varphi(\bar{v},\bar{a},\bar{m}) \in p(\bar{v})$. Then by assumption there is $\bar{m}'$ in M with $\varphi(\bar{m}',\bar{a},\bar{m})$. Projecting to M and to N yields $\varphi(\bar{m}',\bar{a}_0,\bar{m})$ and $\varphi(\bar{0},\bar{a}_1,\bar{0})$ respectively.

So the pp-type p_1, which is as p but with $\bar{a}_0$ replacing $\bar{a}$, is finitely realised in M. Since M is pure-injective (and $\bar{a}_0$ is in M) p_1 is realised in M - say by $\bar{m}_0$. The claim is that $\bar{m}_0$ realises p. For a formula, φ as above, one now has $\varphi(\bar{m}_0, a_0,\bar{m})$ and $\varphi(\bar{0},\bar{a}_1, \bar{0})$. Adding these gives $\varphi(\bar{m}_0,\bar{a},\bar{m})$, as required. ◻

<u>Theorem 6.11</u> [Pr85; 1.7] *Let* M *be a pure-injective model, let* $\bar{a}$ *be arbitrary and suppose that* $\bar{b}$ *is in* $M(\bar{a})$. *Then* $\bar{a}$ *dominates* $\bar{b}$ *over* M.

Proof Since $\bar{b}$ is in $M(\bar{a}) = N(M^\frown\bar{a})$ one has, by 4.10, that $\mathrm{pp}(\bar{b}/M^\frown\bar{a})$ is maximal in $S^+(M^\frown\bar{a})$.

Let $\bar{c}$ depend on $\bar{b}$ over M: it must be shown that $\bar{c}$ depends on $\bar{a}$ over M; so suppose not. Since $\bar{c} \not\perp \bar{b}/M$ there is (5.5 and 2.6(c)) some pp formula ψ and $\bar{m}$ in M, such that $\psi(\bar{c},\bar{b},\bar{m})$ holds but such that, for every $\bar{m}_1$ in M, one has $\neg\psi(\bar{m}_1,\bar{b},\bar{m})$.

Let $\varphi(\bar{v},\bar{a},\bar{m}_1)$ be in the pp-type of $\bar{b}$ over $M^\frown\bar{a}$. Then $\psi(\bar{c},\bar{b},\bar{m}) \wedge \varphi(\bar{b},\bar{a},\bar{m}_1)$ holds; hence so does $\exists\bar{v}\,(\psi(\bar{c},\bar{v},\bar{m}) \wedge \varphi(\bar{v},\bar{a},\bar{m}_1))$. From $\bar{c} \perp \bar{a}/M$ one concludes that there is $\bar{m}_0$ in M such that $\exists\bar{v}\,(\psi(\bar{m}_0,\bar{v},\bar{m}) \wedge \varphi(\bar{v},\bar{a},\bar{m}_1))$ holds. Therefore the set of pp formulas $\{\exists\bar{v}\,(\psi(\bar{w},\bar{v},\bar{m}) \wedge \varphi(\bar{v},\bar{a},\bar{m}_1)) : \varphi(\bar{v},\bar{a},\bar{m}_1) \in \mathrm{pp}(\bar{b}/M^\frown\bar{a})\}$ is finitely satisfied in M. So, by 6.10, there is $\bar{m}_0$ in M realising this set. Thus $\{\psi(\bar{m}_0,\bar{v},\bar{m})\} \cup \mathrm{pp}(\bar{b}/M^\frown\bar{a})$ is consistent.

Maximality of $\mathrm{pp}(\bar{b}/M^\frown\bar{a})$ then gives $\psi(\bar{m}_0,\bar{b},\bar{m})$; but $\bar{m}_0$ is in M - contrary to choice of the formula, ψ, witnessing dependence. Hence $\bar{c} \not\perp \bar{a}/M$, as required. ◻

Example 1 The hypothesis of pure-injectivity of M in 6.11 cannot be dropped.

Take M to be the abelian group $\mathbb{Z}_{(p)}$ and note that $\bar{M}$ is uncountable. So there is a strictly increasing ω-sequence of countable models:

$M=M_0 \prec M_1 \prec M_2 \prec \ldots$, all elementary submodels of $\overline{M}$. Take $b=c \in \overline{M}\setminus \bigcup_i M_i$. Since $\mathbb{Z}_{(p)}$ is superstable, there exists (by "dcc" on forking, as measured by U-rank) $n \in \omega$ such that, for each $i \geq n$, the type of c over M_i is non-forking over M_n. Choose $a \in M_{n+1}\setminus M_n$.

One has $b \in M_n(a)=\overline{M}$. Certainly the type of c over $M_n{}^\frown b(=c)$ forks over M_n (for $c \notin M_n$): that is, c depends on b over M_n. Yet $\mathrm{tp}(c/M_{n+1})$ - hence $\mathrm{tp}(c/M_n{}^\frown a)$ - does not fork over M_n: that is, c is independent from a over M_n.

Thus, although $b \in M_n(a)$, a does not dominate b over M_n.

Lemma 6.12 *Let M be any model and let $\overline{m}_0$ be in M. Then for any parameters $\overline{c},\overline{b}$ one has $\overline{c} \perp \overline{b}/M$ iff $\overline{c} \perp (\overline{b}+\overline{m}_0)/M$.* □

The lemma is actually quite general and the proof is left as an exercise. Now the algebraic and model-theoretic notions above are linked.

Theorem 6.13 [Pr85; 1.9] *Let p and q be types over the pure-injective model M. Then the following conditions are equivalent.*

(i) $q \triangleleft^* p$.

(ii) If $\overline{b}_1$ realises $q_*{}^M$ then there is $\overline{a}_1$ realising $p_*{}^M$ with $\overline{a}_1$ dominating $\overline{b}_1$ over M.

(iii) If $\overline{b}$ realises q then there is $\overline{a}$ realising p with $\overline{a}$ dominating $\overline{b}$ over M: that is, $q \triangleleft p$.

Proof I use the fact (see [Poi85; 19.19]) (and exercise above) that, in order to check (ii)/(iii), it is enough to show that there is some realisation of $q_*{}^M$, respectively q, dominated over M by some realisation of $p_*{}^M$, respectively p.

(i)⇒(iii) Let $\overline{a}$ realise p. By assumption, q is realised in $M(\overline{a})$. So (iii) follows by 6.11.

(iii)⇒(ii) Let $\overline{b}$ realise q. Set $M(\overline{b})=M \oplus N(q_*)$ and put $\overline{b}=(\overline{b}_0,\overline{b}_1)$, where $\overline{b}_1$ realises $q_*{}^M$ (6.4). By assumption, there is $\overline{a}$ (in $\tilde{M}$) realising p and dominating $\overline{b}$ over M. By 6.4, there is $\overline{a}_0$ in M such that $\overline{a}_1=\overline{a}-\overline{a}_0$ realises $p_*{}^M$. It is claimed that $\overline{a}_1$ dominates $\overline{b}_1$ over M.

Suppose then that $\overline{c}$ depends on $\overline{b}_1$ over M. By 6.12, $\overline{c}$ depends on $\overline{b}=\overline{b}_0+\overline{b}_1$ over M. So, by choice of $\overline{a}$, $\overline{c}$ depends on $\overline{a}$ over M. Then, again using 6.12, one concludes that $\overline{c}$ depends on $\overline{a}_1=\overline{a}-\overline{a}_0$ over M, as required.

(ii)⇒(i) Choose $\overline{b}_1$ realising $q_*{}^M$. By assumption there is a realisation, $\overline{a}_1$, of $p_*{}^M$ dominating $\overline{b}_1$ over M. Consider $M(\overline{a}_1,\overline{b}_1)$. Decompose this as $M(\overline{a}_1)\oplus N'$ for some N'; set $\overline{b}_1=(\overline{b}_1',\overline{b}')$ accordingly. It will be shown that the type of $\overline{b}_1'$ over M is $q_*{}^M$ which, by 6.7, will be enough.

By 5.25, $\overline{b}'$ is independent from $\overline{a}_1$ over M; so by choice of $\overline{a}_1$ one has that $\overline{b}'$ is independent from $\overline{b}_1$ over M. Now suppose that the type of $\overline{b}_1'$ over M were not $q_*{}^M$. Since the type of $\overline{b}_1$ over M is $q_*{}^M$, it follows, upon projecting, that $\mathrm{pp}(\overline{b}_1'/M)$ strictly contains the pp-part of $q_*{}^M$. Hence there is a pp formula φ and $\overline{m}$ in M such that $\varphi(\overline{b}_1',\overline{m}) \wedge \neg\varphi(\overline{b}_1,\overline{m})$ holds. This immediately yields $\neg\varphi(\overline{b}',\overline{0})$ and also $\varphi(\overline{b}_1-\overline{b}',\overline{m})$. Since $\overline{b}'$ and $\overline{b}_1$ are independent over M one concludes that there is $\overline{m}_0$ in M with $\varphi(\overline{b}_1-\overline{m}_0,\overline{m})$. Projecting this yields $\varphi(\overline{b}',\overline{0})$ - contradiction, as required. □

Corollary 6.14 [Pr85; 1.10] *Let p and q be unlimited types over 0. Then $N(q)$ is a factor of $N(p)$ iff $q^M \triangleleft p^M$ for some (equivalently, for all) pure-injective models, M, of T.*

Proof This is immediate from 6.7 and 6.13. □

Corollary 6.15 [Pr85; 1.11] *Let M be pure-injective and let $\overline{a},\overline{b}$ be parameters. Then:*

(a) $\overline{b}$ lies in some copy of $M(\overline{a})$ iff $\overline{a}$ dominates $\overline{a}{}^\frown\overline{b}$ over M;

(*b*) *if the types of both* $\bar{a}$ *and* $\bar{b}$ *are unlimited, then* $\bar{b}$ *lies in some copy of* $N(\bar{a})$ *iff* $\bar{a}$ *dominates* $\bar{a}\frown\bar{b}$ *over* 0.

Proof One half of (a), resp. (b), is just **6.11**, resp. **6.9**.

Suppose then that $\bar{a} \rhd \bar{a}\frown\bar{b}/M$; one argues as in **6.13**(ii)⇒(i). Set $M(\bar{a},\bar{b}) = M(\bar{a}) \oplus N'$ and write $\bar{b} = (\bar{b}_2, \bar{b}')$ accordingly. By **5.25** (and **6.12**) $\bar{b}'$ is independent from $\bar{a}$ over M; so, by assumption, it is independent from $\bar{a}\frown\bar{b}$.

By projecting, one has $pp(\bar{b}_2/M\frown\bar{a}) \geq pp(\bar{b}/M\frown\bar{a})$. Suppose, for the converse, that φ is pp and $\bar{m}$ is in M with $\varphi(\bar{b}_2, \bar{a}, \bar{m})$ - that is, with $\varphi(\bar{b}-\bar{b}', \bar{a}, \bar{m})$. Independence over M yields $\varphi(\bar{b}-\bar{m}_0, \bar{a}, \bar{m})$ for some $\bar{m}_0$ in M. Projecting this yields $\varphi(\bar{b}', \bar{0}, \bar{0})$. Combining this with the original formula then gives $\varphi(\bar{b}, \bar{a}, \bar{m})$.

Hence $\bar{b}_2$ and $\bar{b}$ have the same (pp-)type over $M\frown\bar{a}$. So there is an $M\frown\bar{a}$-automorphism taking $\bar{b}_2$ to $\bar{b}$. The copy of $M(\bar{a})$ containing $\bar{b}_2$ maps to a copy of $M(\bar{a})$ containing $\bar{b}$, as required.

The second half of (b) follows from (a) by choosing the pure-injective model M to be independent from $\bar{a}\frown\bar{b}$ over 0 (for each direction, this can be done). ◻

Example 2 Let T be the theory of the abelian group $\mathbb{Z}_4^{\aleph_0}$. Let $p < q < pp(0)$ (ordered by inclusion of their pp-parts) be the types in $S_1(0)$. Note that the non-forking extensions of p dominate those of q and *vice-versa*.

Let a be an element of order 4. Then the hull of a is just the module it generates. In particular the only (non-zero) elements dominated by a over 0 are $a, 2a$ and $3a$. On the other hand $2a$ certainly dominates these elements over 0 (by **6.9**) but also dominates many others. For if c is any element of order 2 then $(a+c)2 = 2a$ - so $a+c$ generates a copy of the hull of $2a$ and so, by **6.9**, $2a$ dominates $a+c$ over 0. It is easily verified, using **6.15**, that all elements dominated by $2a$, apart from itself, are of this form (at least in this example, one does not need "$\bar{a} \rhd \bar{a}\frown\bar{b}$" - just "$\bar{a} \rhd \bar{b}$": how general is this?).

It is implicit above (and it may be checked as an exercise) that if p and q are types over the pure-injective model M, and if M' is any pure-injective elementary extension of M, then one has $p \rhd q$ iff $p^{M'} \rhd q^{M'}$.

Now let p and q be types over a pure-injective model M. If $p \rhd q$ and $q \rhd p$ then set $p \equiv q$ and say that p and q are **RK-equivalent**. The order induced by "$\lhd$" on the resulting RK-equivalence classes is the (generalised, in that **[Las82a]** is concerned with the superstable case) **RK-order**. This order will be discussed further in the next section, after orthogonality has been described. Let us note the following for now and then go on to describe the RK-minimal types.

<u>Corollary 6.16</u> **[Pr85**; 1.12**]** *Let* p *and* q *be types over the pure-injective model* M.
(*a*) p *dominates* q *iff* $p_*{}^M$ *dominates* $q_*{}^M$*, and this will be so iff* $N(q_*)$ *is a direct summand of* $N(p_*)$.
(*b*) p *and* q *are RK-equivalent iff* $p_*{}^M$ *and* $q_*{}^M$ *are RK-equivalent, and that is the case iff* $N(p_*)$ *is isomorphic to* $N(q_*)$.

Proof (a) By **6.13**, $q \lhd p$ is equivalent to $q \lhd^* p$; this, by **6.7**, is equivalent to $q_*{}^M \lhd^* p_*{}^M$ which, in turn, is equivalent to $q_*{}^M \lhd p_*{}^M$.

(b) The first equivalence is by (a); the second follows since two pure-injectives, each purely embeddable in the other, are isomorphic (**1.8**). ◻

A type p (over a pure-injective model) is said to be **RK-minimal** if it is not algebraic but is minimal such in the RK-order. This is the usual definition in the superstable case, but now we encounter a potential problem arising from the generality in which we work. All the above allows infinite tuples and types in infinitely many free variables, but then it seems that a choice must be made in the above definition: whether to retain this generality or to restrict to the finite case. For instance, Th($\mathbb{Z}^{\aleph_0}$) is not superstable and, if p is the pp-type of the element 1 (in a copy of $\mathbb{Z}$), then $N(p)=\overline{\mathbb{Z}}=\mathrm{pi}(\oplus\{\overline{\mathbb{Z}(p)} : p \text{ is prime}\})$ has infinitely many indecomposable direct summands. In determining those types q with $q \triangleleft p$ should we restrict only to types in finitely many free variables or not? For it is not immediately obvious whether or not every summand of $N(p)$ is the hull of finitely many elements. Fortunately, every summand of $N(p)$ is the hull of a single element and, indeed, this is true in general (9.16). Therefore no real problem arises, and one may regard the RK-order as applying just to types in only finitely many free variables.

<u>Theorem 6.17</u> [Pr85; 1.13] *Let p be a type over the pure-injective model M. Then p is RK-minimal iff every non-zero direct summand of $N(p_*)$ is isomorphic to $N(p_*)$.*

Proof $\Rightarrow$ If $N(\bar{c})=N(q)$ is a non-zero direct summand of $N(p_*)$ (by 9.16, this is a typical summand) then, noting that p_* unlimited implies $q=\mathrm{tp}(\bar{c})$ unlimited, 6.16(a) yields that q^M is dominated by p. So RK-minimality of p implies that it is RK-equivalent to q^M. Hence, by 6.16(b), it follows that $N(p_*)\simeq N(q_*{}^M)=N(q)$, as required.

$\Leftarrow$ Suppose that q is a type over M (in finitely many free variables) which is dominated by p. Then, by 6.16(a), $N(q_*)$ is a direct summand of $N(p_*)$. So, by assumption, either $N(q_*)=0$ or $N(q_*)\simeq N(p_*)$. In the first case q is a realised type; in the second case 6.16(b) yields that p and q are RK-equivalent. Thus p is RK-minimal, as required. □

<u>Corollary 6.18</u> [Pr85; 1.14] *Let p be a type over the pure-injective model M. If p_* is irreducible then p is RK-minimal.*

Proof This is immediate from 6.17, since irreducibility of p_* is just indecomposablity of $N(p_*)$. □

<u>Corollary 6.19</u> [Pr85; 1.15] *Suppose that the theory T has no continuous part. If p is a type over the pure-injective model, M, of T then p is RK-minimal iff p_* is irreducible.*

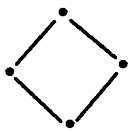
Proof The assumption on T implies that $N(p_*)$ must have an indecomposable direct summand. So the result follows by 6.17. □

Example 3 For an example of an RK-minimal type which is not irreducible, let R be a regular ring which has the property that every finitely generated (so cyclic) right ideal is isomorphic to R itself. For instance, let $R=\mathrm{End}(U_K)/\mathrm{soc}(\mathrm{End}(U_K))$ where U_K is an $\aleph_0$-dimensional K-vectorspace (K a field) and "soc" denotes the socle (sum of all simple right submodules). Another class of examples is provided by existentially closed prime rings [Pr83a]. An example of the first kind is right self-injective, so is equal to its own hull (since R is regular, pure-injective and injective hulls coincide - 16.14).

Take T to be the theory of R_R (or $(R_R)^{\aleph_0}$ if $R \not\equiv R^{\aleph_0}$); $M=\overline{R}$; $p=\mathrm{tp}^M(1_R)$.

From the condition on the right ideal structure of R it follows that p satisfies the condition of 6.17. (For the non-self-injective case, a little argument is needed: note that a finitely generated essential right ideal of R must equal R.) So p is RK-minimal. But, provided R is not right artinian (i.e., (semi-)simple artinian), p will not be irreducible.

Example 4 Let T be the theory of the abelian group $\mathbb{Z}_6{}^{\aleph_0}$. The models of T are (exercise) precisely those groups of the form $\mathbb{Z}_2{}^{(\kappa)}\oplus\mathbb{Z}_3{}^{(\lambda)}$ where $\kappa,\lambda \geqslant \aleph_0$. Let M be any (pure-injective) model. The types over 0 are: $p_0(v)=\langle v=0\rangle$; $p_2(v)=\langle v2=0 \wedge v\neq 0\rangle$; $p_3(v)=\langle v3=0 \wedge v\neq 0\rangle$; $p_6(v)=\langle v2\neq 0 \wedge v3\neq 0\rangle$.



Since $N(p_2)=\mathbb{Z}_2$, $N(p_3)=\mathbb{Z}_3$, $N(p_6)=\mathbb{Z}_6=\mathbb{Z}_2\oplus\mathbb{Z}_3$ it follows from 6.16 that ${p_2}^M \lhd {p_6}^M$ and ${p_3}^M \lhd {p_6}^M$ and that there are no other non-trivial relations between ${p_2}^M$, ${p_3}^M$, ${p_6}^M$ in the RK-order. Since the RK-order is determined by the free parts of types (so types over 0), it follows that the RK-order (with the algebraic types included at the bottom) has the shape shown.

Therefore, by 6.17, $p\in S_1(M)$ is RK-minimal iff $p_*=p_2$ or $p_*=p_3$.

Note (from 6.6) that if $p\in S(A)$, $q\in S(B)$ are such that $G_0(p)=G_0(q)$ then p and q are RK-equivalent in the following strong sense: if p', q' are non-forking extensions of p, q to a pure-injective module $N\supseteq A\cup B$ and if M is any elementary extension of N, then M realises p' iff M realises p_* iff M realises $q_*=p_*$; hence, iff M realises q'.

6.3 Orthogonality and the RK-order

Now we have a description of the RK-order in terms of algebraic conditions for q to dominate p. There is another aspect to this order, illustrated, for example, by the theory of $\mathbb{Z}_6{}^{\aleph_0}$. Here there are two RK-minimal classes and these are in some sense strongly independent (independent as RK-classes). This strong independence is termed orthogonality and in modules it may be described in terms of direct-sum decompositions of pure-injectives.

Let p,q be types over the model M. Say that p and q are **weakly orthogonal**, $p\perp^{\mathrm{w}}q$, if whenever $\bar{a}$ realises p and $\bar{b}$ realises q then $\bar{a}$ and $\bar{b}$ are independent over M. It is equivalent (exercise) to require that $p(\bar{v}_1)\cup q(\bar{v}_2)$ be a complete type.

Let p,q be types over the model M. Then p and q are **orthogonal**, $p\perp q$, if for every elementary extension, M', of M the non-forking extensions of p and q to M' are weakly orthogonal. If M is $(|R|+\aleph_0)^+$-saturated then orthogonality and weak orthogonality over M coincide for types over M (see [Pi83; 9.46]). That they need not coincide for types over arbitrary models is shown by the following example.

Example 1 Take M to be the $\mathbb{Z}$-module $\mathbb{Z}_{(p)}$. Note that $\mathbb{Z}_{(p)}\equiv\mathbb{Z}_{(p)}\oplus\mathbb{Q}$. The U-rank of M is 1 (Ex5.2/7), so every extension of an unrealised 1-type over a model is either non-forking or algebraic.

Choose an element $c\in\bar{M}\setminus M$ and let $(a_i)_{i\in\omega}$ be a sequence in M such that $a_i\in Mp^i\setminus Mp^{i+1}$ and $p^{i+1}|c-\Sigma_{j=1}^{i}a_j$ for all j: that is, the a_i describe that path through the p-branching tree of pp-definable cosets which corresponds to c (cf. §2.$\mathbb{Z}$). Let p be the type of c over M, and let q be the type of the element $(0,1)\in\bar{M}\oplus\mathbb{Q}$. It is claimed that p and q are weakly orthogonal but not orthogonal.

Consider any realisations of p and q. It entails no loss in generality (apply an M-automorphism) to suppose that the realisation of p is the original one c. Let b be the realisation of q. If b and c were not independent over M then (as remarked above) b would be algebraic over $M^\frown c$; but (the chosen copy of) $\bar{M}$ is a model extending $M^\frown c$ and does not contain a non-zero element which, like b, is divisible by all powers of p. This would be a contradiction. Therefore b and c are independent over M. So, by definition, p and q are weakly orthogonal.

On the other hand they are not orthogonal: in fact they are RK-equivalent (clearly this excludes the possibility that they are orthogonal). For $G(p)=G(q)=\bigcap\{\tilde{M}p^i : i\in\omega\}$ (consider the a_i as above) and so, if M' is a pure-injective model extending M and if p',q' are the non-forking extensions of p and q to M', then by 6.6 there is $m'\in M'$ with $p'=m'+q'$. Hence (by 6.12) p' and q' are not weakly orthogonal - as claimed.

Theorem 6.20 [Pr85; 1.16], [Zg84; 11.6], also see [Kuc87; 3.9] *Let* p *and* q *be types over the pure-injective model* M. *Then the following conditions are equivalent.*
(i) p *and* q *are orthogonal.*
(ii) $(p_*)^M$ *and* $(q_*)^M$ *are orthogonal.*
(iii) $N(p_*)$ *and* $N(q_*)$ *have, to isomorphism, no non-zero direct summand (which may be taken to be the hull of a single element) in common.*
(iv) *If* N *is a direct summand of* $\tilde{M}$ *and contains a realisation* $\bar{a}$ *of* p_* *and a realisation* $\bar{b}$ *of* q_* *then, choosing any copies* $N(\bar{a})$, $N(\bar{b})$ *of the hulls of* $\bar{a}$ *and* $\bar{b}$ *in* N, *there is a decomposition of* N *as* $N(\bar{a})\oplus N(\bar{b})\oplus N'$ *for some* N' *(compare §5.3).*

Proof Since conditions (ii), (iii) and (iv) depend only on the free parts of the types considered, we may enlarge M if necessary and assume that it is $|T|^+$-saturated. Therefore we need check only weak orthogonality (exercise: show that pure-injectivity of M is actually enough for "$\perp^w \Rightarrow \perp$").

(i)⇔(ii) By 6.4 there are $\bar{a}_0$ and $\bar{b}_0$ in M with $p=\bar{a}_0+p_*{}^M$ and $q=\bar{b}_0+q_*{}^M$. Therefore $p(\bar{v}_1)\cup q(\bar{v}_2)$ is a complete type iff $p(\bar{v}_1-\bar{a}_0)\cup q(\bar{v}_2-\bar{b}_0)$ is a complete type, and this will be so iff $p_*{}^M(\bar{v}_1)\cup q_*{}^M(\bar{v}_2)$ is complete.

(iv)⇒(iii) This is obvious.

(iii)⇒(ii) Let $\bar{a}$ realise p_* and let $\bar{b}$ realise q_*. If $\bar{a}$ and $\bar{b}$ are not independent then, by 5.6, there is a pp formula linking them. By 4.31 it then follows that the hulls of $\bar{a}$ and $\bar{b}$ have a non-zero direct summand in common.

(ii)⇒(iv) Let $\bar{a}, \bar{b}, N, N(\bar{a}), N(\bar{b})$ be as in the statement of (iv) and suppose that $(p_*)^M$ and $(q_*)^M$ are orthogonal - so $\bar{a}$ and $\bar{b}$ are independent over 0. Two applications of 6.9 show that $N(\bar{a})$ and $N(\bar{b})$ are independent over 0. So by 5.30 there is a direct sum decomposition as described. □

Corollary 6.21 [PP87; 3.5] *Let* p *be any type. Then* p *is non-orthogonal - even RK-equivalent - to a type over* 0 *(namely* p_**).* □

Exercise 1 The definition of (weak) orthogonality makes sense over any set (in place of M). Even when every model is pure-injective (i.e. the t.t. case) weak orthogonality and orthogonality do not coincide). Show this by an example.
[Hint: try $\mathrm{Th}(\mathbb{Z}_2\oplus\mathbb{Z}_4{}^{\aleph_0})$ and consider 1-types over 0.]

Corollary 6.22 [Pr85; 1.17] *Suppose that* p *and* q *are types over a pure-injective model. Suppose also that* p *is RK-minimal. Then* p *is orthogonal to* q *iff* $N(p_*)$ *is not (isomorphic to) a direct summand of* $N(q_*)$.
In particular, if both p *and* q *are RK-minimal then they are orthogonal iff their hulls are non-isomorphic.*

Proof Certainly if p and q are orthogonal then, by 6.20, $N(p_*)$ cannot be a direct summand of $N(q_*)$. Conversely, if p and q are non-orthogonal so, by 6.20, if $N(p_*)$ and $N(q_*)$ have a common direct summand, then this summand may be taken to be the hull of a single element which, by 6.17, must be isomorphic to $N(p_*)$. The second assertion then follows by 6.17. □

Let us consider again the RK-order restricted to RK-equivalence classes of types with discrete hulls. With the results of this section and the last, it is clear that this is a $\wedge$-semi-lattice with $(p/\equiv)\wedge(q/\equiv)$ being given by the largest common direct summand (up to isomorphism) of $N(p)$ and $N(q)$. Then one has $p\perp q$ iff $(p_*/\equiv)\wedge(q_*/\equiv)=0$, where 0 is the class of algebraic types. This may be connected with the notion of spectrum in the case of injective modules (cf. §6.1). An operation of product on stationary types is defined thus (see

[Las82a]): given stationary types p, q (extended in a non-forking way to a common domain), let $\bar{a},\bar{b}$ be independent realisations of p, q respectively. Then $p\otimes q$ is the type of the tuple $\bar{a}^\frown\bar{b}$.

Let T be a theory of modules. Define the **dimension** of T, $\mu(T)$, to be the number of orthogonality classes of weight one types (for weight, see §6.4; I exclude continuous pure-injectives because I don't know exactly how I should count them). For modules, it follows from what has been shown, that there is such a maximal number (every type is non-orthogonal to a type over 0): modules are "non-multidimensional" (for a precise definition of the term, see the texts on stability theory). An example of a "multidimensional" theory is that of a single equivalence relation: as the models get larger, so do the number of dimensions/cardinals needed to specify them up to isomorphism (each new class introduces a new dimension whereas, in modules, we have addition to link new cosets to old ones).

6.4 Regular types

Regular (or even just weight one) types are particularly important in the stability-theoretic classification of (sufficiently saturated) models. In some sense, they form the skeleton, or correspond to the basic components, of a model. That this certainly is the case in modules will be seen here and later. Regular types can also be seen as generalising various notions of critical right ideal, both in their model-theoretic description and in some of their uses (see the end of the section).

A non-algebraic type $p\in S(A)$ is **regular** if, for any model M containing A and for any forking extension, q, and any non-forking extension, p', of p to M, the types p' and q are orthogonal (p is "orthogonal to all its forking extensions").

Recall §4.6 that a type is critical if its pp-part defines a minimal non-zero $\bigwedge\!\!\!\bigwedge$-pp-definable subgroup of its hull. The next result characterises regular types and at the same time shows that regularity of a type is independent of the over-theory.

<u>Theorem 6.23</u> [Zg84; 11.4], [Pr85; 1.18] (also [Sr81; Lemma 11]) *Suppose that p is a type over a model M. Then p is regular iff p_* is critical.*

Proof I give the proof for 1-types: for n-types, just put a bar above everything. We may as well suppose that M is pure-injective.

$\Rightarrow$ Write $M(p)$ as $M\oplus N(p_*)$ and decompose a realisation $a=(a_0,a_1)$ of p in $M(p)$ accordingly. If p_* is not critical then there is $b\in N(a_1)$ realising $p_*{}^+$ but also with $\psi(b)$ true for some pp ψ not in p_*.

Consider the model $M\oplus N(a_1')\oplus N(a_1)$, where a_*' is a realisation of p_* (since p_* is unlimited, there is such a model). If p_1 is the type of a over $M\oplus N(a_1')$ then p_1 is the non-forking extension of p to that model. Let p_2 be the type of $a_0+a_1'+b$ over $M\oplus N(a_1')$. We will see that p_2 is a forking extension of p.

Observe first that the restriction of p_2 to M is $\mathrm{tp}(a_0+(a_1'+b))/M)$, which clearly equals $\mathrm{tp}(a_0+a_1/M)$. So p_2 extends p to $M\oplus N(a_1')$.

To see that p_2 forks over M, note that $\psi(b)$ yields $\psi((a_0+a_1'+b)-(a_0+a_1'))$ and, were p_2 equal to p_1 (the unique non-forking extension) this would give $\psi(a-(a_0+a_1'))$ which, on projecting, would yield $\psi(a_1)$ - contrary to choice of ψ.

Therefore, since p is regular, $a=a_0+a_1$ and $a_0+a_1'+b$ must be independent over $M\oplus N(a_1')$. So, by **6.12**, a_1 and b are independent over $M\oplus N(a_1')$; but $b\in N(a_1)$, so we have a contradiction to **6.11**, as required.

$\Leftarrow$ Let a realise the non-forking extension, p', of p to some model M', which may be assumed to be pure-injective. Set $M'(a) = M' \oplus N(a_1)$ with $a = (a_0, a_1)$ as in **6.4**: by that result, we have $\mathrm{pp}(a_1) = p_*{}^+$.

Let b realise any extension of p to M' and suppose that a and b are not independent over M': so, to establish regularity of p, it must be shown that $\mathrm{tp}(b/M') = p'$. By **6.4**, it will be enough to show that if $M'(a,b) = M' \oplus N(a_1) \oplus N'$ and if $b = (b_0, b_1, b_2)$ accordingly, then $\mathrm{tp}(b_1 + b_2/0) = p_*$.

Since M is an elementary substructure of M' and since the type of b over M is p, it follows (exercise) that the pp-type of $b_1 + b_2$ is at least $p_*{}^+$. Hence $p_*{}^+(b_1)$ holds. Also, since a and b are dependent over M', it cannot be that b_1 is zero: so, by criticality of p_*, it follows that $\mathrm{pp}(b_1) = p_*{}^+$. Then, if φ is pp and $\varphi(b_1 + b_2)$ holds - so $\varphi(b_1)$ holds - it follows that $\varphi(a_1)$ holds. This shows that $\mathrm{pp}(b_1 + b_2) = p_*{}^+$, as required. ◻

Corollary 6.24 [Zg84; 11.5]

(a) *If* $p \in S(A)$ *is regular then* p_* *is irreducible.*

(b) *If* p *(an unlimited 1-type over* 0*) is regular then* p *is irreducible.*

Proof This follows by **6.23** since (4.48) critical types are irreducible. ◻

A regular type need not itself be irreducible. For an example, consider the type over 0 of an element of order 6 in $\mathbb{Z}_3{}^{\aleph_0} \oplus \mathbb{Z}_2$. Of course, irreducible unlimited types need not be regular (rather they are of weight one - see after **6.27** below): consider $\mathbb{Z}_4{}^{\aleph_0}$ or, better, $\mathbb{Z}(p)^{\aleph_0}$.

Corollary 6.25 [Pr85; 1.19] *Let* p *be a 1-type (over a model* M*). Then the following are equivalent.*

(i) p *is regular.*

(ii) $p_*{}^{(M)}$ *is regular.*

(iii) p_* *is critical.*

(iv) *If* $q \in S_1(0)$ *and* $q^+ \supset p_*{}^+$ *then* $N(q)$ *and* $N(p)$ *have no non-zero direct summand in common.*

Proof This is immediate by **6.23**, **6.24** and since $(p_*{}^M)_* = p_*$. ◻

Example 1 Consider the theory of the abelian group $\mathbb{Z}_2{}^{\aleph_0} \oplus \mathbb{Z}_4{}^{\aleph_0}$. The type over 0 of an element of order 4 is easily seen to be not critical, hence not regular; whereas both types of elements of order 2 are critical, so are regular.

Example 2 Take the theory of the abelian group $\mathbb{Z}_6{}^{\aleph_0}$. With the notation of Ex6.2/4, the types p_2 and p_3 are regular, whereas p_6 is not (but for rather different reasons than the type of an element of order 4 in the previous example; that type is at least RK-minimal).

If one is using realisations of regular types to classify models (of a particular sort), then one needs to know that there are "enough" realisations of regular types. The next result gives this in the superstable case.

Proposition 6.26 (Srour) *Suppose that* T *is superstable and let* $M \prec N$, $M \neq N$ *be models of* T. *Then there is* $c \in N \setminus M$ *such that* $\mathrm{tp}(c/M)$ *is regular.*

Proof Consider N/M purely embedded in $\tilde{M}$. By **3.8**, N/M is totally transcendental so, by **4.49**, it realises a critical type. Let $c \in N \setminus M$ be such that its image, a, in N/M is critical. By definition, $\mathrm{tp}(c/M)_* = \mathrm{tp}(a)$. Therefore, by **6.23**, $\mathrm{tp}(c/M)$ is indeed regular. ◻

The original proof was somewhat different. Srour also later found essentially the above short proof. It is presented in [PP87; 6.3], but with a slip, which I take the opportunity to correct. (Notation as in [PP87]) In the second paragraph one should begin by choosing a copy,

$\bar{M}$, of the hull of M independent from c over 0 and then consider the hull of $\bar{M}^\frown c$, rather than the hull of $M^\frown c$. The only other change is that, at the beginning of the fourth paragraph, one replaces the hull of $M^\frown c$ by the hull of $\bar{M}^\frown c$.

Exercise 1 Show that, if $M \prec N$, $M \neq N$ are superstable modules, then there is an element in the difference whose type over M is regular and non-forking over 0.

For the following definition and result, which arose in discussion with Bouscaren, let T be totally transcendental. A non-realised type $p \in S_1(M)$ is **strongly regular** if there is a formula, φ, over M such that the pair (p,φ) satisifes the condition that every element $a \in M(p)\setminus M$ which satisfies φ actually satisfies p. In general, strong regularity is a more stringent condition than regularity: for t.t. modules they coincide.

<u>Theorem 6.27</u> [Pr85; 1.21], [Kuc87; 3.5] *Suppose that T is totally transcendental and let p be a type over the model M. Then the following conditions are equivalent.*
(i) p is regular.
(ii) p is strongly regular.
(iii) p_* is critical.

Proof The equivalence of (i) and (iii) is 6.23. Let a realise p and set $M(a)=M\oplus N$ and $a=(a_0,a_1)$ accordingly. Since T is t.t. there is a pp formula φ which is equivalent modulo T to $p_*{}^+$. So (6.4) p^+ is equivalent to $\varphi(v-a_0)$. Therefore, if p_* is critical then clearly $p(M(a))=\varphi(M(p)-a_0)\setminus M$; so $(p,\varphi(v-a_0)$ is a regular pair. Thus (iii) implies (ii). That (ii) implies (i) is true in general (see [Poi85; 20.10]) and is fairly obvious here - exercise. □

Strong regularity is theory-independent only within the context of t.t. theories. The type of a non-zero element of $\mathbb{Q}_{\mathbb{Z}}$ is strongly regular in the theory of $\mathbb{Q}$ but not in the theory of $\mathbb{Z}_{(p)}$.

I have already mentioned the term "weight" of a type: I define this now. Let p be a type over the set A. Let M be any pure-injective model containing A. The **weight** of p, wt(p), is defined if $N(p_*)$ is discrete, in which case it is the number of indecomposable direct summands in any decomposition of $N(p_*)$. Thus, wt(p)=1 iff p_* is irreducible, and if p is regular then wt(p)=1. If T is superstable then any type in finitely many free variables has its weight defined: indeed its weight is finite (exercise: prove this directly for modules). It is clear that if (the hull of) p_* is discrete, then the number of RK-minimal classes in the fundamental order below that of p is no more than wt(p), but (exercise) it may be less. The usual definition of wt(p) for $p \in S(M)$ (T superstable) is max{n : there exists a realising p and there exist $a_1,\dots,a_n$ with $a_i \perp \{a_j : j\neq i\}/M$ and $a_i \not\perp a/M$ for all i} (exercise: show that this definition gives the same result as that above). Although regular types are the ones usually employed for classifying superstable structures, the module case (see Chapter 10) seems to indicate that weight-one types are almost as useful (and there are many theories with "enough" weight-one types but not enough regular types). But also note §10.T that nice theories of modules (those with m-dimension) have enough regular types, provided we work in T^{eq}.

The weight of a type p depends only on the unlimited type p_*, but in some contexts it is the **algebraic weight**, algwt(p), of a type which it is appropriate to consider - I define this as the number of indecomposable direct summands in a decomposition of the hull $N(p)$ if this is discrete (otherwise the algebraic weight is ∞).

If $T=T^{\aleph_0}$ is not totally transcendental then there is some type, over infinitely many parameters, of infinite weight. In particular, if T is not superstable, then there is such a type. For, let M be a model of T and let φ_i $(i\in\omega)$ be pp formulas with $\varphi_0(M)>\varphi_1(M)>\dots$ (3.1). Choose elements $a_i \in \varphi_i(M)\setminus\varphi_{i+1}(M)$. By the argument of 2.11, there is some $a \in \mathrm{pi}(M^{(\aleph_0)})$ such that, if b_n is the element $(a_0,a_1,\dots,a_{n-1},0,0,\dots)$ of $M^{(\aleph_0)}$, then $\varphi_n(a-b_{n+1})$ holds. Since $\neg\varphi_n(b_{n+1})$ also holds, one has that for each n, a depends on

b_{n+1} over 0 and hence $a_n \not\downarrow a/\{a_0, a_1, \ldots, a_{n-1}\}$ (here, a_i lies in the i-th copy of M). But the a_i form an independent set (exercise). Therefore the hull of a has infinite weight. In general, one cannot get away with only finitely many parameters: consider $\mathbb{Z}_{(p)}^{\aleph_0}$.

Next, we see various conditions which guarantee existence of regular or weight-one (i.e., irreducible) types with some specified properties.

Recall **4.33**, which is a general construction of types which are irreducible and unlimited. With appropriate choice of Ψ (as there) the types maximal with respect to missing Ψ will actually be regular; if Ψ consists of all pp formulas satisfying some finiteness condition then regularity is essentially a consequence of the fact that forking is witnessed by pp formulas - so such a type tends to be very different from its forking extensions (see **9.9** for a way of making this precise).

First I give two direct applications of **4.33**.

Proposition 6.28 [PP87; 6.7] *Suppose that the Morley rank of the theory T is infinite or undefined. Then there is a 1-type over 0 which is non-isolated, irreducible and unlimited. If T is totally transcendental then p may, further, be taken to be regular.*

Proof Let Ψ be the set of all pp formulas, φ, such that the interval $[\varphi, 0]$ (in the lattice of pp-definable subgroups) has finite length. By modularity, Ψ is an ideal. By assumption, **5.12** and **5.18**, "$v=v$" is not in Ψ; so there will be a type p over 0, maximal with respect to the condition $p^+ \cap \Psi = \emptyset$. By **4.33**, any such p is irreducible.

Furthermore, p is non-isolated. For if $\varphi \wedge \bigwedge_1^n \neg\psi_i$ isolated p (where by **9.20** it may be supposed that φ is equivalent to p^+) then, as in the argument of **4.33**, one concludes that, for each i, $\varphi \wedge \psi_i$ is in Ψ. Then $\bigvee_1^n \varphi \wedge \psi_i \leq \Sigma_1^n \varphi \wedge \psi_i \in \Psi$; so the interval $[\Sigma_1^n \varphi \wedge \psi_i, 0]$ has finite length. But then, since $[\varphi, 0]$ has infinite length, there must be θ pp with $\varphi > \theta > \Sigma_1^n \varphi \wedge \psi_i$. This shows that the formula $\varphi \wedge \bigwedge_1^n \neg\psi_i$ is not complete, so cannot isolate p - a contradiction as required.

For similar reasons, p is unlimited (by **4.42**).

If T is t.t. then consider $N(p) = N(p_*)$ - a t.t. module. Take (by dcc on pp-definable subgroups) a critical type q with hull isomorphic to that of p. By **6.23**, q is regular. Furthermore, p non-isolated implies that p is not realised in the prime model, M_0, of T. Since M_0 is an essentially unique direct sum of indecomposable submodules, this means that $N(q) \simeq N(p)$ is not a direct summand of M_0. In particular q is not realised in M_0 - so q is non-isolated. Thus q is non-isolated, regular and unlimited, as required. □

In **6.28** one needs an assumption such as $\mathrm{MR}(T) \geq \aleph_0$: non-$\aleph_0$-categoricity is not enough to ensure a non-isolated *irreducible* type - consider $\mathbb{Q}_{\mathbb{Z}}$.

Proposition 6.29 [PP87; 6.8] *Let T be non-totally-transcendental. Then there is a 1-type over 0 which is unlimited, not finitely generated, contains no minimal pair (in particular, is non-isolated) and regular.*

Proof Let Ψ be the set of all pp formulas φ such that the interval $[\varphi, 0]$ has the dcc. Then Ψ defines an ideal in the lattice of pp-definable subgroups and, since T is not t.t., "$v=v$" is not in Ψ. So, by **4.33**, if p is maximal with respect to the condition $p^+ \cap \Psi = \emptyset$ then p is irreducible and clearly unlimited.

Also p is not finitely generated. For if there were some pp formula φ equivalent to p^+, then any ψ strictly below φ would, by maximality of p^+, have to be in Ψ; so $[\psi, 0]$ would have the dcc. But then $[\varphi, 0]$ would have the dcc - contradiction.

If $\varphi/\psi \in p$ were a minimal pair, then there would be $\varphi' \in p$ with $\varphi' \wedge \psi \in \Psi$ - so it may be assumed that $\psi \in \Psi$ (by **9.1**). But then the interval $[\varphi, 0]$ would have the dcc - contradiction.

Now suppose that $q \in S_1(A)$ is a forking extension of p. So there is $\psi \in \mathcal{G}(q) \setminus \mathcal{G}_0(p)$. By maximality of p^+, there is $\varphi \in p^+$ with $\varphi \wedge \psi \in \Psi$ - so it may be supposed that ψ already is in Ψ. Let $p_1 \in S_1(A)$ be a non-forking extension of p. Take realisations c, b of p_1, q respectively. In order to show that p is regular it must be proved that c and b are independent over A.

If they were not independent then there would be some pp formula φ linking them. But then, since ψ has the dcc, **9.10** implies that p_1, so p, contains a minimal pair - contradicting the last paragraph. (The proof of this in [PP87] is related, but more direct). □

The proof above actually has shown the following, since every forking extension of p (as above) clearly has Morley rank.

Corollary 6.30 [Pr81c] *Let T be non-totally-transcendental. Then there is a (regular unlimited) 1-type over 0 which has U-rank but does not have Morley rank.* □

Corollary 6.31 *Every complete theory of modules with an infinite model has discrete pure-injective models of arbitrarily large cardinality.*

Proof Certainly this is true if the theory has continuous part zero. On the other hand, if $T_C \neq 0$ then, by **3.14**, T is not t.t. and so, by **6.29**, there is an unlimited irreducible 1-type; so clearly (using **4.36**) the result follows. □

As another application of the idea behind **4.33**, I give the result of Pillay which is that, if T is a countable complete theory of modules with only finitely many countable models, then it has just one model (up to isomorphism). In fact the presentation here rather reverses the order of discovery, since **4.33** was generalised from the argument of [Pi84a] and from analogous arguments which were suggested by the corresponding technique in ring theory. The notation $n(\kappa, T)$ is used for the number of models of T of cardinality κ (up to isomorphism).

Theorem 6.32 [Pi84a] *Let T be a countable theory of modules. If T is not $\aleph_0$-categorical then $n(\aleph_0, T) \geqslant \aleph_0$. In fact there exist, in this case, $\aleph_0$ non-isomorphic models of the form $M \oplus A^n$ $(n \in \omega)$ for suitable $M (\models T)$ and A.*

Proof First, I give a proof, taken from [Pr81c], on the basis of results already established. Then I give the original proof of [Pi84a] which is more direct, but appeals to some general stability theory. The result also follows from Ziegler's classification of the models of a theory with m-dimension, as presented in §10.4.

1. If T is totally transcendental then the result follows directly from the classification theorem (**4.63**) for the models of such theories.

If $|D^T(0) = \bigcup\{S_n{}^T(0) : n \in \omega\}| = 2^{\aleph_0}$ then, of course, T must have $2^{\aleph_0}$ countable models to hold all the $2^{\aleph_0}$ n-types (for suitable n). The more precise statement may be deduced using the fact that there is, by **6.29**, an irreducible type p with no minimal pair. It follows (see after **10.24**) that there is some pure-injective model not realising p. Let M_0 be a countable submodel, and then proceed as in the next paragraph.

If $D(T)$ is countable then T has a prime model M_0 (**1.6**). Let p be as in **6.29** and note that, since p is unlimited, if A is any countable pure submodule of $N(p)$ then $M_0 \oplus A^n$ is a model of T for each $n \in \omega$. Moreover, since the types realised in M_0 are all isolated, and since p does not contain a minimal pair, it follows by **9.22** that the indecomposable $N(p)$ is not a direct summand of $\overline{M_0}$. Hence the $M_0 \oplus N(p)^n$ are non-isomorphic, and so (by uniqueness of pure-injective hull) the same is true of the $M_0 \oplus A^n$.

2. The more direct proof proceeds as follows. Let Ψ be the set of all pp formulas ψ such that the interval $[\psi, 0]$ is a finite set. Now, Ψ need not be closed under addition (see the example below), so **4.33** itself cannot be applied. In any case, let p be chosen maximal not

containing any formula from Ψ. Observe that if ψ is pp and not in p^+ then there must be $\varphi \in p^+$ with $\varphi \wedge \psi \leq \bigvee_1^n \psi_i$ for suitable $\psi_1, \ldots, \psi_n$ in Ψ.

If p were isolated, say by $\varphi \wedge \bigwedge_1^m \neg\theta_j$ where $\theta_j < \varphi$ for each j then, for each $j = 1, \ldots, m$, choose $\psi_{j1}, \ldots, \psi_{jn_j}$ in Ψ such that $\theta_j \leqslant \bigvee\{\psi_{ji} : i = 1, \ldots, n_j\}$. It is claimed that the interval $[\varphi, 0]$ is finite. For, if $\theta < \varphi$ then, since $\varphi \wedge \bigwedge_1^n \neg\theta_j$ is supposed to isolate p, it must be that $\theta \leqslant \bigvee_1^m \theta_j$. Hence $\theta \leqslant \bigvee\{\psi_{ji} : j = 1, \ldots, m \quad i = 1, \ldots, n_j\}$ and so $\theta = \bigvee_j \bigvee_i \theta \wedge \psi_{ji}$. But, since each ψ_{ji} is in Ψ, there are only finitely many possibilities for the $\theta \wedge \psi_{ji}$; hence there are only finitely many possibilities for their union θ. Thus $\varphi \in \Psi$ - contradiction. So p is indeed non-isolated.

Moreover p has U-rank. For if q is a forking extension of p, then q represents some pp formula $\theta(v, \bar{y})$ not represented in p. But then there is $\varphi \in p$ such that $\varphi(v) \wedge \theta(v, \bar{0}) \in \Psi$. Clearly, any formula in Ψ has finite Morley rank - so in particular has U-rank. Therefore $\mathrm{UR}(p)$ exists and is even less than or equal to ω.

Now it is a result of Lascar that if T has a non-isolated type with U-rank then T has at least countably many models, and one may simply quote that result. In [Pi84a] Pillay gives a direct proof (which gives more information on what the models look like), using the fact that we have a non-isolated type of U-rank $\leqslant \omega$. I refer the reader there for the proof and references. One should not expect elementary dimension theory to be immediately applicable here since, as is shown by Ex 3 below, the type p produced above need not be irreducible, let alone regular. □

Exercise 2 Show that every non-trivial complete theory of modules has a 1-type of U-rank 1 (necessarily regular) which is non-isolated if T has infinitely many algebraic elements (by "non-trivial" I mean that there is more than one model).
[Hint: consider those formulas ψ such that the index $[\psi:0]$ is finite.]

Example 3 [Pr81c] Let K be an infinite field and set R to be the ring $K[x_1, x_2 : x_i x_j = 0, i, j \in \{1,2\}]$. Let T be the theory of R_R. The definable subgroups have already been described Ex 2.1/6(vi). In particular, T is t.t. with prime model R and has Morley rank 3 - so **6.28** does not apply.

There is, in fact, just one non-isolated 1-type: the type $p(v)$ "at J" which says that $vx_1 = 0 = vx_2$ but which does not place "v" in any of the ideals $\langle x_1 + x_2 k\rangle$ $(k \in \mathrm{P}(K))$. This is the type constructed in the second proof of **6.32** above. Note that it is reducible. Explicitly, it is the type of any element of $R \oplus R$ which lies in $J \oplus J$, has both co-ordinates non-zero, and does not lie in the diagonal submodule.

One has $N(p) \simeq R \oplus R$, $p = p_k \cap p_l$, where $p_k = \mathrm{pp}(x_1 + x_2 k))$ for any $k \neq l$ in $\mathrm{P}(K)$.

Example 4 [Pr81c] The type constructed in **6.28** may have any U-rank between 1 and ω.

(i) Take the theory of the abelian group $\mathbb{Z}_{2^\infty}{}^k$ for some finite k. Here the types constructed in **6.28** and **6.32**(2) coincide, being the type of a torsionfree element (hull $\mathbb{Q}$), and certainly this type has MR = UR = 1 (since k is finite).

(ii) A non-t.t. example is the theory of the abelian group $\mathbb{Z}_{(p)}$. Again, one obtains a type p whose hull is $\mathbb{Q}$ (the type of a non-zero divisible element) and clearly $\mathrm{UR}(p) = 1$, but $\mathrm{MR}(p)$ is undefined.

(iii) Fix $n \in \omega$, $n \geqslant 1$ and let $R = K[x_i, y_j \ (i,j \in \omega) : x_i x_j = 0 = y_i y_j,\ x_{n+i} y_j = 0 \quad (i,j \in \omega)]$ where K is a finite field. Let T be the theory of R_R. It is easily checked that T is ω-stable. In this case there are finitely many choices for the types constructed in **6.28** and **6.32**(2).
Exercise: determine the models for this theory.

(iv) Take the theory of the abelian group $\mathbb{Z}_{2^\infty}{}^{\aleph_0}$. In this case **6.28** and **6.32**(2) give the same type, just as in (i), but this type has rank ω.

(v) Here is an example where both constructions yield a type but they give different types.

Let K be a countable field and let R be the commutative ring $K[x_i\,(i\in\omega): x_i x_j = 0\ (i,j\in\omega)]$. Take T to be the theory of R_R (cf. Ex2.1/6(vi)). If K is finite then the types constructed will coincide and they will have Morley and U-ranks 1. But if K is infinite then there are $\aleph_0$ choices for the type p given by **6.32**(2), and in all cases $\mathrm{MR}(p)=\mathrm{UR}(p)=2$. But there is just one choice for the type q as constructed by **6.28**, and $\mathrm{MR}(q)=\mathrm{UR}(q)=\omega$.

6.1 An example: injective modules over noetherian rings

Eklof and Sabbagh [ES71] showed that if R is a right coherent ring, then the model-completion of the theory of R-modules is the largest theory of injective modules and the models are the "fat" absolutely pure modules. This theory has complete elimination of quantifiers (see §16.1) and so various model-theoretic concepts take on a particularly transparent, algebraic, form. A large number of the results of §5.1-5.2 and §6.1-6.4 were first proved in this setting and the proofs of their generalisations are usually the result of seeing just what is special about the "purely algebraic" case (in particular, one has to learn how pp-types replace right ideals).

If the ring is right noetherian then every model of the aforesaid theory is injective. Indeed, every injective module over a right noetherian ring is totally transcendental. (Exercise 1: give a proof of this. Exercise 2: give another.) In this section I will interpret some of what we have discovered in terms of the largest theory of injective modules over such a ring.

Although I will simply use this case for illustration, one should be aware that a good deal of the algebra which I will simply quote may be derived by a model-theoretic approach: see [Kuc84], [Kuc87]. In particular, Kucera is able to recover, from a stability-theoretic development, many aspects of: primary decomposition for commutative noetherian rings; the Lesieur-Croisot generalisation of this to right noetherian rings (see, e.g., [St75; §VII.1]); results of Lambek and Michler on the connections between right ideals of a right noetherian ring and indecomposable injective modules.

I will begin with a commutative noetherian ring R, then say something concerning the non-commutative case.

Matlis [Mat58] clarified the structure of the injective modules over a commutative noetherian ring. There is a bijection between the prime ideals of R and the indecomposable injectives. It is given by taking the prime P to the injective hull of an element with annihilator precisely P: $P \mapsto E(R/P)$ - denote this injective by E_P.

Matlis shows that if a is a non-zero element of E_P then the annihilator, Q, of a is a P-primary ideal (i.e., $Q\leqslant P$, $P^n\leqslant Q$ for some n and $rs\in Q$, $r\notin Q$ implies $s\in P$). Moreover, if $Q_1\cap\ldots\cap Q_k = I$ is an irredundant primary decomposition of the ideal I of R, then (e.g., [SV72; 4.9]) the injective hull of R/I is isomorphic to the direct sum $E(R/P_1)\oplus\ldots\oplus E(R/P_k)$, where Q_i is P_i-primary.

One may show the following. The pp-definable subgroups of the indecomposable injective E_P are precisely the annihilators of the $\cap$-irreducible P-primary ideals. There is a unique minimal pp-definable subgroup - the annihilator of P - so the type of an element with annihilator P is the unique regular type (**6.23**) for that indecomposable. This minimal pp-definable subgroup has the structure of a 1-dimensional vectorspace over $\mathrm{End}(E_P)/J\mathrm{End}(E_P)$ (**4.53**): Matlis shows that this division ring is just the quotient field of the integral domain R/P. If a is an element of an injective module and its annihilator I is decomposed as above, then the algebraic weight of the type (§6.4) of a is k. Elements a and b have orthogonal types (**6.20**) iff their annihilators have no associated primes in common iff, up to isomorphism, their hulls have no non-zero direct summand in common.

Over non-commutative noetherian rings, there is no longer a bijection between prime ideals of the ring and indecomposable injectives. There is an "intermediate" case - that of fully bounded noetherian (FBN) rings - where one still has this bijection. Over such a ring, the injective hull of R/P, though not necessarily indecomposable, is a finite direct sum of indecomposable injectives, all isomorphic to each other and every indecomposable injective arises in this way. But, in the general right noetherian case, if one wants isomorphism classes of indecomposable injectives to correspond to "primes", then one must be content with "prime" torsion theories (see [Gol75]). In terms of right ideals, the primes are replaced by relatedness-classes of $\cap$-irreducible right ideals: irreducible right ideals I and I' are related if there exist ring elements r and s such that $(I:r)=(I':s)\neq R$ (see [LM73]). There is a bijection between relatedness classes of right ideals and indecomposable injective modules, given by $[I]\mapsto E(R/I)$.

To explain this further, let a and b be non-zero elements of an indecomposable injective E and let their annihilators be respectively I and I'. Since an indecomposable injective is uniform, there are $r,s\in R$ with $ar=bs\neq 0$. Then the annihilator of ar is just $(I:r)$ and the annihilator of bs is $(I':s)$. The proof of the correspondence is now left as an exercise. The reader will appreciate where some of our terminology has come from.

The reader who wishes to pursue these matters, especially this non-commutative case, further, should consult [LM73] and also the relevant papers of Kucera and the author, especially [Pr82] and [Kuc87]. Herzog (personal communication) has some work in progress on "primary decomposition" for types of finite weight.

6.5 Saturation and pure-injective modules

In modules, one typically obtains results for the class of pure-injective modules, where the general case would have a class of saturated models. This section elucidates the relation between saturation and pure-injectivity. The results are taken from [PP87].

Saturation in modules can be resolved into "vertical" and "horizontal" components. The vertical component is pure-injectivity - positive saturation: the horizontal component is a kind of "fatness" (weak saturation). This statement will be made precise.

Recall that if M is $|T|^+$-saturated then M is pure-injective (2.9). The first result is [PP87; 4.1]: a slightly weaker statement is [Poi84; Thm2].

<u>Proposition 6.33</u> *Let $\kappa\geq|T|^+$. Let M be κ-saturated and suppose that N is a pure-injective elementary extension of M. Then N also is κ-saturated.*

Proof Let A be a subset of N, of cardinality strictly less than κ, and let p be a 1-type over A. Without loss of generality (1.3), it may be assumed that A is an elementary submodel of N.

Since M is κ-saturated, there is $A'\leq M$ realising the same type over 0 as A. By hypothesis, the type, p', over A' which corresponds to p over A is realised, by c say, in M. Working inside M, set $N(A'^\frown c)=N(A')\oplus N'$ for some $N'=N(c')$, say. Since A, hence $N(A')$, is a model, N' is unlimited.

Since M is κ-saturated, it contains more than $|A|$ copies of N': so the same is true of N. Therefore there is a copy N'' of N' in N such that $N(A)\oplus N''$ is a summand of N. Hence p is realised in N, as required. □

<u>Lemma 6.34</u> [PP87; 4.2] *If M is superstable and pure-injective then so is every elementary extension of M.*

Proof This is immediate by 3.8. □

<u>Proposition 6.35</u> [PP87; 4.3] ([Gar80; Thm5] for the t.t case) *Let T be superstable. Then any pure-injective model, M, of T is homogeneous: given any finite tuples $\bar a$ and $\bar b$ with the same type over 0, there is an automorphism of M taking $\bar a$ to $\bar b$.*

Proof Let $\bar a,\bar b$ be as above. Set $M=N(\bar a)\oplus N'=N(\bar b)\oplus N''$ for suitable N', N''. Then there is (by 4.15) an isomorphism $N(\bar a)\xrightarrow{f}N(\bar b)$ taking $\bar a$ to $\bar b$.

Superstable pure-injectives are discrete (by 3.14 and 4.37 for example). So M and its summands have essentially unique decompositions as pure-injective hulls of direct sums of indecomposables. When the hulls of $\bar{a}$ and $\bar{b}$ are decomposed there may well be infinitely many summands (consider $a=1$ in $\mathbb{Z}$), but each summand occurs only finitely many times in the decomposition. For otherwise, let us suppose that $N_0^{(\aleph_0)}$ is a direct summand of $N(\bar{a})$: then $M \simeq M \oplus N_0$ (clearly!), so N_0 is unlimited. Choose non-zero elements a_i ($i \in \omega$), one from each of these copies of N_0. By 5.24, the family $\{a_i : i \in \omega\}$ is an independent one (i.e., $a_i \perp A_i/0$ for each i, where $A_i = \{a_j : j \neq i\}$). But, being unlimited and in the hull of $\bar{a}$, each a_i is dependent on $\bar{a}$ over 0 (6.9). This contradicts the fact that, in a superstable theory, every finite tuple has finite weight (see, e.g., [Poi85; 19.10]).

Since each summand of $N(\bar{a}) \simeq N(\bar{b})$ therefore occurs only finitely many times, it follows that $N' \simeq N''$ (for the decomposition of M is essentially unique). Say $N' \xrightarrow{g} N''$ is an isomorphism.

Then $f \oplus g$ is an automorphism of M which takes $\bar{a}$ to $\bar{b}$, as required. □

Superstability is not necessary for the proof of 6.35 (in contradiction with the comment after [PP87; 4.3]!). The conclusion is true provided it is assumed that T has m-dimension in the sense of Chapter 10. This follows by the first part of the proof of 6.35 together with 7.16.

<u>Proposition 6.36</u> *If T has m-dimension then every pure-injective model of T is homogeneous.* □

The model M is $\mathbf{F}^a_\kappa$-**saturated** if M realises all strong 1-types (see §6) over subsets of cardinality strictly less than κ. Recall that the model M is weakly saturated if it realises all n-types over 0 for each $n \in \omega$.

Exercise 1 Show that if M is pure-injective and weakly saturated and if $p \in S(0)$ is unlimited then $\mathrm{pi}(N(p)^{(\aleph_0)})$ is a direct summand of M.

<u>Proposition 6.37</u> [PP87; 4.5] *Suppose that M is superstable. Then M is $\mathbf{F}^a_{\aleph_0}$-saturated iff M is pure-injective and weakly saturated.*

Proof $\Rightarrow$ Clearly, if M is $\mathbf{F}^a_{\aleph_0}$-saturated then M is weakly saturated. To show that M is pure-injective, let $\Phi(v)$ be a set of pp formulas over M which is finitely satisfied in M; it must be shown that Φ is satisfied in M. Since T is superstable there is (3.1) some $\varphi(v,\bar{a})$ in Φ such that, for every $\psi(v,\bar{b})$ in Φ, the index $[\varphi(v,\bar{0}):\varphi(v,\bar{0})\wedge\psi(v,\bar{0})]$ is finite.

Choose c in $\tilde{M}$ satisfying Φ. Since M is $\mathbf{F}^a_{\aleph_0}$-saturated there is d in M with the same strong type over $\bar{a}$ as c. So by 6.43 below one has, for each $\psi(v,\bar{b}) \in \Phi$, that $\psi(c-d,\bar{0})$ holds. Therefore from $\psi(c,\bar{b})$ follows $\psi(d,\bar{b})$; and this for all $\psi(v,\bar{b})$ in Φ. So the element d of M does satisfy Φ.

$\Leftarrow$ Suppose, conversely, that M is pure-injective and weakly saturated. Since, by 6.35, M is also homogeneous, M is (exercise) $\aleph_0$-saturated.

Let $\bar{a}$ be a finite subset of M and let $c \in \tilde{M}$. It must be shown that there is $d \in M$ with the same strong type as c over $\bar{a}$. Let $q = \mathrm{tp}(c/M)$. Since M is pure-injective there is $d' \in M$ satisfying q^+. Let $p = \mathrm{tp}(c/\bar{a})$.

Consider $p(v) \cup \{\varphi(v-d') : \varphi \in \mathcal{G}_0(q)\}$. This set is consistent, being realised by c, and is over $\bar{a}^\frown d'$ - a finite set. So, M being $\aleph_0$-saturated, there is $d \in M$ realising it.

From $p(d)$ one has $\mathrm{tp}(d/\bar{a}) = \mathrm{tp}(c/\bar{a})$. Since c and d both satisfy $\{\varphi(v-d') : \varphi \in \mathcal{G}_0(q)\}$ it follows from 6.43 below that c and d actually have the same strong type over $\bar{a}$. □

<u>Corollary 6.38</u> *Suppose that M is a pure-injective superstable module. Then M is weakly saturated iff M is $\aleph_0$-saturated.* □

<u>Corollary 6.39</u> *Suppose that M is a pure-injective module with m-dimension. Then M is weakly saturated iff it is $\aleph_0$-saturated.* □

These corollaries follow by (the proof of) **6.37** and **6.36**, since $F^a_{\aleph_0}$-saturated implies $\aleph_0$-saturated.

One may as whether superstability is necessary in **6.37**. The following example shows that one needs some hypothesis for the direction "⇒". Take T to be a non-t.t. theory satisfying $T=T^{\aleph_0}$ and with only countably many n-types for each $n\in\omega$. For examples see §7.2. Since $T=T^{\aleph_0}$, strong types do not really differ from types, so the countable saturated model (which exists since T is small) is actually $F^a_{\aleph_0}$-saturated. But, since T is not totally transcendental, it is not pure-injective. In fact, we have the following (which was pointed out to me by Rothmaler).

<u>Proposition 6.40</u> *Suppose that every $F^a_{\aleph_0}$-saturated model of T is pure-injective. Then T is superstable.*

Proof If T is not superstable, then its unlimited part is not totally transcendental. So there is an unlimited pure-injective summand, N, of a model of T which is not Σ-pure-injective: that is, $N^{(\aleph_0)}$ is not pure-injective. Let M be a sufficiently large $F^a_{\aleph_0}$-saturated model of T that $N^{(\aleph_0)}$ purely embeds in M (actually, $F^a_{\aleph_0}$-saturated is enough). Consider the model $M\oplus N^{(\aleph_0)}$.

Any finite subset of this model lies in $M\oplus N^{(k)}$ for some finite k. Since that is isomorphic to M, all strong types over finite sets are realised. Hence $M\oplus N^{(\aleph_0)}$ is $F^a_{\aleph_0}$-saturated. But it is not pure-injective. □

One may note that the last part of the proof of **6.37** shows that every pure-injective $\aleph_0$-saturated module is $F^a_{\aleph_0}$-saturated.

<u>Corollary 6.41</u> [PP87; 4.6] *Suppose that M is superstable and $F^a_{\aleph_0}$-saturated. Then every elementary extension of M also is $F^a_{\aleph_0}$-saturated.*

Proof By **6.37** M is weakly saturated and pure-injective. If N is an elementary extension of M then certainly N is weakly saturated (from the definition) and, by 6.34, also is pure-injective. So by **6.37**, N is $F^a_{\aleph_0}$-saturated. □

6.6 Multiplicity and strong types

Let p be a type. Recall that the multiplicity of p, mult(p), is the number of non-forking extensions of p to any model. Recall also that p is said to be stationary if mult(p)=1. It has already been noted (5.4) that if $G_0(p)=G(p)$ then p is stationary and that the converse is false (Ex5.1/2). The precise determination of those types which are stationary is not straightforward, even in the abelian group case.

Rothmaler [**Rot83c**; Thm4] characterises the stationary abelian groups. He distinguishes between the condition that every type be stationary and the strictly weaker one that every non-algebraic type be stationary and determines the abelian groups with each property. For the list of possibilities, the reader is referred to [**Rot83c**], but Rothmaler points out that the example $\mathbb{Z}_2\oplus\mathbb{Q}$ is indicative of the general stationary theory of abelian groups which is not closed under products (he also shows [**Rot83c**; Thm1] that it is enough to look at 2-types).

In this section, strong types are characterised and then this is used to estimate mult(p). It is also shown that the dimensions in a model of different strong extensions of a regular type are closely related.

The strong type of $\bar{c}$ over A, stp($\bar{c}/A$), is the set $\{cE- : E$ is an equivalence relation, definable over A, with only finitely many classes$\}$ together with tp($\bar{c}/A$). One has (see [Pi83; 4.35]): if $p\in S(A)$, $A\le M\models T$ and $\bar{c},\bar{d}$ realise non-forking extensions of p to M, then $\mathrm{tp}(\bar{c}/M)=\mathrm{tp}(\bar{d}/M)$ iff $\mathrm{stp}(\bar{c}/A)=\mathrm{stp}(\bar{d}/A)$. So, to estimate the multiplicity of p, one may estimate the number of strong types extending p. The following definitions and results should come as no surprise.

Given $\bar{c}$ and A, let $\mathrm{stp}^+(\bar{c}/A)=\{\bar{c}+\varphi(\bar{v}) : \varphi(\bar{v})\in \mathcal{G}_0(\mathrm{tp}(\bar{c}/A))\}$ – the set of cosets of groups in $\mathcal{G}_0(\mathrm{tp}(\bar{c}/A))$ to which $\bar{c}$ belongs. Clearly $\mathrm{stp}^+(\bar{c}/A)\le\mathrm{stp}(\bar{c}/A)$; it will be seen that $\mathrm{stp}^+(\bar{c}/A)$ determines $\mathrm{stp}(\bar{c}/A)$.

Lemma 6.42 *Given parameters $\bar{c}$, A and a pure-injective model M of T containing A, there is $\bar{m}$ in M with $\mathrm{stp}^+(\bar{m}/A)\ge\mathrm{stp}^+(\bar{c}/A)$.*
If $\bar{m}$ is any such tuple and if $p=\mathrm{tp}(\bar{c}/M)$ is a non-forking extension of $\mathrm{tp}(\bar{c}/A)$, then $p=\bar{m}+p_*{}^M$.

Proof Since M is pure-injective the first statement is clear. So let $\bar{m}$ be in M with $\mathrm{stp}^+(\bar{m}/A)\ge\mathrm{stp}^+(\bar{c}/A)$. Let $q=p-\bar{m}=\mathrm{tp}(\bar{c}-\bar{m}/M)$; it will be shown that $q=p_*{}^M$. It will suffice to show that the restriction of q to 0 is p_* (i.e., that p_* is the type of $\bar{c}-\bar{m}$ over 0). For then the fact that $G(q)=G(p^M)=G_0(p)$ (since $q=p-\bar{m}$), which in turn in turn equals $G(p_*)$ (by 6.4), implies that both q and $p_*{}^M$ are non-forking extensions to M of the same stationary type - so are equal.

Now, $\varphi(\bar{v})\in p_*$ implies $\varphi(\bar{v})\in G(p)$ (6.4), so there is $\psi(\bar{v},\bar{y})$ represented in p - say $\psi(\bar{c},\bar{m}')$ holds - with $\psi(\bar{v},\bar{0})$ equivalent to $\varphi(\bar{v})$. From $\psi(\bar{c},\bar{m}')$ and $\mathrm{stp}^+(\bar{m}/A)\ge\mathrm{stp}^+(\bar{c}/A)$ it follows (by definition of stp^+) that $\psi(\bar{m},\bar{m}')$ holds. Thus $\psi(\bar{c}-\bar{m},\bar{0})$ - that is, $\varphi(\bar{c}-\bar{m})$ - holds.

Conversely, $\varphi(\bar{c}-\bar{m})$ implies $\varphi(\bar{v})\in\mathcal{G}_0(p)$ and so $\varphi(\bar{v})\in p_*$, as required. □

Proposition 6.43 [PP87; 3.6] *Tuples $\bar{c}$ and $\bar{d}$ with the same type, p, over a set A have the same strong type iff $\mathrm{stp}^+(\bar{c}/A)=\mathrm{stp}^+(\bar{d}/A)$: that is, iff $\psi(\bar{c}-\bar{d})$ holds for every $\psi\in\mathcal{G}_0(p)$.*

Proof The direction "⇒" is immediate from the definitions of stp and $G_0(-)$. So suppose that $\mathrm{stp}^+(\bar{c}/A)=\mathrm{stp}^+(\bar{d}/A)$. Choose a pure-injective model, M, containing A and independent from $\bar{c}^\frown\bar{d}$ over A. As mentioned already, it will be enough to show that $p=\mathrm{tp}(\bar{c}/M)=\mathrm{tp}(\bar{d}/M)=q$.

Let $\bar{m}$ in M with $\mathrm{stp}^+(\bar{m}/A)\ge\mathrm{stp}^+(\bar{c}/A)=\mathrm{stp}^+(\bar{d}/A)$ be as in 6.42. Then, by that result, $p=\bar{m}+p_*{}^M$ and $q=\bar{m}+q_*{}^M$. Also, $\bar{c}$ and $\bar{d}$ have the same type over A and they are independent from M over A, so $p_*{}^M=q_*{}^M$ (since, for example p_* "is" the connected component, $G_0(\mathrm{tp}(\bar{c}/A))$, of $G(\mathrm{tp}(\bar{c}/A))$). Thus the result is proved. □

Referring back to the proofs of **5.2** and **5.11**, we see now what the completions of the (generally incomplete) type $p^+(\bar{v})\cup\{\neg\psi(\bar{v},\bar{m}) : \psi\notin\mathcal{G}_0(p),\ \bar{m}\in M\}$ are - they are simply the strong types extending p and we may see that, in the proof of **5.11**, the choice of element $\bar{c}$ simply was a choice of strong type extension of p.

What is the multiplicity of a type p?, that is, how many strong type extensions does it have? Clearly, by 6.43, it is the number of cosets of $G_0(p)$ in $G(p)$ which are "consistent with p".

If the order of $G_0(p)$ in $G(p)$ is finite then, since $G_0(p)$ certainly is one excluded coset (for then $G_0(p)$ is just $G(p)\wedge\varphi$ for some φ), one has $\mathrm{mult}(p)<[G(p):G_0(p)]$. It is easy to construct examples where $\mathrm{mult}(p)$ is much less that the index of $G_0(p)$ in $G(p)$.

Corollary 6.44 [PP87; 3.6] *Let p be a type over A.*

(i) *If $[G(p):G_0(p)]$ is finite then its value strictly bounds the multiplicity of p.*

(ii) *If $[G(p):G_0(p)]$ is infinite then $\mathrm{mult}(p)\geqslant 2^{\aleph_0}$, with equality if R is countable.*

Proof (i) This has just been discussed.

(ii) Let $\theta(\bar{v},\bar{a}) \in p^+$. Consider the tree of cosets, contained in $\theta(\tilde{M},\bar{a})$, of pp formulas in $\mathcal{G}_0(p)$. Since the index $[G(p):G_0(p)]$ is greater than one, there is a coset C of (say) $\varphi(\bar{v}) \in \mathcal{G}_0(p)$ which is not definable over A. It follows that if $\psi(\bar{v}) \in \mathcal{G}_0(p)$ with $\psi \to \varphi$ then every coset of ψ which is contained in C is undefinable over A. For if $\psi'(\bar{v},\bar{b})$ with $\bar{b}$ in A defined such a coset, then the formula $\exists \bar{w}(\psi'(\bar{w},\bar{b}) \wedge \varphi(\bar{v}-\bar{w}))$ would define C. Also, since $[G(p):G_0(p)]$ is infinite, the tree of cosets of formulas $\psi \in G_0(p)$ with $\psi \to \varphi$ is of infinite depth. Hence there are $2^{\aleph_0}$ branches through it, and hence $\text{mult}(p) \geq 2^{\aleph_0}$, as required. ◻

Necessarily one has $\text{mult}(p) \leq 2^{|R|}$, and it is a general result (see [Pi83; 5.21]) that if $\text{mult}(p) \geq \aleph_0$ then $\text{mult}(p) \geq 2^{\aleph_0}$.

<u>Proposition 6.45</u> [PP87; 6.9] *Let p be a regular type over* 0.

(a) *Let $a_0, a_1, \ldots, a_n$ be an independent set of realisations of p, all with the same strong type. Then $\{a_1-a_0, \ldots, a_n-a_0\}$ is an independent set of realisations of p_*.*

(b) *Let b realise p and let $\{a_1, \ldots, a_n\}$ be an independent set of realisations of p_*, this set being independent from b over 0. Then $\{b, a_1+b, \ldots, a_n+b\}$ is an independent set of realisations of p and, moreover,* $\text{stp}(a_i+b) = \text{stp}(b)$ *for each i.*

Proof (i) By 6.43 one has $\varphi(a_i-a_0)$ for each i and each $\varphi \in p_*{}^+$. On the other hand, if φ is pp and $\varphi \notin p_*{}^+$ then, since $a_i \mathop{\downarrow} a_0$, one has $\neg\varphi(a_i-a_0)$ and also $\varphi \notin \mathcal{G}_0(\text{tp}(a_i))$. Therefore the a_i-a_0 do realise p_*.

That they are independent follows from general considerations. Suppose inductively that $\{a_1-a_0, \ldots, a_k-a_0\}$ is an independent set. Choose a model, M, containing $a_0, \ldots, a_k$ and independent from a_{k+1} over 0. Then, since $a_0 \in M$, clearly $(a_{k+1}-a_0) \mathop{\downarrow} M/0$ so the conclusion follows.

(ii) That all the a_i+b realise p and have the same strong type as b follows from 6.42 and 6.43. Independence follows as in (i). ◻

<u>Corollary 6.46</u> [PP87; 6.10] *Let p be a regular type over* 0. *Let M be a model of T and suppose that $a \in M$ realises p. Set $n = \dim(p_*, M)$ (the maximum number of independent realisations of p_* in M). Then $\dim(\text{stp}(a), M)$ is n or* $n+1$.

Proof Since p is regular so is p_* (6.25). Therefore (see, e.g., [Poi85; Chpt 20]) $\dim(p_*, M)$ is well-defined. By 6.45(a), one has $\dim(\text{stp}(a), M) \leq n+1$.

To show that $\dim(\text{stp}(a), M) \geq n$, take an independent set, $\{b_1, \ldots, b_n\}$ of realisations of p_* in M. Let B denote a maximal subset of $\{b_1, \ldots, b_n\}$ such that a is independent from B over 0. Then it follows, since p has weight one, that $B \supseteq \{b_1, \ldots, b_n\} \setminus \{b_i\}$ for some i. So, by 6.45(b), $\{a\} \cup \{a+b : b \in B\}$ is an independent set of realisations of $\text{stp}(a)$ in M. Hence $\dim(\text{stp}(a), M) \geq n$, as required. ◻

Examples where $\dim(p_*, M) = n$ and $\dim(\text{stp}(a), M) = n+1$ are not difficult to find: take p to be the type of $1 \in \mathbb{Z}_{(p)}$ in the theory of that group and let $M = \mathbb{Z}_{(p)} \oplus \mathbb{Q}^n$ (note that p_* is the type of $1_{\mathbb{Q}}$). The other case is obtained if $p = p_*$.

CHAPTER 7 SUPERSTABLE MODULES

In Chapter 4 we obtained a fairly complete classification for the models of a totally transcendental theory of modules. We cannot expect to have as complete results for superstable modules, but we can at least hope to say a lot about them. For instance, we can classify the pure-injective models although, of course, that is something we are able to do in circumstances way beyond the superstable case. Indeed, it is quite noticable that the gap between the superstable and arbitrary cases appears to be much narrower in modules than in general stable structures.

The difficulties in superstable modules lie, not in the (t.t.) unlimited part, but inside the prime-pure-injective model. For example, Vaught's Conjecture is open for superstable modules (whereas it followed immediately for t.t. modules from the classification in §4.6). Even for modules of U-rank 1, Vaught's Conjecture was confirmed only recently (1986) by Buechler, and his proof is far from trivial.

The first section is concerned with the uncountable spectrum: given an uncountable cardinal λ, how many models are there (up to isomorphism) of cardinality λ? Shelah showed that for stable theories there is a severely limited number of possible spectrum functions $\lambda \mapsto n(\lambda, T)$. Ziegler showed which possibilities can be realised in modules (one need look no further than abelian groups). A proof, which does not depend on his classification result (§10.4), is given here.

A module has U-rank 1 iff every infinite pp-definable subgroup is of finite index in the whole module. It turns out (§2) that every model of a U-rank 1 theory of modules is the direct sum of a unidimensional totally transcendental part and an elementary submodule of the prime-pure-injective model. This is not yet strong enough to give us Vaught's Conjecture. An example is presented of a superstable, non-totally-transcendental, theory of modules with only countably many countable models, showing that a superstable module need not have $2^{\aleph_0}$ types (thus ruling out a simple solution to Vaught's Conjecture). The "module-theoretic" part of Buechler's proof (of Vaught's Conjecture for modules of U-rank 1) is presented: to deal with the "geometry of types" part would take us too far afield. Some partial results and examples relating to the conjecture are included.

In the third section, it is shown that a theory of modules of U-rank n has no more than n dimensions: that is, the number of isomorphism types of unlimited indecomposable pure-injective modules is bounded above by n. This gives one proof of the fact that if every module has finite Morley rank then the ring is of finite representation type (cf. §11.4). Also, the technique used to prove this is used again in §8.4.

7.1 Superstable modules: the uncountable spectrum

In [Zg84; §10] all the possible uncountable spectrum functions (see the introduction to this chapter for terminology) for complete theories of modules were described, and all were shown to be realised over the ring of integers. This description follows from Ziegler's results as presented in §10.4 (where it is left as an exercise). Here I give a direct proof (due to Pillay, and taken from [PP87]).

Let M be pure in N. Say that M is **relatively pure-injective** in N if every pp-type over M which is realised in N is realised also in M. Let M be an elementary substructure of N. Say that M is **relatively $F^a_{\aleph_0}$-saturated** in N if, for every finite tuple $\bar{a}$ in M and element b of N, the strong type of b over $\bar{a}$ is realised in M (see §6.6 for strong types). The proof of the first result, which is a relative version of (part of) **6.37**, is essentially as there and is left as an exercise.

<u>Proposition 7.1</u> [PP87; 5.2] *Suppose that T is superstable and let $M \prec N$ be models of T. If M is relatively $F^a_{\aleph_0}$-saturated in N then M is relatively pure-injective in N.* □

Proposition 7.2 [PP87; 5.3] *Let* M *be relatively pure-injective in* N. *Then* N *decomposes as* $M\oplus K$ *for some* K.

Proof Again, the proof is essentially that of the non-relative result (2.8), and so it is left as an exercise to show that the inclusion $M\leqslant N$ is split. □

Proposition 7.3 [PP87; 5.4] *Suppose that the countable theory* T *is superstable. Let* N *be a model of* T. *Then there is an elementary substructure* M *of* N *such that:*

(*i*) $|M|\leqslant 2^{\aleph_0}$; *and*

(*ii*) $N=M\oplus K$ *where* K *is totally transcendental.*

Proof First one obtains an elementary substructure M of cardinality no more than $2^{\aleph_0}$ and such that M is relatively $F^a_{\aleph_0}$-saturated in N. This is possible (exercise) since, for each $n\in\omega$, there are no more than $2^{\aleph_0}$ n-types over 0 and, for any finite $\bar{a}$ in M, there are at most $2^{\aleph_0}$ elements of N with different strong types over $\bar{a}$. (So realise all strong types over 0 in some $M'\prec N$ with $|M'|\leqslant 2^{\aleph_0}$; then realise all strong types in N over M' in some $M''\prec N$, ...)

Having obtained such a model M, one has, by 7.1, that M is relatively pure-injective in N so is, by 7.2, a direct summand of N: say $N=M\oplus K$. By 3.8 K is t.t., as required. □

The result above may be refined in the U-rank 1 case (see 7.14).

Let T be superstable and countable. By 3.14 and 4.A.14 the t.t. module K above has an essentially unique expression as a direct sum of indecomposable submodules, each of which is itself t.t. and T-unlimited.

Recall that $\mathcal{I}_*(T)$ denotes the set of (isomorphism classes of) unlimited indecomposables in $\mathcal{I}(T)$. If $N_i\in\mathcal{I}_*(T)$ $(i\in I)$ then clearly each module of the form $\oplus_I N_i^{(\kappa_i)}$ is t.t. and in $\mathcal{P}(T)$. Also, whenever M is a model of T and K is such that $M\oplus K$ also is a model of T, the module K may be decomposed as a direct sum of the form just given.

Set $\kappa=|\mathcal{I}_*(T)|$, say $\mathcal{I}_*(T)=\{N_i : i<\kappa\}$. Since each member of $\mathcal{I}(T)$ is the hull of a 1-type and since (T_U is ω-stable so) there are only countably many 1-types over 0 modulo T_U, one has $1\leqslant\kappa\leqslant\aleph_0$ (of course $\kappa\geqslant 1$). Here $T_U=\mathrm{Th}(\oplus\{N_i^{\aleph_0} : N_i\in\mathcal{I}_*(T)\}$ is the unlimited part of T (cf. §4.5).

Theorem 7.4 [Zg84; 10.1] *Suppose that* T *is countable, superstable and not* ω-*stable. Let* λ *be a cardinal, say* $\lambda=\aleph_\alpha$. *Then:*

(*i*) *if* $\aleph_0<\lambda\leqslant 2^{\aleph_0}$ *then* $n(\lambda,T)=2^\lambda$;

(*ii*) *if* $\lambda>2^{\aleph_0}$ *then* $n(\lambda,T)=2^{2^{\aleph_0}}$ *or* $=2^{2^{\aleph_0}}+|\alpha|$ *or* $=2^{2^{\aleph_0}}+|\alpha|^{\aleph_0}$, *according as* $\kappa=1$, $1<\kappa<\aleph_0$, *or* $\kappa=\aleph_0$.

Proof [PP87; 5.5] (i) This is by Shelah [She78; VIII, 1.7, 1.8].

(ii) By 7.3 and the remarks above, any model N of T is of the form $M\oplus K$, where M is a model of T of cardinality no more than $2^{\aleph_0}$ and where K decomposes as $\oplus\{N_i^{(\kappa_i)} : i<\kappa\}$ with the κ_i determined uniquely by K.

As a module of cardinality bounded by $2^{\aleph_0}$, there are at most $2^{2^{\aleph_0}}$ possibilities for M (exercise). On the other hand, by [She78; VIII, 1.7, 1.8] and since T is not superstable, for $\lambda>2^{\aleph_0}$ one has $n(\lambda,T)\geqslant 2^{2^{\aleph_0}}$. Then if $\kappa=1$, respectively $1<\kappa<\aleph_0$, resp. $\kappa=\aleph_0$, there are correspondingly 1 (note $\lambda>2^{\aleph_0}$), $|\alpha|$ (if α is infinite: if α is finite it is absorbed), $|\alpha|^{\aleph_0}$ possibilities for K (count the number of possibilities for the frequency of occurrence (a cardinal, note) of each N_i in K).

Thus the result follows. □

The three cases in (ii) are Ziegler's 3, 4 and 5 of [Zg84; 10.1]. His first two cases occur if T is totally transcendental, and are easily computed from the description of the models of such a theory (4.63). His case 6 corresponds to T being non-superstable: here one has (by Shelah) the maximal possible number of models (2^λ of cardinality λ). Because modules are non-multidimensional (6.21) one spectrum function is not seen in modules (cf. [Lac78]). Also see [PaSt87].

Exercise 1 Let $R=\mathbb{Z}$. Show that the following modules have, among them, theories of each of the three kinds in 7.4(ii):
$\bigoplus\{\mathbb{Z}_{(p)} : p$ is prime$\}$;
$\mathbb{Z}_2^{\aleph_0} \oplus \bigoplus\{\mathbb{Z}_{(p)} : p$ is prime, $p>2\}$;
$\mathbb{Z}_{(2)} \oplus \bigoplus\{\mathbb{Z}_{p^\infty}{}^{\aleph_0} : p$ is prime$\}$.

The proof of the following, which partly generalises the existence of hulls, is left as an exercise.

Proposition 7.5 [PP87; 5.6] *Suppose that N is superstable and that M is an elementary substructure which is relatively $F^a_{\aleph_0}$-saturated (respectively, relatively pure-injective) in N. Let A be a subset of N. Then there is N', an elementary substructure of N, which contains $M\cup A$ and which is prime over $M\cup A$ among all those $N''\succ M$ with $A\subseteq N''$, with $N''\prec N$ and with N'' relatively saturated (resp., relatively pure-injective) in N. Moreover N' is unique up to isomorphism over $M\cup A$.* ▫

7.2 Modules of U-rank 1

We have a fairly complete analysis of the pure-injective models of any superstable theory of modules, but our results tell us only a limited amount about the structure of non-pure-injective superstable modules. In fact the detailed analysis of superstable modules seems to be very hard, even in the U-rank 1 case.

In this section we consider the structure of (non-t.t.) modules of U-rank 1. I begin by describing the shape of the lattice of pp-definable subgroups then we move on to obtain a broad description of the models. After that, I present Buechler's result that a U-rank 1 theory of modules with fewer than $2^{\aleph_0}$ models has no more than $\aleph_0$ models (Vaught's Conjecture for modules of U-rank 1). This latter part is not self-contained, since it depends on some rather involved model theory. The section finishes with some further discussion of non-totally-transcendental small theories of modules of U-rank 1.

U-rank was discussed in §5.2. From the definition there, it is immediate that the theory T has U-rank 1 iff T is not the theory of a finite structure and if the U-rank of every 1-type over $\emptyset$ is 1. Such a theory, having U-rank, is of course superstable.

Proposition 7.6 [PP87; 7.1] *Let M be any module. Then $\mathrm{UR}(M)=1$ iff, for every pp formula φ in one free variable, exactly one of $[M:\varphi(M)]$ and $[\varphi(M):0]$ is finite.*

Proof If both were infinite then, with notation established earlier (§2.2), $G_0(v=v)>G_0(\varphi)>0$ and so PP_0 would have length at least two - so by **5.13** we would have $\mathrm{UR}(T)\geqslant 2$ - contradiction. (Of course, if both were finite then M would be finite and so have U-rank zero.)

The argument reverses. So the result follows. ▫

Example 1 Abelian groups of U-rank 1 include: $\mathbb{Z}$; $\mathbb{Z}_{(p)}{}^n$ $(n\in\omega)$; $\mathbb{Z}_p{}^{\aleph_0}$; $\mathbb{Z}_4\oplus\mathbb{Z}_2{}^{\aleph_0}$; $\bigoplus\{\mathbb{Z}_p : p$ prime$\}$. (Exercise: verify all this and determine which are t.t..)

Example 2 There is a "canonical example" (see [Zg84; 10.3(3)]) of a module of U-rank 1 which is not totally transcendental but whose theory has only countably many countable models.

Take a finite field K and let R be the polynomial ring over K in countably many indeterminates, factored by the square of the ideal generated by the indeterminates: $R = K[x_i\ (i \in \omega) : x_i x_j = 0\ (i,j \in \omega)]$. Let M be the module $\bigoplus \{y_i R : i \in \omega\}$ where the action of R is defined by $y_i x_j = 0$ iff $i > j$.

Then one may see that the pp-definable subgroups are of two kinds. There are those which are of finite index - they lie in the sequence $M > \mathrm{ann}_M x_0 > \mathrm{ann}_M x_1 > \ldots > \mathrm{ann}_M x_i > \ldots$ and their intersection (in any given model) will be denoted by S. Then there are the algebraic ones - those of the form $Mx_0, Mx_1, \ldots$ together with their finite sums - the sum of them all is denoted by A and it consists of all the algebraic elements. It is easy to see (and just a bit more difficult to prove - see below) that M is the prime model of its theory.

It follows that M has U-rank 1, and so (**7.7** below) there is a unique unlimited (irreducible) type p. The space of indecomposable components has $\aleph_0$ isolated points - namely the $y_i R$ (each of these is a t.t. module) - and just one limit point - the hull of p (also t.t.). It follows that the pure-injective models have the form $\overline{M} \oplus N(p)^{(\kappa)}$, where $\kappa \geqslant 0$ is arbitrary.

One may verify directly that there are only $\aleph_0$ types in any finite number of variables, but it is no more difficult to see that there are only countably many models, up to isomorphism. This may be shown as follows (a few details are left to the reader).

By **7.14** below, the countable models have the form $M' \oplus N(p)^{(\kappa)}$ where κ takes any value between 0 and $\aleph_0$ and M' is an elementary submodel of the hull, $\overline{M}$, of the prime model M. By **7.13** below, M' does not realise p.

Therefore, to count the countable models of the theory of M, it is enough to count the isomorphism types of countable submodels of $\overline{M}$. We see that there is, in fact, only one. Let $M' \prec \overline{M}$ be countable. I show that M' is isomorphic to M by verifying the back-and-forth property (e.g., see [**Poi85**]). It is enough to show that if $\overline{a}$ is a finite sequence of elements of M', if b is in M' and if $\overline{c}$ in M has the same type over 0 as $\overline{a}$, then there is d in M such that $\mathrm{tp}^M(d/\overline{c}) = \mathrm{tp}^{M'}(b/\overline{a})$.

Since every finite subset of M lies in a finitely generated direct summand of M, we may (inductively) assume that $\overline{c}$, so $\overline{a}$, has been expanded to a sequence which generates a t.t. pure submodule (so a direct summand) of M, resp. M' (for every n-type realised in M is isolated, so the type of the generators of the hull of $\overline{c}$ is isolated over c: hence the same is true of $\overline{a}$). Thus $\overline{a} \mapsto \overline{c}$ defines an isomorphism between direct summands of M' and M. Now, b splits as the sum of two elements - one, b', algebraic over $\overline{a}$ and the other, b'', with type over $\overline{a}$ determined by its type over zero. It follows that if d' denotes the corresponding algebraic element over $\overline{c}$ and d'' is an element in a complement of $\overline{c}R$ in M with the same type over zero as b'' then, setting $d = d' + d''$, one has $\mathrm{tp}^M(d/\overline{c}) = \mathrm{tp}^{M'}(b/\overline{a})$, as required. (Such an element d'' exists because, by the description of the pp-definable subgroups and **7.13**, every 1-type realised in $\overline{M}$ is isolated and, since the hull of $\overline{c}$ can be taken to be an "initial segment" of M, the location of b'', so d'', is specified by the x_i's which it annihilates.)

Since the theory above is not totally transcendental (there is an infinite descending chain of pp-definable subgroups), this gives us our first example of a small theory of modules which is not totally transcendental. One says that a countable complete theory is **small** if, for each $n \in \omega$, there are only finitely many n-types over $\emptyset$.

Problem I don't know whether there would be any significant corollaries, but: if T is a complete theory of modules with only countably many 2-types is T small?

One may note what happens if we modify this example by replacing K with an infinite field. Of course, the example no longer is of U-rank 1 - indeed, it is not even superstable.

Nevertheless, the structure of pp-definable subgroups is the same and, although there are now $2^{\aleph_0}$ countable models, still, there are only countably many n-types for each n.

Similarly, if one considers the theory of $M^{\aleph_0}$, one finds a (non-superstable) theory with only countably many countable models (the same argument shows that there is, up to isomorphism, only one countable elementary submodel of $\mathrm{pi}(M^{(\aleph_0)})$, and the non-isolated indecomposable is just as before).

In this chapter we discover a good deal about modules of U-rank 1. It will be seen that all of them, whether pure-injective or not, split into a limited part and an unlimited t.t. part. The structure of the limited part will be partly elucidated; indeed both the base ring and the module (which we may assume to be faithful) are tied down quite considerably.

Recall that the dimension of T, $\mu(T)$, is the number of non-isomorphic unlimited indecomposables in $\mathcal{I}(T)$ - that is $|\mathcal{I}_*(T)|$. In the superstable case one has "enough" regular types (by **6.23** or **6.26**), so $\mu(T)$ also equals the number of non-orthogonality classes of regular types.

In the remainder of this section, unless otherwise stated, T is a complete theory of modules of U-rank 1, which is not totally transcendental.

Also I fix some notation as we go along. If M is any model of T then denote by S the connected component of M, and denote by A the algebraic closure of 0. Although S depends on M, A may be regarded as fixed. Observe that every algebraic subset of M is contained in an algebraic subgroup (immediate from **2.12**). In the light of **7.6**, we have that every pp-definable subgroup of M either contains S or is contained in A. (Exercise: give an example where S does not contain A.)

Lemma 7.7 [PP87; 7.2] *Let T have U-rank 1. Then there is a unique unlimited, necessarily irreducible, 1-type, p, over 0.* □

This type is, of course, that which says of "v" that "$v \in S$ and $v \notin A$". Since S and 0 are the only connected $\bigwedge$-pp-definable subgroups (by **7.6**) it is immediate that p is the only unlimited (non-zero) type over 0; so it must be irreducible. Observe that this is another illustration of the construction of **4.33**.

Example 3 Let us identify this type and its hull for the example described at the beginning of the section. The type $p(v)$ says that "v" annihilates every x_i but is not divisible by any element of the radical. So, if a realises p, then $N(a) = aR \simeq R/J \simeq K$. Clearly, this type is irreducible and contains no minimal pair.

Say that T is **unidimensional** if T has continuous part zero and $\mu(T)=1$. The next corollary is immediate.

Corollary 7.8 [PP87; 7.3] *If T has U-rank 1 then T is unidimensional.* □

This corollary is generalised (by a different proof) in the next section (**7.23**). There is a partial converse - if a theory of modules is unidimensional then it is superstable; it then follows (exercise: use **6.29**) that it is of finite U-rank, and examples such as $\mathbb{Z}_{2^n}{}^{\aleph_0}$ show that any finite U-rank (even Morley rank) may be attained. In fact, it has been shown by Hrushovsky [Hru86; Chpt3] that *any* unidimensional theory (of modules or not) is superstable (in the general case also, it is necessarily of finite U-rank).

Proposition 7.9 [PP87; 7.4] *Suppose that the theory T of modules is unidimensional (i.e., there is an indecomposable $N \in \mathcal{I}(T)$ such that every unlimited $N' \in \mathcal{P}(T)$ has the form $\mathrm{pi}(N^{(\kappa)})$ for some κ). Then T is superstable.*

Proof Let $T' = T_U$ be the theory of $N^{\aleph_0}$. First, it is shown that if T were not superstable then neither would be T'. For this, it will be sufficient to show that if φ, ψ are pp formulas with $\mathrm{Inv}(T,\varphi,\psi)$ infinite then $\mathrm{Inv}(T',\varphi,\psi)$ also is infinite.

Let κ be greater than the cardinality of any limited direct summand of a model of T (cf. end of §4.5). Since $\mathrm{Inv}(T,\varphi,\psi)$ is infinite there is a model M of T with $\mathrm{Inv}(T,\varphi,\psi) > \kappa$, so it must be that $\mathrm{Inv}(T',\varphi,\psi)$ is infinite.

So it will be enough to show that T' is superstable; equivalently, since $T' = T'^{\aleph_0}$, that T' is totally transcendental.

If T' were not t.t., then 6.29 provides some irreducible type p with no minimal pair. But by 10.16 and since $T_C = 0$ (by hypothesis), $|\mathcal{I}(T')| = 1$ implies that there is an irreducible type, q, with a minimal pair. But then 9.12 gives $N(p) \not\cong N(q)$ - contradiction. □

Returning to the main topic of this section, let us consider the hull of the type p which was produced in 7.7. By 7.6, we see that $N(p)$ has no non-trivial, improper pp-definable subgroup. (So p is an example of a generic regular type in the sense of Poizat [Poi83a].) In particular, $\mathrm{End}N(p)$ is the division ring $D(p) = \mathrm{End}N(p)/J\mathrm{End}N(p)$. When we come to examine the detailed structure of the models of T, we will prove Buechler's result: if T is a non-t.t. of modules of U-rank 1 and with only countably many types, then $D(p)$ is a finite field (observe, consider $\mathbb{Q}$, that one does require T to be non-t.t.). The fact that T is assumed to be non-t.t. is used in the form that there exists an infinite strictly descending chain of pp-definable subgroups (3.1). This is now used to derive some more information about the unique unlimited type, p, and its hull.

<u>Proposition 7.10</u> [PP87; 7.6] *Let T and p be as before (so, in particular, T is not totally transcendental). Then p is not finitely generated and hence is not isolated. Moreover, p contains no minimal pair.*

Proof Suppose that $\varphi \in p^+$, $\psi \in p^-$ and $\varphi > \psi$. Since each interval $[v=v, \varphi(v)]$ and $[\psi(v), v=0]$ has the dcc (by 7.6) and since T is not t.t., there must be φ' with $\varphi > \varphi' > \psi$: indeed, there must be such a pp formula of finite index in the model and hence in p^+. Both statements now follow. □

<u>Lemma 7.11</u> [PP87; 7.7] *Suppose that the module $M \oplus N(p)^{(\kappa)}$ is a model of T. Then M is a model of T.*

Proof One checks the invariants $\mathrm{Inv}(T,\varphi,\psi)$ (2.18). Consider first the case that $\mathrm{Inv}(T,\varphi,\psi)$ is finite. Since $N(p)$ is unlimited, it must be that $\mathrm{Inv}(N(p),\varphi,\psi) = 1$. So by 2.23, $\mathrm{Inv}(M,\varphi,\psi) = \mathrm{Inv}(T,\varphi,\psi)$, as desired.

Now suppose that $\mathrm{Inv}(T,\varphi,\psi)$ is infinite. By 7.6 it follows that φ is of finite index and ψ is finite. So, continuing the argument of 7.10, there is an infinite strictly descending chain between φ and ψ: $\varphi > \theta_1 > \ldots > \theta_n > \ldots > \psi$. Each index $[\theta_i : \theta_{i+1}]$ is finite so, by the first paragraph, $\mathrm{Inv}(M,\theta_i,\theta_{i+1}) = \mathrm{Inv}(T,\theta_i,\theta_{i+1}) > 1$ for each i. Therefore $\mathrm{Inv}(M,\varphi,\psi)$ is infinite. □

This result has the corollary that there is a prime pure-injective model. In fact such a model exists under considerably wider circumstances - see 10.24 - but a direct proof here is easy. A model omits a type if it does not realise that type.

<u>Corollary 7.12</u> [PP87; 7.8] *T has a prime pure-injective model M_0, which omits p.*

Proof Let M be any pure-injective model of T. Then by 5.23 and 7.7, one may write M as $M_0 \oplus N(p)^{(\kappa)}$, where $N(p)$ is not a component of M_0 (since $N(p)$ is t.t., $N(p)^{(\kappa)}$ already is pure-injective (3.2)). By 7.11, M_0 is a model of T. By unidimensionality, M_0 has no unlimited direct summand so, by 9.5 (or directly), each component of M_0 occurs a fixed, finite, number of times in every pure-injective model of T. Hence M_0 is a direct summand of every pure-injective model, as required. □

It follows that every pure-injective model has the form $M_0 \oplus N(p)^{(\kappa)}$ for some cardinal κ. But we can do better than this: *every* model has the form $M' \oplus N(p)^{(\kappa)}$ for some κ and some elementary substructure, M', of M_0.

Lemma 7.13 [PP87; 7.9] *Let M be any model of T. Then M elementarily embeds in the prime pure-injective model M_0 iff M omits p.*

Proof By what has just been shown, it will be enough to establish that if a model, M, omits p then so does $\bar{M}$. So suppose that $\bar{M}$ contains a realisation, a, of p. Since a lies in the hull of M and p is unlimited, it follows (6.9) that $a \not\perp M/0$. So $\mathrm{UR}(\mathrm{tp}(a/M)) < \mathrm{UR}(\mathrm{tp}(a/0) = 1$. Hence a is algebraic over M and so lies in (the model) M. Thus the claim is established. □

All this gives us the following structure theorem.

Theorem 7.14 [PP87; 7.10] *Let T be a complete theory of modules which has U-rank 1 but which is not totally transcendental. Let M be any model of T. Then M has the form $M' \oplus N(p)^{(\kappa)}$, where M' is an elementary substructure of the prime pure-injective model, M_0, of T, and where p is the unique unlimited type over 0.*

Proof Let $\{a_i : i<\kappa\}$ be a maximal independent set of realisations of p in M. If a realises p, and if b lies in any copy of the hull of a, then (6.9) b depends on a over 0 and so, as in the previous proof, b is algebraic over a.

Therefore M contains a (in fact, every) copy of the hull of a_i for each $i<\kappa$. Since $a_i \perp A_i/0$, where $A_i = \{a_j : j\neq i\}$, it follows by 6.9, that $N(a_i) \perp \mathrm{N}_i/0$, where $\mathrm{N}_i = \bigcup\{N(a_j) : j\neq i\}$. So by 5.31, the sum $\bigoplus\{N(a_i) : i<\kappa\}$ is a pure submodule of M. But this submodule, being t.t., is pure-injective. So $M = M' \oplus \bigoplus\{N(a_i) : i<\kappa\}$ for some M'.

Any realisation of p in M' would be independent over 0 from $\{a_i : i<\kappa\}$ (5.24) - contradicting maximality of this set. So M' omits p and hence, by 7.13, is an elementary substructure of M_0, as required. □

Example 4 Consider the example at the beginning of the section. We saw that every countable elementary substructure of $\bar{M}$ is isomorphic to M. So the above result says that the countable models are just the modules of the form $M \oplus (R/J)^{(\kappa)}$ $(\kappa \geqslant 0)$. The fact that, if a realises p then the hull of a is algebraic over a, is obvious, since a even generates its hull.

By use of this structure theorem most questions about a theory such as T may be reduced to questions about elementary submodels of the prime pure-injective model. In particular this is true of Vaught's Conjecture: if T is countable and $n(\aleph_0, T) > \aleph_0$ then $n(\aleph_0, T) = 2^{\aleph_0}$. A major motivation in proving the results above was the hope that a detailed analysis of the models would lead to a proof of Vaught's Conjecture for modules of U-rank 1.

Recall that a countable complete theory T is said to be small if, for each $n\in\omega$, it has only countably many n-types. Clearly Vaught's Conjecture is valid for countable theories which are not small (for one needs $2^{\aleph_0}$ models in order to realise all $2^{\aleph_0}$ types (1.6) in finitely many free variables). So, for Vaught's Conjecture, we may restrict to small theories.

I now give a characterisation of small theories of modules, which is due to Herzog.

Theorem 7.15 [Her87] *Let T be a countable complete theory of modules. Then T is small iff the following conditions are satisfied.*

(i) $\mathcal{I}(T)$ is countable;

(ii) every $N\in\mathcal{I}(T)$ realises, for each $n\geqslant 1$, only countably many n-types over 0;

(iii) *every element, a, of a model has finite algebraic weight - that is, $N(a)$ is a direct sum of finitely many indecomposables.*

Proof First, we see that (i), (ii) and (iii) together imply few types. So let $\bar{a}$ be any finite tuple in the monster model. By (iii) it follows that the hull $N(\bar{a})$ is a direct sum of finitely many indecomposables: $N(\bar{a}) = N(\bar{a}_1) \oplus ... \oplus N(\bar{a}_k)$, where $\bar{a}$ is split as $(\bar{a}_1,...,\bar{a}_k)$ and the $N(\bar{a}_i)$ are indecomposable. The pp-type of $\bar{a}$ is the intersection of the pp-types of the $\bar{a}_i$. By (i) and (ii), there are only countably many possibilites for each $\mathrm{pp}(\bar{a}_i)$. Hence there are only countably many possibilities for $\mathrm{pp}(\bar{a})$, as required.

For the converse, assume that T is small. Then (i) and (ii) are immediate. So let us take $\bar{a}$ in the monster model, and suppose that $N(\bar{a}) = \mathrm{pi}(\oplus\{N_i : i\in\omega\})$ has infinite algebraic weight.

Now, the direct sum $\oplus\{N_i : i\in\omega\}$ is purely embedded in the pure-injective direct product $\prod\{N_i : i\in\omega\}$, so we may think of $N(\bar{a})$ as purely embedded in this product. In particular, $\bar{a}$ may be represented as an element $(\bar{a}_i)_i$ of the direct product. For each $i\in\omega$ there is, by 4.10(d), a pp formula $\varphi_i(\bar{v},\bar{w})$ linking $\bar{a}$ and $\bar{a}_i$: $\varphi_i(\bar{a}_i,\bar{a}) \wedge \neg\varphi_i(\bar{0},\bar{a})$. By projecting, we obtain $\varphi_i(\bar{a}_i,\bar{a}_i)$ and hence $\varphi_i(\bar{0},\bar{a}-\bar{a}_i)$. So $\varphi_i(\bar{0},\bar{a}_j)$ holds for each $j\neq i$ and so, from $\neg\varphi_i(\bar{0},\bar{a})$, we obtain $\neg\varphi_i(\bar{0},\bar{a}_i)$.

Let J be any subset of ω and consider the type of the element $\bar{a}_J = (\bar{b}_i)_i$, where $\bar{b}_i=\bar{a}_i$ if $i\in J$ and $\bar{b}_i=\bar{0}$ otherwise. Let $i\in\omega\setminus J$: then, by the above paragraph, $\varphi_i(\bar{0},\bar{b}_j)$ holds for each $j\in J$, so $\varphi_i(\bar{0},\bar{a}_J)$ holds. On the other hand, if $i\in J$, then $\bar{b}_i=\bar{a}_i$ and so $\neg\varphi_i(\bar{0},\bar{a}_J)$ holds. Thus the type of $\bar{a}_J$ determines J. Hence there are $2^{\aleph_0}$ types over 0, and T is not small - contradiction, as required. □

This allows us to deduce that T is small iff $T^{\aleph_0}$ is small. Herzog gives a direct proof of that, but I allow myself to use some later results and also extract the following point (the proof that I give is not really very different from Herzog's).

Corollary 7.16 *Let $\bar{a}$ be a finite tuple and suppose that $N=N(\bar{a})$ has a direct summand of the form $\mathrm{pi}(N^{(\aleph_0)})$. Then $\text{m-dim}\,N=\infty$ (see §10.2). In particular, the theory of N is not small.*

Proof The proof of the above result shows this. First, it may be assumed (see Exercise 4.1/10) that N is actually $\mathrm{pi}(N^{(\aleph_0)})$. We use the notation of the proof above. Let $\{J(r) : r\in\mathbb{Q}\}$ be a densely ordered (by inclusion) set of subsets of ω (existence of such is left as an exercise!). Then, by the above proof, the pp-types of the corresponding elements $\bar{a}_{J(r)}$ are densely ordered also (inversely to the ordering of the $J(r)$). By **10.6**, the first conclusion follows. The second follows also, since a densely ordered set has $2^{\aleph_0}$ cuts - so there are $2^{\aleph_0}$ types. □

Corollary 7.17 [Her87a] *The complete theory T is small iff $T^{\aleph_0}$ is.*

Proof The direction "⇐" is clear (by **2.33**), so assume that T is small: we verify the conditions of **7.15**. The first and second are by **4.39**. Suppose that $\bar{a}$ is in the monster model of $T^{\aleph_0}$ with $N(\bar{a})=\mathrm{pi}(\oplus\{N_i : i\in\omega\})$ with the N_i indecomposable. If there are infinitely many isomorphism types among the N_i then (take one of each), we contradict smallness of T (using **7.15**(iii) and **4.39**). Otherwise, there are only finitely many isomorphism types represented among the N_i, and then **7.16** contradicts smallness of T. □

In **[Saf81]** Saffe undertook a deep analysis of the models of arbitrary complete theories of U-rank 1 and it seemed at first that he had established the validity of Vaught's Conjecture for them. It turned out, however, that there was an error in the proof, which has not to date been compensated for. Nevertheless, Saffe's ideas have been taken up and developed, especially by Buechler, who showed that if the "geometry" induced by algebraic closure is of a certain nice form then Vaught's Conjecture follows for the theory **[Bue8?]**, **[Bue86]**. I say a little about this. First, the discussion has to take place inside T^{eq} (see §10.T).

A (possibly infinitely) definable subset of the monster model is said to be **weakly minimal** if it has Shelah degree 1: in modules, one may as well say that it has U-rank 1 (by **5.18**). The theory is weakly minimal if the monster model has Shelah degree 1: a type is weakly minimal if the set of its realisations in the monster model is weakly minimal (i.e., if the type has Shelah degree 1).

Now let T be a weakly minimal theory with fewer than $2^{\aleph_0}$ countable models: there is a dichotomy, depending on whether or not T satisfies the Saffe **condition**:

(S) if A is a finite set of parameters and if p is a non-isolated weakly minimal type over A, then p has finite multiplicity.

In **[Bue8?]**, Buechler proves a structure theorem for the models of a weakly minimal theory which satisfies (S) and has fewer than $2^{\aleph_0}$ models: this structure theorem is strong enough to derive that such a theory has only countably many countable models.

In **[Bue86]**, Buechler deals with the case where (S) is not satisfied. He works with a theory T which is small, unidimensional, weakly minimal and not ω-stable. He shows that, if algebraic closure induces a locally modular geometry on the models and if Saffe's condition (S) is not satisfied, then it has $2^{\aleph_0}$ countable models. Local modularity is a condition on the "geometry" of the model.

Buechler was able to show **[Bue86a]** that a U-rank 1 small theory of modules is even modular (a certain geometry is just projective geometry over a finite field) and so, in the case where condition (S) is not satisfied, his theorem from **[Bue86]** applies, to give $2^{\aleph_0}$ models. What I do here is prove this fact ("modularity") about modules (I define the geometry below), but I make no attempt to indicate the proof of Buechler's results from **[Bue8?]** and **[Bue86]**. The interested reader will probably find **[Bue87]** a good place to start.

It will be shown that, for a small U-rank 1 theory of modules (with the notation already establised) $D(p)$ is a finite field and, for elements of models of T, forking independence is the same as linear independence (modulo algebraic elements) over $D(p)$ (the latter does not use smallness). Buechler's original proof of the first fact depended on the deep results which he had developed in **[Bue8?]**, **[Bue86]**. The proof given here is simpler and is due independently to myself (after Buechler wrote to me to inform me of his result) and to Rothmaler. The proof of the second part has also undergone considerable simplification. The proof that I give here is due to Pillay; I also indicate another method of proof, found by the author (after hearing Buechler talk on his proof, and with input from Herzog and Pillay), which is similar to Buechler's original proof of that part. Also Rothmaler found a similar simplified proof. Buechler's paper **[Bue86a]** contains more on the history of the result.

Our discussion will be aided by the notion of the finitiser of a theory of modules. This ideal was first introduced by Rothmaler for another purpose (see **[Rot87]**), but it is central to the proof here.

Let T be any complete theory of modules; fix a model M of T. Define the **finitiser** of T to be the set of all ring elements, r, such that Mr is finite: $\text{fin}(T) = \{r \in R : Mr \text{ is finite}\}$. Since the condition "$|Mr| = n$" is an elementary one, $\text{fin}(T)$ does not depend on the model, M, chosen. Also define the **annihilator** of T, $\text{ann}(T)$, to be set of elements of R which annihilate one (and hence, every) model of T. Clearly $\text{ann}(T)$ is an ideal, and is such that T is "faithful" over $R/\text{ann}(T)$ (in particular, one may always assume that one is working over $R/\text{ann}(T)$).

<u>Proposition 7.18</u> [Rot87] *Let T be any complete theory of R-modules, and let T_u be its unlimited part. Then the finitiser* $\mathrm{fin}(T)$ *of T is an ideal of R. Moreover* $\mathrm{fin}(T)=\mathrm{ann}(T_u)$.

Proof Since $M(r+s)$ is contained in $Mr+Ms$, which is finite if Mr and Ms are, it follows that $\mathrm{fin}(T)$ is closed under addition. Closure under left multiplication follows since $Mtr \subseteq Mr$. Furthermore, if Mr is finite then so is its image, Mrt, under multiplication by t; so $\mathrm{fin}(T)$ is closed under right multiplication also and hence is an ideal of R.

Now recall that T_u may be defined as the theory of W, where $W=M'/M$, M' and M being models of T with M' M-saturated. Let r be in $\mathrm{fin}(T)$. Since Mr is an algebraic set, $M'r$ is contained in M and hence $Wr=0$. Suppose, for the converse, that $r \in \mathrm{ann}(T_u)=\mathrm{ann}(W)$. If Mr were infinite, then the partial type saying that "v" is divisible by r but is not in Mr, would be consistent, and hence realised in M': this would contradict $M'r \leqslant M$. So Mr is finite, as required. □

Let us return to the case of T non-t.t. and of U-rank 1. By 7.6, if r is any element not in the finitiser, I, of T then Mr is infinite and hence, by 7.6, the annihilator of r in M is finite. We note that if $M \prec M'$ is a pair of models then M'/M is naturally an R/I-module; in particular, the action of R on any direct sum of copies of $N(p)$ is precisely the action of R/I.

<u>Theorem 7.19</u> [Bue86a; Thm A(i)] *Suppose that T, as above, has only countably many n-types for each $n \in \omega$. Then R/I is a finite field.*

Proof Let M be any model of T. From the comments above, M/A is an R/I-module (for A, see the conventions before 7.7). First we see that M/A is torsionfree over R/I, in the sense that if $m \in M$ and $r \in R\setminus I$ are such that $mr \in A$, then already $m \in A$. One has that if $mr \in A$ then there is φ algebraic such that $\varphi(mr)$ holds. Therefore m belongs to the subset of M defined by $\varphi(vr)$. I claim that this set is finite and hence m itself is algebraic. The claim follows from the fact that $\mathrm{ann}_M r$ is finite since $r \notin I$, and hence, under the endomorphism of $M_{\mathbb{Z}}$ which takes m to mr, the finite set $\varphi(M)$ has finite pre-image (defined by $\varphi(vr)$).

It follows that R/I is a domain. For, let r,s be any elements of $R\setminus I$, and take m to be any element of $M\setminus A$. By the above paragraph, $mr \notin A$ and, again, $mrs \notin A$. So $rs \neq 0$.

The next step is to show that R/I is finite: we argue by contradiction. Suppose that $0,1,r_1,r_2,\dots,r_n,\dots$ are representatives of distinct cosets of I in R. Let us also take M to be the prime-pure-injective model.

Let $m \in M\setminus A$. Set m_0 to be m. Since M is the prime pure-injective model, $m \notin S$ (by 7.13, since $v \in S \wedge v \notin A$ defines the unlimited type p). Hence there is φ_0 pp, of finite index in M, such that $m \notin \varphi_0(M)$. Consider the elements $m, mr_1, mr_2, \dots$: since this list is infinite, there exist $i \neq j$ with $mr_i - mr_j$ in $\varphi_0(M)$. Set $s_1 = r_i - r_j$ - an element of $R\setminus I$. Set m_1 to be ms_1.

Since $s_1 \notin I$, $ms_1 \notin A$: so now repeat the argument with m_1 in place of m. One obtains that there is a pp formula φ_1 of finite index, such that ms_1 is not in $\varphi_1(M)$ and, replacing φ_1 with $\varphi_0 \wedge \varphi_1$ if necessary, it may be supposed that $\varphi_1(M) < \varphi_0(M)$. Then one obtains $s_2 \in R\setminus I$ with $ms_1s_2 \in \varphi_1(M)\setminus A, \dots$.

Thus one obtains an infinite descending chain $M > \varphi_0(M) > \dots > \varphi_n(M) > \dots$ of pp-definable subgroups (of finite index) and elements $ms_1, ms_1s_2, \dots$ with $ms_1\dots s_n$ lying in $\varphi_{n-1}(M)\setminus\varphi_n(M)$. These elements all are definable over m. So by the usual argument (cf. 3.1) there are $2^{\aleph_0}$ 1-types over m and hence $2^{\aleph_0}$ 2-types over 0. (In particular, there must be $2^{\aleph_0}$ countable models to contain all these 2-types.)

Finally we recall that a finite domain is a field (to see that every non-zero element has an inverse, consider the list of positive powers; then apply Wedderburn's Theorem, which says that a finite division ring is a field). □

As a consequence, the structure of the unlimited part of any model of T is that of a vectorspace over R/I. In particular, $D(p) = R/I$, and the hull, $N(p)$, is finite.

The next stage is to characterise forking independence of non-algebraic elements. The short proof given is due to Pillay. I also give a proof of the author's which is similar to Buechler's original proof: it is longer (modulo what has been proved already) but it is very different, and the method may have some other application.

Theorem 7.20 [Bue86a; Thm A(ii)] *Let T be a U-rank 1 theory of modules. Let $a, \overline{b}$ be in the monster model of T, with a non-algebraic. Then a depends on $\overline{b}$ over 0 iff there exist $r, s_1, \ldots, s_n$ in R, with r not in I, such that $ar - \Sigma_i b_i s_i$ is an algebraic element (in other words iff, modulo algebraic elements, a is a non-trivial linear combination of the b_i).*

Proof One direction is easy: namely, if the right-hand condition is satisfied then let ψ define a finite set containing $ar - \Sigma_i b_i s_i$. The formula $\psi(vr - \Sigma_i b_i s_i)$, which is satisfied by a, defines a finite set (since $\mathrm{ann}_M r$ is finite); so a is indeed algebraic over $\overline{b}$.

For the converse, suppose that $a \not\downarrow \overline{b}/0$. By **5.6** and 7.6 there is a pp-formula $\varphi(v, \overline{w})$ such that $\varphi(a, \overline{b})$ holds and such that $\varphi(M, \overline{0})$ is finite. Consider the type of $a^\frown\overline{b}$ over 0. Let $a'^\frown\overline{b}'$ realise a non-forking extension of this type to any pure-injective model M which contains $a^\frown\overline{b}$; one has, in particular, $\varphi(a', \overline{b}')$. Let M' be a model containing $M^\frown a'^\frown\overline{b}'$, and set $M' = M \oplus U$, where U is, of course, a direct sum of copies of the R/I-vectorspace $N(p)$. By **6.4**, one may suppose that a' and $\overline{b}'$ have the form $a + c, \overline{b} + \overline{d}$ for some $c, \overline{d}$ in U. We have $\varphi(a' - a, \overline{b}' - \overline{b})$ - that is $\varphi(c, \overline{d})$ - therefore c is algebraic over $\overline{d}$. Since the R-module structure of U is simply its R/I-vectorspace structure, this implies that c is linearly dependent on $\overline{d}$; therefore there exist s_i in R such that $c = \Sigma_i d_i s_i$. This is $(a' - a) = \Sigma_i (b_i' - b_i) s_i$, which re-arranges to $a' - \Sigma b_i' s_i = a - \Sigma b_i s_i$. The term on the right is in M, but $a' - \Sigma b_i' s_i$ is independent from M over 0, so it must be that $a' - \Sigma b_i' s_i$ is algebraic over 0. Thus, $a - \Sigma b_i s_i$ is an algebraic element, as required. □

I now sketch an alternative proof, by a rather different method. In outline, it is as follows.

(i) We begin by supposing that $a \not\downarrow \overline{b}/0$.

(ii) This means that there is some pp relation $\varphi(a, \overline{b})$ with $\varphi(M, \overline{b})$ finite.

(iii) The formula $\varphi(u, \overline{v})$ has, in matrix notation, the form $\exists \overline{w}\ (u\ \overline{v}\ \overline{w}).H = 0$, where $H = \begin{pmatrix} \overline{r} \\ S \\ T \end{pmatrix}$ and $\overline{r}$ is a row vector. If G is a matrix with entries from R, then denote by $\widetilde{G}$ the matrix over R/I which is obtained when the entries of G are replaced by their equivalence classes modulo I.

(iv) If the rank of $\widetilde{H}$ is strictly less than the number of its columns, we can remove the dependent columns: the resulting pp formula still witnesses that a is algebraic over $\overline{b}$.

(v) So one may assume that the rank, k, of $\widetilde{H}$ equals the number of columns of $\widetilde{H}$.

(vi) By considering the U-ranks of image groups of matrices, one shows that the rank of $\widetilde{T}$ being k would contradict $\varphi(v, \overline{b})$ being finite.

(vii) Since, therefore, the rank of $\tilde{T}$ is strictly less than the rank of $\tilde{H}$, there is a linear combination of columns of $\tilde{H}$ which, modulo I, looks like $\begin{pmatrix} r \\ \bar{s} \\ \bar{0} \end{pmatrix}$, where r is non-zero, as is the column $\bar{s}$: thus one has a relation between a and $\bar{b}$ of the required form.

I now expand some of these steps where, for the sake of clarity, I suppose that $\bar{b}$ is just a single element b: so H is a $(2+l)\times k$ matrix $\begin{pmatrix} \bar{r} \\ \bar{s} \\ T \end{pmatrix}$, where $\bar{r}, \bar{s}$ are rows.

(iv) Partition H into columns: $H = (\kappa_1 \dots \kappa_k)$ where κ_j is the column $\begin{pmatrix} r_j \\ s_j \\ t_{1j} \\ \vdots \\ t_{lj} \end{pmatrix}$.

Suppose that κ_1 is linearly dependent, modulo I, on the other columns; say $\kappa_1 = \sum_{j=2}^{k} \kappa_j y_j + \bar{x}$, where $y_j \in R$ and $\bar{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_{l+2} \end{pmatrix}$ with every x_i in I (since R/I is a field, it is quite justified to take the coefficient of κ_1 to be 1).

Consider the pp formula $\varphi'(u,v)$ which is obtained by deleting the first column from H: $\exists w_1, \dots, w_l \bigwedge_{j=2}^{l} vr_j + us_j + \sum_{i=1}^{l} w_i t_{ij} = 0$. Certainly $\varphi'(a,b)$ holds. So it will be enough to show that $\varphi'(u,0)$ is finite.

Suppose then that $\varphi'(a',0)$ holds: say $\bar{d} = (d_1, \dots, d_l) \in M^l$ is such that $\bigwedge_{j=2}^{k} a' r_j + \sum_{i=1}^{l} d_i t_{ij} = 0$. Then $a' r_1 + \sum_{i=1}^{l} d_i t_{i1} = (a' \; 0 \; \bar{d})\kappa_1 =$ $\sum_{j=2}^{k} (a' \; 0 \; \bar{d})\kappa_j y_j + (a' \; 0 \; \bar{d})\bar{x} = \sum_{j=2}^{k} (a' r_j + \sum_{i=1}^{l} d_i t_{ij}).y_j + a' x_1 + \sum_i d_i x_{i+2} =$ $0 + m$, where m is an element of the finite set $\sum_{i=1}^{l+2} M x_i$. Thus $a' r_1 + \sum_{i=1}^{l} d_i t_{i1}$ is an element of some finite set, B, which depends only on φ'. Therefore, there are finitely many elements $a_1', a_2', \dots$ such that, for every a' in $\varphi'(M,0)$, one has $\varphi(a'-a_i',0)$ for some i. Since $\varphi(M,0)$ also is finite, finiteness of $\varphi'(M,0)$ follows.

(vi) Suppose, for a contradiction, that the rank of T is equal to k. Then the set $MT = \{\bar{m} \in M^k : \exists \bar{m}' \in M^l \text{ with } \bar{m}'T = \bar{m}\}$ has U-rank k. For, by invertibility of T modulo I, there exists a matrix G over R with $(GT)^{\sim}$ having the form $\begin{pmatrix} I_k \\ 0 \end{pmatrix}$, where I_k is the $k\times k$ identity matrix. Hence GT has the form $\begin{pmatrix} r_1' & & & \\ & \cdot & & \\ & & \cdot & \\ & & & r_k' \\ & X & & \end{pmatrix}$ for some $r_1', \dots, r_k'$ in $R\setminus I$ and some matrix X with entries in I. Then $MT \geqslant M^l GT = Mr_1' \oplus \dots \oplus Mr_k'$ - which has U-rank k.

Now, $\varphi(v,b) = \{a' \in M : a'\bar{r} + b\bar{s} \in MT\} = (M\bar{r} + b\bar{s}) \cap MT \leqslant M^k$. Since $\varphi(v,b)$ is finite and $\mathrm{UR}(MT) = \mathrm{UR}(M^k)$, it must be that $M\bar{r} + b\bar{s}$ is finite. Hence, each r_i is in I (otherwise Mr_i, hence $M\bar{r} + b\bar{s}$, would be infinite). Therefore $\bigcap_{i=1}^{k} \mathrm{ann}_M r_i$ is infinite and then, for every m in this intersection one has, from $\varphi(a,b)$, that $\varphi(a+m,b)$ holds - contradicting finiteness of $\varphi(v,b)$. □

Going back to the case of a small countable non-t.t. U-rank 1 theory of modules, let M be any model of it. There is defined on M a geometry: the "objects" of the geometry are the algebraically closed subsets of M. Theorem **7.20** shows that this geometry is just (infinite-dimensional) projective geometry over a finite field (by **7.19**). The "zero" of the geometry is just A - the algebraic closure of the empty set. One sees that the geometry is projective (modular) from the fact that every two algebraically closed sets have non-trivial intersection: specifically, **7.20** shows that the closed subsets generated by single elements are just the submodules of the form $bR+A$ for $b \in M$, and any two such always intersect in at least A. It follows that Buechler's **[**Bue86; Thm B**]** applies and therefore Vaught's Conjecture is correct for modules of U-rank 1 **[**Bue86a; Thm B**]**.

It is unfortunate that, as yet, there is no way known of finishing off this proof without an appeal to the above rather deep and difficult results. There is also the question of whether these results may be extended to higher U-rank. It is not clear whether or not the case of modules of finite U-rank is any easier than the general case, for it has been shown that Vaught's Conjecture for finite U-rank may be reduced to the case of certain abelian structures and these may be turned into modules (though perhaps not of the same U-rank).

A number of people have given some thought to modules of higher U-rank, and some progress has been made, especially by Herzog and Rothmaler. But, because to describe this would take some space and also because no definitive results have appeared as yet, this is not, perhaps, the time or place to give details.

Let me at least mention, however, that Herzog and Rothmaler recently pointed out that Rothmaler's proof of **7.20** shows that there is no small non-t.t. module over a right noetherian ring. For Rothmaler's proof shows that if $\varphi(M)$ is of finite index in M (M small and non-t.t.) then $\varphi(M)$ contains $\mathrm{ann}_M I$ (he does this by simpifying the form of a general pp formula; alternatively, one may do linear algebra with the corresponding matrices modulo I, as above). But if I is finitely generated as a right ideal, then $\mathrm{ann}_M I$ is pp-definable (as opposed to $\bigwedge$-pp-definable) and so, by **7.6**, it is of finite index in M. Therefore, by **7.6**, M has the dcc on pp-definable subgroups, so is totally transcendental.

One hopes, for example, to obtain results which give more information on the structure of the prime pure-injective model. An apparently reasonable conjecture was that the prime pure-injective model has infinitely many components, each of which is totally transcendental. The example which follows quoshes this conjecture. Another conjecture (see **[**PP87; §7**]**) was that the prime pure-injective model would be **atomic** (all types realised are isolated) - for then Vaught's Conjecture would follow from **7.14** and the fact that there is, up to isomorphism, just one countable atomic model. But an example of F. Piron, which follows the first example, shows that this need not be so (although Piron does prove that it is so under the special hypotheses that the lattice of pp-definable subgroups is a chain and that every pp-definable coset contains an element algebraic over the parameters used to define it **[**Pir87; Chpt 4**]**).

Example 5 We begin with the ring R and module M as defined in Ex 7.2/2. Recall that $\mathcal{I}(\mathrm{Th}(M))$ has $\aleph_0$ isolated t.t. points. To get our example, we stitch these together. Let B be the submodule = subspace of M generated by the $y_0x_0 - y_ix_i$ ($i \in \omega$) and set M' to be M/B. Thus, retaining the same notation for cosets of B as for elements, we have the extra relations $y_ix_i = y_jx_j$ for all $i,j \in \omega$. We will see that M' is not totally transcendental, is of U-rank 1 and has only countably many types. Furthermore $\overline{M}'$ is indecomposable.

Since B is infinitely generated, M' cannot be interpreted back in M, so it must be shown directly that U-rank $M' = 1$. It may be checked that (essentially) the only "new" formulas are those of the form $x_k | v x_l$ - this formula defines $\oplus\{y_{l+i}R + A : i \geqslant 0\}$, which was already pp-definable (remember that we are not distinguishing sets and elements from their images mod B). Hence there are no new pp-definable subgroups. Therefore U-rank $M' = 1$ and clearly M' is not totally transcendental. Furthermore, the theory of M' has only countably many countable models - since one may show that every type realised in the prime pure-injective model is isolated. The details of checking these points are left to the reader.

To show that M' is indecomposable, it will be enough (see below) to show that every two algebraic elements are linked. So let a,b be elements of A - the set of algebraic elements of M'.

Consider b: since it is algebraic, it is a sum of terms of the form $y_j x_k \beta_{jk}$ $(j \leqslant k)$ where $\beta_{jk} \in K$. Fix one such expression, and (+) look for the least "j" which occurs. If, for this value of j, there is some term $y_j x_k \beta_{jk}$ with $k \neq j$, then go to (*) below; otherwise, only $y_j x_j \beta_{jj}$ occurs with that value of j, so replace this term by $y_{j+1} x_{j+1} \beta_{jj}$ (to which it is equal), and then go back to (+).

Either we go to (*) eventually, or else b is just $y_0 x_0 \beta_{00}$. Similarly try to write a in the form (*) below.

(*) $b = y_j p_j(\bar{x}) + y_{j+1} p_{j+1}(\bar{x}) + \ldots$ where the $p_t(\bar{x})$ are polynomials in the x_i $(i \in \omega)$ with constant term 0, and where x_k, for some $k > j$, occurs in $p_j(\bar{x})$. We assume that the $p_t(\bar{x})$ have no redundant terms, in the sense that x_s does not appear in p_t if $s < t$. Replacing $y_j x_j$ by $y_{j+1} x_{j+1}$ if the former appears, it may be supposed that x_j does not occur in p_j.

Suppose that each of a,b may be brought to the form (*). Let $y_l x_m \alpha$ be the "leading term" in the expansion of a (i.e., l is the least index of a "y" and $m > l$). Write $a = q_1(\bar{y})x_1 + q_2(\bar{y})x_2 + \ldots$ with the $q_i(\bar{y})$ K-linear terms in the y's (again, assume no redundant terms, so at least the first l terms will be zero). Let $\varphi(u,v)$ be $\exists w_j, w_{j+1}, \ldots, z_1, z_2, \ldots$ $((v = w_j p_j(\bar{x}) + \ldots) \wedge (\bigwedge_{i \geqslant 1} w_{j+i}\, x_j = 0) \wedge (u = z_1 x_1 + z_2 x_2 + \ldots) \wedge (z_m x_l \alpha^{-1} = w_j x_j))$ - so $\varphi(a,b)$ holds. If $\varphi(0,b)$ held, then we would have $z_1 x_1 + z_2 x_2 + \ldots = 0$. Therefore "$z_m x_m$" would be in the K-space generated by $y_0 x_0$. It must be that z_m, when written out, has no term in any y_k for $k < m$ - for $y_k x_m \notin \langle y_0, x_0 \rangle$ for such k. In particular, no y_k for $k \leqslant l$ appears; hence $z_m x_l = 0$. Therefore, $w_j x_j = z_m x_l \alpha^{-1} = 0$. Now, since $w_j \in \operatorname{ann} x_j$, one has $w_j = y_{j+1}\delta_{j+1} + y_{j+2}\delta_{j+2} + \ldots + d$ where $d \in A$. So $w_j p_j(\bar{x})$ has no term in y_j (of course, since it might have $y_{j+1} x_{j+1}$, it may be that it can be written with $y_j x_j$ appearing, but there is no way in which $y_j x_k$ can appear). More generally, since also $w_{j+i} x_j = 0$ $(i \geqslant 1)$, it follows that the term $y_j x_k$ cannot appear in (any expression of) $w_j p_j(\bar{x}) + w_{j+1} p_{j+1}(\bar{x}) + \ldots$. This is a contradiction.

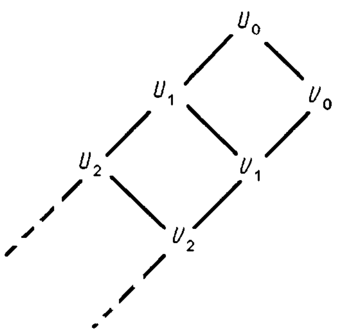
If we fail to bring each of a,b to the form (*) then either a,b both are K-multiples of $y_0 x_0$ - so certainly are related - or a (say) may be written as $y_0 x_0 \alpha$ and b may be written in the form (*) above. In that case, let $\varphi(u,v)$ be $\exists w_j, w_{j+1}, \ldots ((v = w_j p_j(\bar{x}) + \ldots) \wedge (\bigwedge_{i \geqslant 1} w_{j+i}\, x_j = 0) \wedge (v\alpha^{-1} = w_j x_j))$. Then $\varphi(a,b)$ holds. The formula $\varphi(0,b)$ would give $w_j x_j = 0$, and we finish as before.

To complete the proof, it has to be shown that this is enough to establish that $\bar{M}'$ is indecomposable. Observe first that if a is algebraic and $a = (a', a'') \in X \oplus Y \prec^{+} \tilde{M}$, then a' and a'' are algebraic also (exercise: use 2.12). It follows that there exists a indecomposable summand, N_0, of $\bar{M}'$ which contains an algebraic element. Since, as we have just seen, every two algebraic elements are linked, it also follows that N_0 contains all the algebraic elements. Set $\bar{M}' = N_0 \oplus N$. We have $N \cap A = 0$. Therefore $NI = 0$ and so the structure of N as an R-module is simply its R/I-vectorspace structure. Therefore it is a sum of copies of $N(p)$. By 7.13 this implies that $N = 0$, as required.

Thus $\mathcal{I}(\mathrm{Th}(M')) = \{\bar{M}', R/I\}$.

Example 6 (Piron [Pir87]) The original example was an abelian structure: I describe it and then describe the module version of it. Actually, Piron's thesis [Pir87; Chpt3] gives a method of converting abelian structures into modules which exhibit similar properties.

The abelian structure is a K-vectorspace M (K a finite field), together with predicates U_i, V_i $(i \in \omega)$ for subspaces. The theory of the structure says that these subspaces are arranged as follows, with $\dim_K(U_i/U_{i+1}) = 1$ and $\dim_K(U_i/V_i) = n$ (for some fixed $n \in \omega$).



Piron proves that there are no pp-definable subgroups other than those shown, and there are just countably many countable models. Then he shows that there is a non-isolated type (*viz.* that which says "I am in each U_i but not in V_0 (hence in no V_i)") realised in the prime pure-injective model.

To convert an abelian structure $(M, (U_i)_i)$, say, into a module, Piron introduces a variable for each U_i: $R = K[x_i\,(i \in \omega) : x_i x_j = 0]$. Then define the R-module M' to have underlying vectorspace $M \oplus X = M \oplus (\bigoplus\{M/U_i : i \in \omega\})$, with the action of x_i on M being the natural projection to M/U_i and with x_i having the zero action on X. Of course, one wants the resulting module to reflect the properties of the original abelian structure. Piron shows [Pir87; 3.5] that if $(M, (U_i)_i)$ has U-rank 1 then so does the corresponding module, providing the following conditions are met:

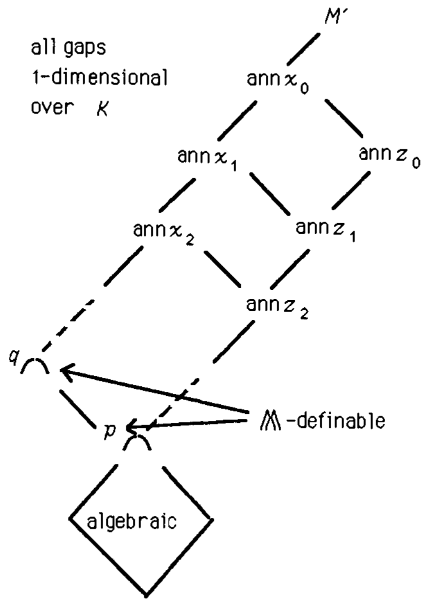
(i) the abelian structure has no pp-definable subgroups other than M, 0 and the U_i; (ii) $U_i \cap U_j \neq 0$ for each i, j; (iii) for each i, there exists j such that $U_i > U_j$.

Also, he shows that, if N and M are abelian structures satisfying these conditions then, if their "module versions" are elementarily equivalent, so are they [Pir87; 3.7]. Furthermore, the number of countable models of the theory of M bounds the number of countable models of the theory of its "module version" [Pir87; 3.8].

Let me give a module version of Piron's example: it is got by modifying the "canonical example" (Ex7.2/2) and is just slightly shifted from what one obtains if the process described above is applied (I leave that as an exercise to the reader). Let R be the ring $K[X_i : i \in \omega]/\langle X_i : i \in \omega\rangle^2$ where K is a finite field. Define M to be the module $\bigoplus\{y_j R : j \in \omega\}$ with the relations $y_j x_i = 0$ iff $j > i$.

Now define R' to be $K[X_i, Z_i : i \in \omega]/\langle X_i, Z_i : i \in \omega\rangle^2$ and let M' be the module $M \oplus \bigoplus\{N_j : j \in \omega\}$ where $N_j = M / \bigoplus\{(y_i + y_{i+1})R : i > j\}$. The action of x_i on M' is just as the R-action on the first component and is annihilating on the N_j's. The action of z_j is, on the first component, the canonical projection $M \twoheadrightarrow N_j$ and is the zero action on each N_j.

One may verify that the non-algebraic pp-definable subgroups of $M'_{R'}$ are as shown



Note that the algebraic pp-definable subgroups all are contained in $M'J$, so are contained in the intersection of all the non-algebraic ones shown.

Now, the unique unlimited (irreducible) type is that which says: " I am in $M'J$ but I am not algebraic" (the first is said by all the $vx_i = 0 = vz_j$).

But there is another non-isolated irreducible type. It says "$vx_i = 0$ $(i \in \omega)$ and $vz_j \neq 0$ $(j \in \omega)$" (actually, it is neg-isolated by "$vz_0 \neq 0$"). This type, q, is limited: the index of $G_0(q)$ in $G(q)$ is $|K|$. Hence the multiplicity of q is $|K|-1$ (cf. Saffe's condition (S)): a non-forking extension of q to a model is determined by specifying a non-zero coset of $\mathcal{G}_0(q) = \bigcap_j \mathrm{ann}\, z_j$ in $\mathcal{G}(q) = \bigcap_i \mathrm{ann}\, x_i$. Of course, q is realised in the prime pure-injective model $\overline{M'}$ (for there is only one unlimited irreducible type; or by **10.1**).

Piron [Pir87; 3.18] makes the following conjecture: let T be a countable complete theory of R-modules, not totally transcendental, of U-rank 1 and with fewer than $2^{\aleph_0}$ countable models. Then every model of T has the form: $M_a \oplus U_1 \oplus \ldots \oplus U_k \oplus N(p)^{(\kappa)}$, where p is the unlimited irreducible type, the U_i are indecomposable and not isomorphic to $N(p)$ and M_a is atomic and infinite. He derives the validity of Vaught's Conjecture, assuming such a structure theorem.

Example 7 With a final example, I indicate an obstruction to extending the results in the U-rank 1 case to the general superstable case. An important ingredient in the proof of **7.14** is that if a is a realisation of the unlimited type, then the hull of a is algebraic over a and therefore is in every model which contains a. Thus the unlimited part splits off. This is no longer true in higher U-rank. Suppose that T has U-rank 2: by **7.23** below, T has either one or two unlimited indecomposables. If just one, then the unlimited indecomposable has U-rank 2, and it need not be the case that, if a has unlimited hull, then the hull of a is present in every model which contains a. For, although the hull of a is isolated over a in its own theory, it need not be isolated over a in T. The example illustrates this - actually it has U-rank 3 but it illustrates the point and, as Herzog points out, it may be modified so that the result has U-rank 2.

We start with the "canonical example": $R = K[x_i (i \in \omega) : x_i x_j = 0]$, $M = \bigoplus \{y_j R$ $(j \in \omega) : y_j x_i = 0$ iff $j > i\}$, where K is a finite field. Introduce a new variable: $R' = R[z : z^2 = 0,\ zx_i = 0\ (i \in \omega)]$, then define M' to be $M \oplus (bR)^{(\aleph_0)}$, where $bx_i = 0\ (i \in \omega)$, $bz = a$ (say) and $Mz = 0$.

One may check that $\mathrm{UR}M' = 3$ and (use **7.15**) that M' is small. Fix b and set $a = bz$: I claim that the pp-type of b over a is not finitely generated. Now, $\mathrm{pp}(b/a) \vdash bx_i = 0$ for each $i \in \omega$. On the other hand, if $c_i \in \mathrm{ann}_M x_{i-1} \setminus \mathrm{ann}_M x_i$, then $(c_i + b)z = a$ but $c_i + b \in \mathrm{ann}_{M'} x_i$. It follows (consider the lattice of pp-definable subgroups) that $\mathrm{pp}(b/a)$ is not equivalent to a single formula.

One may note that the prime model in this example is the submodule of M' generated by the $y_i + b$ $(i \in \omega)$.

Herzog pointed out that if, rather than introducing the new variable z, one replaces the field K by (e.g.) $\mathbb{Z}_4$, then the result has U-rank 2 and displays the same essential features.

If one puts in just one copy of bR, then one obtains another example with U-rank 1 but with non-atomic prime-pure-injective model.

7.3 Modules of finite U-rank

The first result here is that a theory with finite U-rank is finite-dimensional: more precisely the U-rank bounds the number of dimensions. This result for modules contrasts markedly with the general case (take T to be the theory of n disjoint predicates), although Lascar has shown that it is true for ω-stable groups [Las85]. An algebraic application of this result is **11.29**.

<u>Lemma 7.21</u> *Suppose that p is an irreducible type with $p \geqslant p_1 \cap \ldots \cap p_n$ in the lattice of pp-types (definition in §8.1). Then there is q related to p and $i \in \{1,\ldots,n\}$ with $q \geqslant p_i$. If p and all the p_i are finitely generated then q may be taken to be finitely generated.*

Proof Let N_i be the hull of a realisation $\bar{a}_i$ of p_i and let N be the hull of a realisation $\bar{a}$ of p. Since $p_1 \cap \ldots \cap p_n \leqslant p$ there is (**2.8**) a morphism $N_1 \oplus \ldots \oplus N_n \xrightarrow{f} N$ taking $(\bar{a}_1,\ldots,\bar{a}_n)$ to $\bar{a}$. If f_i denotes the canonical injection $N_i \longrightarrow N_1 \oplus \ldots \oplus N_n$ followed by f, then certainly there is some i such that $f_i\bar{a}_i \neq 0$ (we may assume that p is not the zero type). Set q to be the type in $N(\bar{a})$ of one such $f_i\bar{a}_i$. Since p is irreducible, q is related to p and certainly $q \geqslant p_i = \mathrm{tp}(\bar{a}_i)$.

The second assertion of the proposition will not be used yet, so I call on the fact (**8.4**) that if p and all the p_i are finitely generated then one may replace the hulls by finitely presented modules, and then use **8.5**. Since the codomain of f is finitely presented, one concludes (**8.4**) that q is finitely generated. □

<u>Lemma 7.22</u> *Suppose that T is superstable and that $p_1,\ldots,p_n$ are unlimited irreducible types over 0 which are mutually orthogonal (i.e., $N(p_i) \not\simeq N(p_j)$ for $i \neq j$). Then there are $\{q_1,\ldots,q_n\}$ such that for each $i \in \{1,\ldots,n\}$ there exists (a unique) j_i with q_i related to p_{j_i} and such that the chain $q_1 > q_1 \cap q_2 > \ldots > q_1 \cap \ldots \cap q_n$ is strictly decreasing.*

Proof Since the unlimited part T_U of T is t.t. the poset of unlimited (pp-)types has acc. For each i set $Y_i = \{p \in S(0) : N(p) \simeq N(p_i)\}$. Set X_1 to be the disjoint union $Y_1 \cup \ldots \cup Y_n$ and choose q_1 maximal in X_1 - say $N(q_1) \simeq N(p_1)$ for notational convenience. Set $X_2 = Y_2 \cup \ldots \cup Y_n$ and choose q_2 maximal in X_2; .... It must be shown that $q_1 > q_1 \cap q_2 > \ldots > q_1 \cap \ldots \cap q_n$.

If this were not so, then for some $j \in \{1,\ldots,n\}$ one would have $q_1 \cap \ldots \cap q_j = q_1 \cap \ldots \cap q_j \cap q_{j+1}$; that is, $q_{j+1} \geqslant q_1 \cap \ldots \cap q_j$. Then, by **7.21**, there would be q related to q_{j+1} and $i \in \{1,\ldots,j\}$ with $q \geqslant q_i$ and hence, since $q_i \perp q_{j+1}$, with $q > q_i$. Since $q \sim q_{j+1} \in Y_{j+1}$ we have $q \in X_i$ (for $i < j+1$). But q_i was chosen maximal in X_i - so one has a contradiction, as required. □

<u>Theorem 7.23</u> [Zg84; 8.12] *Suppose that* $\mathrm{UR}(T) = n$. *Then T has no more than n dimensions.*

Proof Suppose that $p_1,\ldots,p_k$ are mutually orthogonal unlimited irreducible 1-types (so we're supposing that $\mu(T) \geqslant k$). Then, by **7.22**, one may find a chain of unlimited 1-types of length k-1 (note that $q_1 \cap ... \cap q_j$ is realised in the unlimited module $N(q_1) \oplus ... \oplus N(q_j)$). Taking account of the zero type, one sees that PP_0 has a chain of length k: so **5.13** implies that $UR(T) \geqslant k$, as required. ◻

For a sharper result, see **11.39** below.

It is an immediate corollary of this that a ring, all of whose modules have finite Morley rank (hence finite, so even bounded (consider a model of T^*), U-rank), is of finite representation type (**11.29**).

The theorem **7.23** will also be derived in §10.4 (as **10.20**) as an direct corollary of a result of Ziegler. The argument used to establish **7.23** will be used in §8.4 for another purpose.

Actually, Lascar **[Las85]** has shown that any ω-stable group of finite Morley rank, n (say), has no more than n dimensions. This follows quite easily by working in T^{eq}: for then one may factor by a minimal connected normal definable subgroup and so split the original group as the "sum" of a group of U-rank 1 and another of U-rank n-1. Since any type is non-orthogonal to at least one of these components, one reduces, by induction, to the case of U-rank 1, which may be shown to be 1-dimensional (see **[Las85]** or **[Poi87**; 2.13**]**).

Exercise 1 Show that if $UR(T)=2$ then there are only finitely many unlimited 1-types. Show that the conclusion may fail if $UR(T)=3$.

CHAPTER 8 THE LATTICE OF PP-TYPES AND FREE REALISATIONS OF PP-TYPES

One idea that I have been emphasising in these notes is that pp-types generalise right ideals, at least in their role as annihilators. This viewpoint will be even more explicit in later chapters. In this chapter, we systematically study the lattice of pp-types, bearing in mind its "quantifier-free version" - the lattice of right ideals.

We begin (§1) by noting that the poset P of pp-types is a modular lattice, and the meet and join operations are explicitly described. Then we see that a pp-type is irreducible (i.e., has indecomposable hull) iff it is meet-irreducible in P (justifying the terminology). A pp-type p may be irreducible because there is another pp-type q with the property that every pp-type strictly above p is above q: in that case, we say that p is neg-isolated, since it must then be equivalent to its pp-part together with the negation of a single pp formula. The distinction between those pp-types which are neg-isolated and those which are not turns out to be significant.

A pp-type is finitely generated iff it is realised in a finitely presented module. Half of this is shown in §2, but the proof is not completed until section 3. We see that the finitely generated pp-types form a sublattice of the lattice of all pp-types. Furthermore, a finitely generated pp-type is irreducible, respectively neg-isolated, in the one lattice iff it is so in the other. All this will be of use to us when, in Chapter 11, we restrict our attention to finitely generated modules over right artinian rings.

Given a right ideal I, it is easy enough to find a "free realisation" of an element with annihilator exactly I: the element $1+I$ in the module R/I. So, if pp-types generalise right ideals, there should be a corresponding notion of free realisation of a pp-type. The third section is devoted to showing that there is such a notion. In particular, it is proved that if φ is a pp formula then there is a finitely presented module M_φ and an element (or tuple) a_φ such that the pp-type of a_φ is generated by φ. Such free realisations fall short of being "minimal", but they are at least economical and this seem to suffice for most purposes. In all of this section, it seems to be easier to work with the matrices corresponding to pp formulas: in particular, pp-types correspond to something like right ideals in the ringoid of rectangular matrices over R.

A source of many problems in non-commutative ring theory is the fact that the lattice of right ideals of a ring may be very different from the lattice of left ideals. Rather remarkably, the same is not true for left and right pp-types. I show in the fourth section that there is a duality between the lattice of (right) pp-types and that of pp-types for left modules. I give an algebraic application of this (there should be more): a right pure-semisimple ring has, for each positive integer n, only finitely many modules of length n, up to isomorphism.

8.1 The lattice of pp-types

As usual, T^* denotes the largest complete theory of modules (§2.6). Let $\mathrm{P}\ (=\mathrm{P}_1=\mathrm{P}_1(R))$ denote the poset of all pp-1-types over 0 modulo T^*. In other words, P is the poset of all possible pp-1-types in R-modules, ordered by inclusion. More generally, for $\alpha\geqslant 1$, let P_α be the poset of all pp-types in α free variables. Much of the first result has already been noted and used: it says that P is a complete modular lattice.

Proposition 8.1

(a) P_α *is a modular lattice with operations* $\wedge$ *and* $\vee$ *given by:*
$p\wedge q = p\cap q = p+q$;
$p\vee q=\langle p\cup q\rangle$ (*where* "$\langle\ \rangle$" *denotes deductive closure*).

(b) *If* M *is a module and if* p *and* q *are* α*-types then* $(p\vee q)(M)=p(M)\cap q(M)$. *If* M *is pure-injective, or if* p *and* q *are finitely generated, then* $(p\cap q)(M)=p(M)+q(M)$.

(c) *If* p *and* q *are* α*-types, realised by* $\bar{a}$ *and* $\bar{b}$ *respectively, and if* $\bar{a},\bar{b}$ *are direct-sum independent over* 0, *then* $\bar{a}+\bar{b}$ *realises* $p\cap q$.
In particular, if $\bar{a}$ *is in* M, $\bar{b}$ *is in* M' *and if* $M\oplus M'$ *is pure in* N, *then* $\mathrm{pp}(\bar{a}+\bar{b})=\mathrm{pp}(\bar{a})\cap\mathrm{pp}(\bar{b})$, *where pp-types are taken in* N.

(d) *If* p_i $(i\in I)$ *are pp-*α*-types, then their intersection (which is therefore their meet) also is in* P_α. *In particular,* P_α *is complete.*

Proof Let p and q be α-types. If M is any module then clearly $(p\cup q)(M)=p(M)\cap q(M)$.

Most of (b) has been seen already in **2.3**(iii). Just note that if both p, q are finitely generated, say by φ, ψ respectively, then $(p+q)(M)=(\varphi+\psi)(M)=\varphi(M)+\psi(M)$ (by **2.2**) $=p(M)+q(M)$.

Now suppose that M is pure-injective. Then $p\longmapsto p(M)$ defines an order-reversing map from P_α onto the set of subgroups of $M^{(\alpha)}$ pp-definable in M. The latter is, by **2.2**, a sublattice of the lattice of all subgroups of $M^{(\alpha)}$. If M is a $(|T^*|+|\alpha|)^+$-saturated model of T^* then M realises every α-type and so, using $M\equiv M^{\aleph_0}$ and **2.12**, this map will actually be an isomorphism. Hence P_α is a modular lattice with $p\vee q=\langle p\cup q\rangle$ and $p\cap q=p+q$.

Parts (c) and (d) follow directly from **2.10**. ◻

In consequence, one may define the closure of a set, Φ, of pp formulas (in α free variables) by: $\langle\Phi\rangle=\bigcap\{p\in\mathrm{P}_\alpha : \Phi\subseteq p\}$. Of course $\langle\Phi\rangle$ is just the closure modulo T^* (equivalently modulo the theory of R-modules) of Φ under conjunction and pp-implication. An algebraic description of $\langle\Phi\rangle$ in terms of Φ, corresponding to generation by "+" and "$-\times r$" in right ideals, is given in §3.

If T is a complete theory then one may make the relativised definition: $\mathrm{P}_\alpha{}^T$ is the sub-poset of P_α consisting of all pp-α-types modulo T. From the proof above, one sees that there is a natural order-preserving surjection $\mathrm{P}_\alpha\twoheadrightarrow\mathrm{P}_\alpha{}^T$, given by sending p to its pp-deductive closure modulo T; in other words, p is sent to the unique $q\in S_\alpha{}^T(0)$ for which $q(M)=p(M)$, where M is a $(|T|+|\alpha|)^+$-saturated model of T. This may be regarded as the result of the "localisation" from T^* to T.

There is a problem of notation here. I have previously used "$p\wedge q$" to mean the conjunction - effectively, union - of p and q: on the other hand we have just seen that in P_α "$p\wedge q$" is the intersection of the sets p and q! Therefore I make the convention that <u>the operation of meet in P_α will be denoted by "$\cap$" (and "$\bigcap$" for the infinite operation)</u> (I retain the use of "$\vee$" for join in P_α.)

The next result justifies the terminology "irreducible" when applied to pp-types. Recall that an element p of a lattice is said to be $\cap$-irreducible if whenever $p=q\cap r$ then either $p=q$ or $p=r$. It is stated explicitly in [Kuc84; III.3.10] and [Pr83; 2.2], also see [Fis75; 7.12].

<u>Proposition 8.2</u> *The pp-*α*-type* p *is irreducible iff it is* $\cap$*-irreducible in the lattice* P_α.

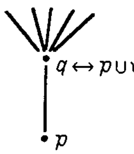
Proof $\Rightarrow$ If $p=q\cap r$ $(p,q,r\in\mathrm{P}_\alpha)$ and if $\bar{a}$ realises p then one may (as in the proof of **2.3**(iii)) find $\bar{b},\bar{c}$ in $N(\bar{a})$ with $q(\bar{b})$, $r(\bar{c})$ and $\bar{a}=\bar{b}+\bar{c}$. Since $p\leqslant q,r$ there are (**2.8**) endomorphisms f,g of $N(\bar{a})$ taking $\bar{a}$ to $\bar{b}$ and $\bar{c}$ respectively. Thus $f+g$ fixes $\bar{a}$ so, by **4.16**, must be an automorphism of $N(\bar{a})$. Since $N(\bar{a})$ is supposed to be indecomposable and so has local endomorphism ring (**4.27**), it follows that at least one of f,g is an automorphism. That is, $p=q$ or $p=r$.

$\Leftarrow$ If $\bar{a}$ is a realisation of p and $N(\bar{a})$ decomposes as $N(\bar{b})\oplus N(\bar{c})$ with $\bar{a}=\bar{b}+\bar{c}$, then one has $p=\mathrm{pp}(\bar{b})\cap\mathrm{pp}(\bar{c})$. Irreducibility of p implies $\mathrm{pp}(\bar{b})=p$ (say). But, by **4.28**, this can happen only if $\bar{c}=\bar{0}$ - as required. ◻

This generalises the fact that a right ideal I is $\cap$-irreducible in the lattice of right ideals of the ring R iff the injective hull of R/I is indecomposable.

Two rather different concepts share the name "irreducibility": the difference has both model-theoretic and algebraic significance (see Chapters 11 and 12). Let me draw the distinction between them. We will say that a type p is **neg-isolated** (by $\psi_1,\dots,\psi_n$) if there are finitely many pp formulas $\psi_1,\dots,\psi_n$ such that p is equivalent to $p^+\cup\{\neg\psi_1,\dots,\neg\psi_n\}$. This is a notion "dual" to that of being finitely generated: in conjunction they are equivalent to isolation (9.20).

<u>Proposition 8.3</u> [Pr83; 2.10] *Let $p\in \mathrm{P}$ be irreducible. Then precisely one of the following circumstances obtains.*



(a) There is a pp-type q such that if $p'>p$ in P then $p'\geqslant q$. In this case there is a pp formula ψ such that $q=\langle p\cup\{\psi\}\rangle$. Hence p (or rather $\tilde{p}$) is neg-isolated by ψ (or, we may say, "by q") since $\tilde{p}$ is then equivalent to $q\wedge\neg\psi$.

(b)...p is the (properly infinite) meet of all the pp-types strictly above it in P: $p=\bigcap\{p'\in\mathrm{P} : p'>p\}$ - and then p is not neg-isolated.

Proof Define q to be the meet of all the pp-types strictly above p. If $q=p$, then we are in the second case. Otherwise $q>p$ and so q certainly has the property ascribed to it in (a); moreover if ψ is any pp formula in q but not in p then the pp-type generated by p together with ψ lies below q and strictly above p, so is equal to q. Hence $\tilde{p}$ is equivalent to $p^+\wedge\neg\psi$.

Now suppose that the irreducible pp-type p is neg-isolated. Say p is equivalent to $p^+\wedge\bigwedge_1^n\neg\psi_i$ for certain pp formulas $\psi_1,\dots,\psi_n$. For $i=1,\dots,n$ choose q_i in P, maximal with respect to containing p but not containing ψ_i. Then the meet $q_1\cap\dots\cap q_n$ contains p and, by construction, contains no ψ_i; hence $q_1\cap\dots\cap q_n=p$. Irreducibility of p then implies (8.2) that $p=q_1$ (say). By maximality of $q_1=p$, if $p'>p$ then $\psi_1\in p'$: hence $\tilde{p}$ is equivalent to $p\wedge\neg\psi_1$, so we are in case (a) with $q=\langle p\cup\{\psi_1\}\rangle$. ▫

If the *pp-type* p is as in (a) above, then I will say that it is **isolated** (in P_α); and in case (b) I will say that it is **non-isolated** (in P_α). Isolation and neg-isolation have been defined with respect to T^*: clearly, one may define a relative version for any complete T closed under products (P^T replaces P). So we have that an irreducible type is neg-isolated iff its pp-part is isolated in the lattice of pp-types (modulo a given complete theory).

So now the term "isolation" may refer to the types in the space $S^T(0)$, indecomposables in the space $\mathcal{I}(T)$ or pp-types in the lattice P^T. In general, though not in the t.t. case, these do not correspond to each other, but there are implications between them and these are detailed in §9.3.

Example 1 The types realised in finitely generated members of $\mathcal{I}_{\mathbb{Z}}$ (i.e., the hulls of finitely generated pp-types) are neg-isolated - they are even isolated. The non-zero types realised in $\overline{\mathbb{Z}(p)}$ are not neg-isolated: infinitely many negations are required to say that an element is torsion-free. On the other hand, those in $\mathbb{Z}_{p^\infty}$ are neg-isolated (but not isolated). Similarly for $\mathbb{Q}$. The latter points can be seen directly or by using the algebraic criterion of 9.29.

In the theory of $\overline{\mathbb{Z}(p)}^{\aleph_0}$, the non-zero elements of $\overline{\mathbb{Z}(p)}$ now have neg-isolated types.

There is a precise sense in which the lattice P of pp-types generalises the lattice $\mathrm{Latt}(R)$ of right ideals. Define $\pi:\mathrm{P}\longrightarrow\mathrm{Latt}(R)$ by $p\mapsto p\cap R$ (regarding p as a set of matrices, as in §2.1, *viz.* $\pi p=\{r\in R : \text{"}vr=0\text{"}\in p\}$). Also define $i_*:\mathrm{Latt}(R)\longrightarrow\mathrm{P}$ to take the right ideal I to the pp-type of the image of 1_R in R/I, and define $i^*:\mathrm{P}\longrightarrow\mathrm{Latt}(R)$ by $I\mapsto \mathrm{pp}^{E(R/I)}(1+I)$. Observe that the $\bigwedge$-atomic part of a type is a collection of formulas of the sort "$vr=0$" so, under the identification of pp formulas with the corresponding matrices (§2.1), such a $\bigwedge$-atomic part "is" a set of ring elements r and it is easily checked that this set is a right ideal. In particular the interval $[i^*I, i_*I]$ consists of all those pp-types whose $\bigwedge$-atomic part is I.

It is easy to see that π is onto; that i_*, i^* are 1-1; that π, i_*, i^* are order-preserving; and that i^*, respectively i_* is right, resp. left, adjoint to π. This last is so since i_*I is the smallest pp-type whose $\bigwedge$-atomic part is I, and i^*I is the largest such pp-type.

Exercise 1 Describe $P_1(\mathbb{Z})$.

8.2 Finitely generated pp-types

If T is totally transcendental then every pp-1-type is finitely generated: that is, P^T has the acc. Indeed this property characterises such theories (**3.1**). This is reflected in the structure theorem (**3.14**) for t.t. pure-injective modules. That theorem exactly generalises the structure theorem which one has for injective modules when Latt(R) has the acc (**1.11**). This suggests that it might be useful to focus on finitely generated pp-types.

A more specific reason for looking at finitely generated pp-types is that they are precisely those we need to consider when dealing with finitely generated modules over artinian rings: for, as will be seen shortly, any pp-type realised in a finitely presented module is finitely generated. In §11.3 an analogue of the theory of hulls (§4.1) is developed within the category of finitely presented modules over artinian rings. By the unqualified term **finitely generated pp-type**, I mean one which is finitely generated modulo T^*.

<u>Proposition 8.4</u> [Pr83; 2.4] *Let p be a pp-n-type. Then p is finitely generated (modulo T^*) iff p is realised in a finitely presented module.*

Proof $\Rightarrow$ Suppose that φ is a pp formula equivalent to p (in T^* and hence in every theory of R-modules). Let $\bar{a}$ in M have pp-type p. Now $\varphi(\bar{v})$ has the form $\exists\bar{w}\,\theta(\bar{v},\bar{w})$ for some $\bigwedge$-atomic θ. So there is $\bar{b}$ in M with $M \models \theta(\bar{a},\bar{b})$.

Let A be the submodule of M generated by (the entries of) $\bar{a}$ and the chosen witnesses, $\bar{b}$, of the existential quantifiers in φ. Since θ has no quantifiers, and since $\theta(\bar{a},\bar{b})$ is true in M, it is also true in A (it is simply a conjunction of linear relations in $\bar{a}\frown\bar{b}$). Thus $A \models \varphi(\bar{a})$. Since φ proves p in T^* and hence in every module, we deduce that the pp-type of $\bar{a}$ in A is p.

Now A is finitely generated, so it remains to show that it may be taken to be finitely presented. Of course if R is right noetherian, and in particular if R is right artinian, then A necessarily is finitely presented. The general case needs more work, and is deferred until 8.15.

$\Leftarrow$ Let $\bar{a}$ be in the finitely presented module M and set $p = \mathrm{pp}^M(\bar{a})$. Let $\bar{c}$ be a tuple of generators for M. Since M is finitely presented, there is a conjunction, $\theta(\bar{w})$, of equations such that $\theta(\bar{c})$ holds and such that all relations between the entries of $\bar{c}$ are consequences (modulo the theory of R-modules) of $\theta(\bar{c})$. (In algebraic terms, if $\bar{d}$ lies in M' and if $\theta(\bar{d})$ holds then the map $\bar{c} \mapsto \bar{d}$ well-defines a morphism $M \longrightarrow M'$.)

Write $\bar{a}$ as an R-linear combination $\bar{c}K$ (for some matrix K) of $\bar{c}$. Let $\varphi(\bar{v})$ be the formula $\exists\bar{w}(\,\bar{v}=\bar{w}K \wedge \theta(\bar{w}))$ (so if H_θ is the matrix (§2.1) corresponding to θ then H_φ is $\begin{pmatrix} I & 0 \\ -K & H_\theta \end{pmatrix}$). The claim is that φ generates p.

So let $\bar{b}$ in M' (M' any module - e.g. a model of T^*) be such that $M' \models \varphi(\bar{b})$. Then there is $\bar{d}$ in M' satisfying $\theta(\bar{d})$ and such that $\bar{b} = \bar{d}K$. Since $\bar{c}$ is free satisfying $\theta(\bar{w})$, there is a well-defined morphism $M \xrightarrow{f} M'$ given by $\bar{c} \mapsto \bar{d}$. Applying f to the equation(s) $\bar{a} = \bar{c}K$ one obtains $f(\bar{a}) = \bar{d}K = \bar{b}$. Hence (**2.7**) $\mathrm{pp}^{M'}(\bar{b}) \geq \mathrm{pp}^M(\bar{a}) = p$.

That is, every pp-type containing φ contains all of p. Since $\varphi\in p$ it follows that φ is equivalent to p, as required. ◻

The proof of the result above has the following useful corollary (cf. 2.8(v)).

Proposition 8.5 [Pr83; 1.9] *Suppose that* $\bar{a}$ *is in the finitely presented module* M. *Let* $\bar{b}$ *in* M' *be such that* $\mathrm{pp}^M(\bar{a})\leqslant \mathrm{pp}^{M'}(\bar{b})$. *Then there is a morphism* $M\longrightarrow M'$ *taking* $\bar{a}$ *to* $\bar{b}$. ◻

As an exercise, one may write down a proof cast in "purely algebraic" terms, using matrices. In the situation of 8.5, one could (and, if one were a topos-theorist, one probably would) say that a morphism $M\longrightarrow M'$ is an "element" of M' (also cf. §10.T).

When dealing exclusively with finitely generated modules over an artinian ring (§11.3), we will have to confine attention to the finitely generated pp-types (and so, for example, will lose direct use of the compactness theorem). So we should look at the sub-poset, $\mathrm{P}_n{}^{\mathrm{f}}$, of P_n which consists of the finitely generated pp-types ordered by inclusion (of course this makes sense only for finite n). Fortunately, P^{f} is a well-behaved sublattice of P.

Proposition 8.6 [Pr83; 2.1] $\mathrm{P}_n{}^{\mathrm{f}}$ *is a* (0,1)-*sublattice of* P_n.

Proof Clearly the bottom element 0, which is the pp-type of the constant tuple $\bar{1}$ in R^n, and the top element 1, which is the pp-type of the zero tuple, are common to P_n and $\mathrm{P}_n{}^{\mathrm{f}}$.

Suppose that p and q, equivalent to φ and ψ respectively, are finitely generated pp-types. Then, by 8.1, one has that $p\vee q$ is generated by $\{\varphi,\psi\}$ (so is equivalent to $\varphi\wedge\psi$), and that $p\cap q$ is equivalent to $\varphi+\psi$. So $\mathrm{P}_n{}^{\mathrm{f}}$ is closed under the lattice operations of P, as required. ◻

Exercise 1 Show that P^{f} need not be a complete lattice.

The intersection of two finitely generated right ideals need not be finitely generated (unless the ring is right coherent): 8.6 says that the situation for pp-types is much better. One should observe that, in the proof, the formula defining $p\cap q$ involves an existential quantifier (cf. 15.41). Next, some of the results of the previous section are relativised to P^{f}.

Proposition 8.7 [Pr83; 2.3] *If* p *is a finitely generated pp-*n*-type then the following conditions are equivalent:*

(i) p *is irreducible;*

(ii) p *is* $\cap$*-irreducible in* P_n;

(iii) p *is* $\cap$*-irreducible in* $\mathrm{P}_n{}^{\mathrm{f}}$.

Proof (i)⇔(ii) This is 8.2.

(ii)⇒(iii) This is trivial.

(iii)⇒(ii) Suppose that $p=q\cap r$ where $q,r\in\mathrm{P}_n$ and q,r strictly contain p. Take φ in $q\setminus p$ and ψ in $r\setminus p$. Set q_1 to be generated by $p\cup\{\varphi\}$ and similarly $r_1=\langle p\cup\{\psi\}\rangle$: then q_1 and r_1 are finitely generated. Also q_1 and r_1 strictly contain p and their intersection is p (for $p\leqslant q_1\cap r_1\leqslant q\cap r$). So p fails to be $\cap$-irreducible in $\mathrm{P}_n{}^{\mathrm{f}}$, as required. ◻

Proposition 8.8 [Pr83; 2.11] *Suppose that* p *is an irreducible* n*-type over* 0 *whose pp-part is finitely generated. Then the following conditions are equivalent:*

(i) p^+ *is isolated in* P_n *(in the sense of* 8.3*);*

(ii) p^+ *is isolated in* $\mathrm{P}_n{}^{\mathrm{f}}$ *(in the same sense);*

(iii) p *is isolated, in the usual sense, in* $S_n(0)$.

These conditions imply that the hull of p *is an isolated point of* $\mathcal{I}(T^*)$.

Proof Suppose that p^+ is equivalent to φ.

(i)$\Rightarrow$(ii) Suppose that p is isolated in P_n by the pp-type q; then, with the notation of 8.3(a), q is equivalent to $\varphi\wedge\psi$, hence is finitely generated. Certainly q isolates p in the smaller lattice.

(ii)$\Rightarrow$(iii) Let $q\leftrightarrow\varphi\wedge\psi$ isolate p in $P_n{}^f$. Then p is equivalent to $p^+\wedge\neg\psi$, so is equivalent to $\varphi\wedge\neg\psi$. Hence p is isolated in the space of n-types modulo T^* over 0.

(iii)$\Rightarrow$(i) Suppose that (modulo T^*) p is equivalent to $\varphi\wedge\neg\psi$ where, by 9.20, it may be supposed that p^+ is equivalent to φ, and where the presence of just one negated formula is justified by the second paragraph of the proof of 8.3. Set $q=\langle\varphi\wedge\psi\rangle$. If p' is an n-type strictly containing p, then φ is in p' and, since $p\neq p'$ and $\varphi\wedge\neg\psi$ isolates p, it must also be that ψ is in p'. Thus $p'\geqslant q$; so q isolates p in P_n, as required.

Finally if (modulo T^*) p is equivalent to $\varphi\wedge\neg\psi$ then any point $N\in\mathcal{I}(T^*)$, with $\varphi(N)>\psi(N)$, contains a realisation of p (namely, any element of $\varphi(N)\setminus\psi(N)$) and so is the hull of p. Thus (φ/ψ) is an isolating neighbourhood for $N(p)$ in $\mathcal{I}_R=\mathcal{I}(T^*)$, as required. □

In 9.20 it is shown that, for modules, isolation in the usual sense (i.e., in the space of types) is equivalent to positive isolation (being finitely generated) plus negative isolation (being neg-isolated).

8.3 pp-types and matrices

The analogy between right ideals and pp-types may be pursued further.

Given a set of elements of the ring R the right ideal generated by this set is formed by closing under the operations of addition and right multiplication. One may ask whether there is an analogous description of the pp-type generated by a set of pp formulas. In fact there is and, as might be expected, the matrix description of pp-formulas is the most appropriate framework for such a characterisation.

In the course of describing how a pp formula generates a pp-type, the analogue of the quotient R/I (I a right ideal) is encountered. It is a notion of "free (though not necessarily minimal) realisation" of a pp formula (or pp-type).

First I note some general points about the sets of matrices which correspond to pp-types. Then pp-types, regarded as sets of matrices, are characterised in terms of closure under certain operations. The results of this section come from [Pr83]: they find application to pure-semisimple rings in §8.4.

Recall that if $\varphi(\bar{v})$ is a pp formula then it may be expressed using matrix notation as $\exists\bar{w}\,(\bar{v}\;\bar{w})H_\varphi=0$ where H_φ is a rectangular matrix over (i.e., with entries in) R. This matrix has a natural decomposition as $\begin{pmatrix}R_\varphi\\ S_\varphi\end{pmatrix}$ corresponding to the variables $\bar{v}$, $\bar{w}$: thus $\kappa(\bar{v})\ (=l(\bar{v}))=\rho(R_\varphi)$ and $\kappa(\bar{w})=\rho(S_\varphi)$, where $\kappa(H)$ is the number of columns of the matrix H and $\rho(H)$ is the number of its rows. I will use such decompositions of matrices without further comment: moreover, I make use of certain obvious conventions concerning matching of matrices when operations are to be performed. Furthermore, "0" will denote any zero matrix and "I" will denote any identity matrix. Finally, "${}_mH_n$" indicates that $\rho(H)=m$ and $\kappa(H)=n$.

Example 1 $R=\mathbb{Z}$. Consider the following pp formulas.

(i) $\varphi(v)$ is $\exists w\,(v=wn)$ (i.e. "$n|v$"). Write this in more standard form as $\exists w\,(v-wn=0)$ and note that $H_\varphi=\begin{pmatrix}1\\ -n\end{pmatrix}$.

(ii) $\psi(v)$ is $vn=0$. Then $H_\psi=(n)$.

(iii) $\psi'(v_1, v_2)$ is $v_1 = v_2 n$. Write this as $v_1 - v_2 n = 0$ to see that $H_{\psi'} = \begin{pmatrix} 1 \\ -n \end{pmatrix}$.

Observe that $H_\varphi = H_\psi$: this emphasises the fact that, in order to recover pp formulas from matrices, one must specify the number of variables which are to remain free.

(iv) $\psi'(v)$ is $\exists w(v = wn) \wedge \exists w(v - wm)$. This may be re-written as

$\exists w_1, w_2 (v - w_1 n = 0 \wedge v - w_2 m = 0)$. Thus $H_{\varphi'} = \begin{pmatrix} 1 & 1 \\ -n & 0 \\ 0 & -m \end{pmatrix}$. Re-writing introduces a possibility of inessential variation, which I ignore.

(v) $\varphi''(v)$ is $\exists w(v - wn = 0 \wedge v - wm = 0)$. Note that this is a stronger assertion than $\varphi'(v)$. Then $H_{\varphi''} = \begin{pmatrix} 1 & 1 \\ -n & -m \end{pmatrix}$. Observe that $H_{\varphi''} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} . H_{\varphi'}$. This illustrates **8.10** below.

Now, let p be any pp-n-type. Define the corresponding set of matrices $I(p) = \{H \in R_\omega : \rho(H) \geqslant n$ and $H = H_\varphi$ for some $\varphi \in p\}$. Here R_ω is the ringoid ("ring with many objects") of all finite rectangular matrices with entries from R, equipped with the partial operations of matrix addition and multiplication. So we should expect $I(p)$ to be something like a right ideal of R_ω (but it is not quite this under the usual definition of ideal of a ringoid (see [Mit72])). I have deliberately excluded from $I(p)$ those matrices with insufficiently many rows. No information is lost in doing this since a smaller matrix may be fleshed out with zero entries, and some awkwardness will thereby be avoided. If p is generated by the single formula φ, write $I(\varphi)$ for $I(p)$.

As usual, these notions are extended to types via their pp-parts.

In this way, any pp-type may be replaced by a certain set of matrices together with some record of the number of free variables in the type. I will give an algebraic characterisation of such sets of matrices. I have already suggested that these sets of matrices should be thought of as generalised annihilators. Nevertheless, it should come as no surprise that there are complications which are masked in the "purely algebraic" (1×1) case. These complications are due to the presence of quantifiers and are not, in any way, on account of the fact that more than one free variable is allowed (compare §10.T).

I comment briefly upon what are, and what are not, permissible operations on sets of the form $I(p)$. It is easily seen that $I(p)$ is closed under right multiplication by elements of R_ω, but that it is closed under neither addition nor "concatenation".

Example 2 $R = \mathbb{Z}$. Let p be the pp-type of the element 12 of $\mathbb{Z}$. Since 4 and 6 divide 12, both $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -6 \end{pmatrix}$ are in $I(p)$. It is, however, easy to check that neither $\begin{pmatrix} 1 \\ -4 \end{pmatrix} + \begin{pmatrix} 1 \\ -6 \end{pmatrix}$ nor $\begin{pmatrix} 1 & 1 \\ -4 & -6 \end{pmatrix}$ lie in $I(p)$.

It is clear that if $H \in I(p)$ then any matrix obtained from H by deleting and/or permuting columns (i.e. equations) still lies in $I(p)$.

Furthermore, if $\begin{pmatrix} R \\ S \end{pmatrix}$ and $\begin{pmatrix} R' \\ S' \end{pmatrix}$ are in $I(p)$ (so $\rho(R) = n = \rho(R')$) then $\begin{pmatrix} R & R' \\ S & 0 \\ 0 & S' \end{pmatrix}$ is in $I(p)$ (note that here "R" is a matrix!).

Also, $I(p)$ is closed under a variant of Cohn's "determinantal sum" [Co71]: suppose that $\begin{pmatrix} R \\ S \end{pmatrix}$ and $\begin{pmatrix} R' \\ S \end{pmatrix}$ in $I(p)$ have the same size - then from $\overline{v}R + \overline{w}S = 0$ and $\overline{v}R' + \overline{w}'S = 0$ one obtains $\overline{v}(R + R') + (\overline{w} + \overline{w}')S = 0$ - thus $\begin{pmatrix} R + R' \\ S \end{pmatrix} \in I(p)$.

I will make use of obvious notations such as ${}_{n}R_{\omega} = \{H \in R_{\omega} : \rho(H) = n\}$; ${}_{\geqslant n}R_{\omega} = \{H \in R_{\omega} : \rho(H) \geqslant n\}$.

The next definition is the key one. Let $n \geqslant 1$, and suppose that $I \subseteq {}_{\leqslant n}R_{\omega}$. Then I is an n-closed set of matrices if it satisfies the following closure conditions.

(i) If $H \in I$ and $G \in R_{\omega}$ then $HG \in I$.

(ii) If $\begin{pmatrix} {}_{n}I_{n} & \\ & G \\ 0 & \end{pmatrix}.H \in I$ where $G \in R_{\omega}$, then $H \in I$.

(iii) If $\begin{pmatrix} R \\ S \end{pmatrix}$ and $\begin{pmatrix} R' \\ S' \end{pmatrix}$ are in I with $\rho(R) = n = \rho(R')$ then $\begin{pmatrix} R & R' \\ S & 0 \\ 0 & S' \end{pmatrix}$ is in I.

Note that if in condition (ii) one sets $H = \begin{pmatrix} {}_{n}R \\ S \end{pmatrix}$ and $G = \begin{pmatrix} G' \\ G'' \end{pmatrix}$; then the condition becomes:

(ii)′ If $\begin{pmatrix} {}_{n}R + G'S \\ G''S \end{pmatrix}$ is in I (for some G', G'') then $\begin{pmatrix} R \\ S \end{pmatrix}$ is in I.

It will be seen that (i) and (ii) are the important conditions. Condition (iii) simply corresponds to the replacement of two pp formulas by their conjunction.

Lemma 8.9 [Pr83; 1.1] *Let $I \subseteq R_{\omega}$ be an n-closed set. Then the following hold.*

(a) $H \in I$ and $K \in R_{\omega}$ implies $\begin{pmatrix} H \\ K \end{pmatrix} \in I$.

(b) $H \in {}_{\geqslant n}R_{\omega}$ and $\begin{pmatrix} H \\ 0 \end{pmatrix} \in I$ implies $H \in I$.

Proof In each case, condition (ii) is used:

(b) $\begin{pmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}.\begin{pmatrix} R \\ S \end{pmatrix} = \begin{pmatrix} R \\ S \\ 0 \end{pmatrix}$;

(a) $\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix}.\begin{pmatrix} {}_{n}R \\ S \\ K \end{pmatrix} = \begin{pmatrix} R \\ S \\ 0 \end{pmatrix}$, then use (b). □

Let $n \geqslant 1$ and suppose $I \subseteq {}_{\geqslant n}R_{\omega}$. The n-closure, $\langle I \rangle^{n}$, of I is the smallest n-closed set containing I. When convenient, and especially when $n = 1$, explicit reference to n is omitted. The next result provides a useful description of the matrices in the closed set generated by a given matrix.

Lemma 8.10 *Let K, H be matrices. Then $K \in \langle H \rangle$ iff there exist matrices G and X such that $\begin{pmatrix} I & \\ & G \\ 0 & \end{pmatrix}.K = HX$.*

Proof One direction is immediate from the definitions. For the other, it will be sufficient to show that, given H, the set, I, of all matrices , K, such that there exist G, X with $\begin{pmatrix} I & \\ & G \\ 0 & \end{pmatrix}.K = HX \ldots (*)$ is closed. The conditions are checked in turn.

(i) If K satisfies (*) and Y is a matrix (matching K), then $KY \in I$ since

$$\begin{pmatrix} I & \\ & G \\ 0 & \end{pmatrix}.KY = H.XY.$$

(ii) Suppose that $\begin{pmatrix} I & G_1 \\ 0 & G_2 \end{pmatrix}.K$ is in I. So there exists $\begin{pmatrix} I & G_3 \\ 0 & G_4 \end{pmatrix}$ such that

$$\begin{pmatrix} I & G_3 \\ 0 & G_4 \end{pmatrix}.\begin{pmatrix} I & G_1 \\ 0 & G_2 \end{pmatrix}.K \in HR_\omega.$$

Thus $\begin{pmatrix} I & G_1+G_3G_2 \\ 0 & G_4G_2 \end{pmatrix}.K \in HR_\omega$, and so $K \in I$.

(iii) Suppose that $\begin{pmatrix} I & G_1 \\ 0 & G_2 \end{pmatrix}.\begin{pmatrix} R' \\ S' \end{pmatrix} = HX'$ and $\begin{pmatrix} I & G_3 \\ 0 & G_4 \end{pmatrix}.\begin{pmatrix} R'' \\ S'' \end{pmatrix} = HX''$. Then one has

$$\begin{pmatrix} I & G_1 & G_3 \\ 0 & G_2 & G_4 \end{pmatrix}.\begin{pmatrix} R' & R'' \\ S' & 0 \\ 0 & S'' \end{pmatrix} = \begin{pmatrix} HX' & HX'' \end{pmatrix} = H\begin{pmatrix} X' & X'' \end{pmatrix}$$ and so

$\begin{pmatrix} R' & R'' \\ S' & 0 \\ 0 & S'' \end{pmatrix} \in I$, as required. □

It is a triviality that the closure of a directed union is the directed union of the corresponding closures; so the lemma may be applied to not-necessarily-finitely-generated pp-types. Of course, a finitely generated closed set is just the same as a singly generated closed set (by closure condition (iii)).

One may note how **8.10** generalises the description of a cyclic right ideal in terms of a generator.

<u>Theorem 8.11</u> [Pr83; 1.4] *Let p be any pp-n-type. Then $I(p)$ is an n-closed set.*

Proof The closure conditions are checked in turn.

(i) Suppose that $H \in I(p)$ and $G \in R_\omega$. Since $H \in I(p)$ the formula $\exists \overline{w}\,(\overline{v}\ \overline{w}).H = 0$ is in p, so certainly the formula $\exists \overline{w}\,(\overline{v}\ \overline{w}).HG = 0$ is in p. Thus $HG \in I(p)$.

(ii) Suppose that $\begin{pmatrix} I & G_1 \\ 0 & G_2 \end{pmatrix}.\begin{pmatrix} R \\ S \end{pmatrix} \in I(p)$. Then the formula $\exists \overline{w}\,(\overline{v}\ \overline{w}).\begin{pmatrix} R+G_1S \\ G_2S \end{pmatrix} = 0$ is in p. That is, $\exists \overline{w}\,(\overline{v}R + (\overline{v}G_1 + \overline{w}G_2)S = 0)$ is in p. So

$\begin{pmatrix} R \\ S \end{pmatrix} \in I(p)$.

(iii) Suppose that $\begin{pmatrix} R \\ S \end{pmatrix}$ and $\begin{pmatrix} R' \\ S' \end{pmatrix}$ are in $I(p)$. Then both formulas

$\exists \overline{w}\,(\overline{v}\ \overline{w}).\begin{pmatrix} R \\ S \end{pmatrix} = 0$ and $\exists \overline{w}\,(\overline{v}\ \overline{w}).\begin{pmatrix} R' \\ S' \end{pmatrix} = 0$ are in p. This gives

$\exists \overline{w}, \overline{w}_1\,(\overline{v}R + \overline{w}S + \overline{w}_1 0 = 0 \wedge \overline{v}R' + \overline{w}0 + \overline{w}_1 S' = 0)$ in p. Hence $\begin{pmatrix} R & R' \\ S & 0 \\ 0 & S' \end{pmatrix} \in I(p)$, as required. □

In order to prove the converse to this theorem, I will show how closed sets may be used to present pp-types, just as right ideals of R may be used to present isomorphism types. For the sake of clarity, I will describe this for the case of 1-types. Because pp-types are closed under conjunction, there is no real difference between the finitely generated and 1-generated cases.

Let ${}_mH_k$ be a matrix over R. Then H defines, by left multiplication, a morphism of free right R-modules (elements of which are regarded as column vectors):

$H^*: R^k \longrightarrow R^m$ by $\begin{pmatrix} r_1 \\ \cdot \\ \cdot \\ r_k \end{pmatrix} \mapsto H.\begin{pmatrix} r_1 \\ \cdot \\ \cdot \\ r_k \end{pmatrix}$. Let $\pi_H: R^m \twoheadrightarrow M_H$ be the cokernel of H^*. Let $i: R \longrightarrow R^m$ be the canonical embedding to the first coordinate. Define $f_H = \pi_H i: R \longrightarrow M_H$, let $a_H = f_H(1)$ and let $\overline{b}_H = \pi_H(e_2,\dots,e_m)$ where e_j is the element of R^m which has "1" at the j-th coordinate and "0"'s elsewhere.

Let H^j denote the j-th column of H: so $H = (H^j)_j = (He_j)_j$.

Example 3 Consider the pp-type of the element "2" in $\mathbb{Z}_8$. It is generated by the formula $v2 = 0 \wedge 4|v$: so $H = \begin{pmatrix} 2 & 1 \\ 0 & -4 \end{pmatrix}$. The morphism from R^2 to R^2 defined by H is given by taking $\begin{pmatrix} r \\ s \end{pmatrix}$ to $\begin{pmatrix} 2 & 1 \\ 0 & -4 \end{pmatrix}.\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 2r+s \\ -4s \end{pmatrix}$. Let e_1 and e_2 be the canonical generators of the second copy of R^2: suppose that the canonical projection π_H takes these respectively to a and b in M_H. The relations between a and b are generated by the images under H^* of the canonical generators of the first copy of R^2: by $a2=0$ and $a=-b4$. Therefore M_H is generated by b - an element of order 8 - and a is indeed the element "2" in this copy of $\mathbb{Z}_8$.

Proposition 8.12 [Pr83; 1.6,1.7]

(i) φ_H is in the pp-type of a_H in M_H.

(ii) If c is an element of the module M and if $M \models \varphi_H(c)$, then there is a morphism from M_H to M which takes a_H to c.

That is, a_H in M_H is a "free realisation" of φ_H (the pp formula φ with $H_\varphi = H$).

Proof (i) It will be sufficient to prove that $(a_H\ \overline{b}_H).H = 0$. Now $(a_H\ \overline{b}_H) = \pi(e_1,e_2,\dots,e_m) = \pi I_m$ where I_m is the $m\times m$ identity matrix. Therefore $(a_H\ \overline{b}_H).H = \pi I_m.H = \pi H = \pi(H^j)_j = (\pi H^j)_j = 0$ (note that it does make sense to have π acting on a matrix, by acting on its columns as elements of R^m).

(ii) Since c satisfies φ_H, there is $\overline{d}$ in M with $(c\ \overline{d}).H = 0$. Define the morphism $f': R \oplus R^{m-1} \longrightarrow M$ by taking $(e_1,(e_2,\dots,e_m))$ to $(c,\overline{d})$. Then $H^j = (e_1,(e_2,\dots,e_m)).H^j$ is mapped to $(c\ \overline{d}).H^j$, which is zero. Thus $\ker f'$ contains $\Sigma_1^k H^jR = \ker\pi$. Hence f' factorises through π, by a morphism from M_H to M which takes $(a_H\ b_H)$ to $(c\ \overline{d})$, as required. □

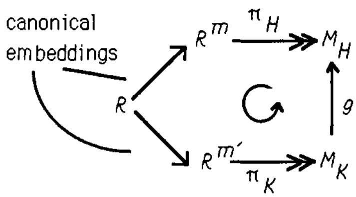
The next theorem says that every (finitely generated) closed set of matrices "is" a pp-type.

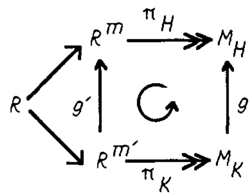
Theorem 8.13 [Pr83; 1.5,1.6] *Let ${}_mH_k$ be a matrix over R. Let p be the pp-type of a_H in M_H. Then $I(p) = \langle H\rangle$.*

Proof The inclusion $I(p) \supseteq \langle H\rangle$ has just been seen in **8**.12(i).

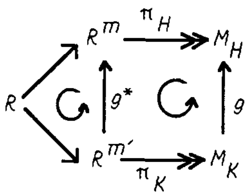
For the converse, let ${}_{m'}K_{k'}$ be in $I(p)$: it will be shown, using the criterion of **8**.10, that $K \in \langle H\rangle$.

By the universal property described in **8**.12 there exists a morphism g, taking a_K to a_H, making the first diagram commute. By projectivity of $R^{m'}$, there is a lifting of g to g' making the square commute (second diagram):





that is, $g\pi_K = \pi_H g'$. We want the morphism g' also to make the triangle commute.



The original choice may be modified to ensure this - define g^* to have the same action as g' on $e'_2,\dots,e'_{m'}$ (e'_i is the i-th canonical generator of $R^{m'}$) and to take e'_1 to e_1. This is well-defined, since $R^{m'}$ is free, and still one has $g\pi_K = \pi_H g^*$ (observe that each side takes e'_1 to a_H). Thus we have the following diagram

Suppose that the matrix of g^* is $\begin{pmatrix} {}_1A_1 & {}_1B_{m'-1} \\ {}_{m-1}C_1 & {}_{m-1}D_{m'-1} \end{pmatrix}$: that is $g^*.(r_1,\dots,r_{m'})^T = G^*\cdot(r_1,\dots,r_{m'})^T$ (where "^{T}" denotes transpose). Apply this to e'_1 - one must obtain e_1: therefore $\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}A\\C\end{pmatrix}$. Thus $A = I$ and $C = 0$.

Now, $0 = \pi_K K^j$, so $0 = g\pi_K.K^j = \pi_H g^*.K^j$ and hence $g^*K^j \in \ker \pi_H$. That is, G^*K^j is in $\ker \pi_H = \operatorname{im} H$ - say $G^*K^j = HX^j$ for some column X^j. Putting these together yields

$G^*K = (G^*K^j)^j = (HX^j)^j = H(X^j)^j = HX$ (say). Thus one has the matrix equation

$\begin{pmatrix} 1 & \\ & G \\ 0 & \end{pmatrix}.K = HX$. Hence $K \in \langle H \rangle$, as required. □

It follows that one may think of a_H in M_H as being a "free realisation" of the pp-type generated by φ_H (and $\overline{b}_H$ is a "free" witness to the existential quantifiers in φ_H). Of course, there are many pp formulas which are equivalent to φ, and the matrices which correspond to these formulas may be very different. In particular, a_H need not be a "minimal realisation" of φH. Over right artinian rings one does have minimal realisations: I don't know whether these exist in general.

Given a pp formula φ, write a_φ and M_φ for a_H and M_H, where $H = H_\varphi$.

<u>Corollary 8.14</u> [Pr83; 1.14] *Let* φ *and* ψ *be pp formulas. Then* $M_\varphi \models \psi(a_\varphi)$ *iff* $T^* \vdash \varphi \to \psi$. □

<u>Corollary 8.15</u> *A pp-type is finitely generated iff it is realised in a finitely presented module.*

Proof The direction $\Leftarrow$ is 8.4. The converse is immediate from 8.13. □

<u>Corollary 8.16</u> [Pr83; §1] *The closed subsets of* R_ω *form a lattice under intersection and closure-of-union. The correspondence* $p \mapsto I(p)$ *defines a lattice isomorphism from the lattice* P(R) *of pp-types to the lattice of closed subsets of*

R_ω. *Under this isomorphism, finitely generated pp-types correspond exactly to the finitely generated closed sets.*

Proof Since any pp-type or closed set of matrices is the directed union of its finitely generated sub-pp-types, resp. closed subsets, the result follows from the finitely generated case, which has just been established. □

One particular consequence of **8.16** is that the intersection of two finitely generated closed sets is finitely generated (exercise: show this directly; in particular, write down $H(\varphi + \psi)$).

Of course, the above results hold equally for n-types and n-closed sets.

Corollary 8.17 [Pr83; 1.15] *Let* $\varphi(\bar{v})$ *and* $\psi(\bar{v})$ *be pp formulas. Then the following are equivalent:*
(*i*) $T^* \vdash \varphi \to \psi$;
(*ii*) $\mathrm{Th}(\mathcal{M}_R) \vdash \varphi \to \psi$;
(*iii*) $\mathrm{Th}(\text{mod-}R) \vdash \varphi \to \psi$.
Furthermore, the following are equivalent:
(*i*) $T^* \vdash \varphi \leftrightarrow \psi$;
(*ii*) $\mathrm{Th}(\mathcal{M}_R) \vdash \varphi \leftrightarrow \psi$;
(*iii*) $\mathrm{Th}(\text{mod-}R) \vdash \varphi \leftrightarrow \psi$.

Proof The equivalence of (i) and (ii) is, for each case, in **2.30**. Therefore, to establish the corollary, it will be enough to show that if $\mathrm{Th}(\text{mod-}R)$ proves $\varphi \to \psi$ then so does $\mathrm{Th}(\mathcal{M}_R)$. But, since M_φ is finitely presented, this is just **8.14**. □

It should be observed that the correspondence, **8.16**, between pp-types and closed sets of matrices, together with the "computational" description **8.10**, of the closed set generated by a matrix, provides an explicit description of the consequences of any given pp formula.

I leave to the reader the exercise of generalising all this to types in infinitely many variables.

8.4 Duality and pure-semisimple rings

One major difficulty encountered when dealing with modules over non-commutative rings is that there need be little connection between the lattices of right and left ideals of the ring. For example, a ring may be artinian on the right yet fail to have even Krull dimension on the left. The situation for pp-types turns out to be much better. In this section I will show that there is a duality between the lattices of pp-types for right modules and that for left modules. This duality will be applied to show that if every right module over a ring is a direct sum of indecomposable modules (i.e., the ring R is right pure-semisimple - see §11.1) then, for each integer n there are, up to isomorphism, only finitely many indecomposable R-modules of length n.

At the base of this duality between pp-types is one defined for matrices; I describe this now.

Let H be a matrix over R. The matrices $H^{\Leftarrow}$ and $H^{\Rightarrow}$ are defined as follows.

$$H^{\Leftarrow} = \begin{pmatrix} 1 & \\ & H \\ 0 & \end{pmatrix} \text{ and } H^{\Rightarrow} = \begin{pmatrix} 1 & 0 \\ & H \end{pmatrix}.$$

Here "1" denotes the 1x1 identity matrix and "0" denotes a zero matrix of appropriate size.

Example 1 Let n be an integer. Then the 1x1 matrix (n) corresponds to the property of being annihilated by n. The matrix $(n)^{\Leftarrow}$ corresponds to divisibility by n in left modules (our convention is that tuples from left modules are written as column vectors). The matrix $(n)^{\Leftarrow\Rightarrow}$ is, of course, different from (n) but it is easily seen (exercise) that they correspond to equivalent pp formulas.

Lemma 8.18 [Pr87; 2.1] *Let* H *be a matrix. Then* $H^{\Leftarrow\Rightarrow}$ *(and* $H^{\Rightarrow\Leftarrow}$*) generate the same pp-type as does* H.

Proof $H^{\Leftarrow\Rightarrow}$ has the form $\begin{pmatrix}1 & 0\\ 1 & H'\\ 0 & H''\end{pmatrix}$ where $H=\begin{pmatrix}H'\\ H''\end{pmatrix}$. To see (by 8.10) that H is in $\langle H^{\Leftarrow\Rightarrow}\rangle$ consider:

$$\begin{pmatrix}1 & 0\\ 0 & 0\\ 0 & -I\end{pmatrix}.\begin{pmatrix}H'\\ H''\end{pmatrix}=\begin{pmatrix}1 & 0\\ 1 & H'\\ 0 & H''\end{pmatrix}.\begin{pmatrix}H'\\ -I\end{pmatrix}.$$

For the converse, one has: $\begin{pmatrix}1 & -1 & 0\\ 0 & 0 & -I\end{pmatrix}.\begin{pmatrix}1 & 0\\ 1 & H'\\ 0 & H''\end{pmatrix}=\begin{pmatrix}0 & -H'\\ 0 & -H''\end{pmatrix}=H.(0\ -I)$. ◻

This "duality" for matrices induces a duality for pp-types as follows. Given $p\in \mathrm{P}(R)=\mathrm{P}(R_R)$, consider the pp-type $q\in \mathrm{P}({}_RR)$ which satisfies $I(q)=\langle\{H^{\Leftarrow}: H\in I(p)\}\rangle$. I show that this is indeed a duality by showing that if $\langle K\rangle\le\langle H\rangle$ in $\mathrm{P}(R)$ then $\langle K^{\Leftarrow}\rangle\ge\langle H^{\Leftarrow}\rangle$ in $\mathrm{P}({}_RR)$: that is, if $K\in\langle H\rangle$ then $H^{\Leftarrow}\in\langle K^{\Leftarrow}\rangle$. The following lemma is used.

Lemma 8.19 [Pr87; 2.2] *Let* X *and* Y *be matrices. Then:*

(*i*) $(XY)^{\Leftarrow}=X^{\Leftarrow}.\begin{pmatrix}1 & 0\\ 0 & Y\end{pmatrix}$

(*i*)^o $(XY)^{\Rightarrow}=\begin{pmatrix}1 & 0\\ 0 & X\end{pmatrix}.Y^{\Rightarrow}$

Proof One has, if $X=\begin{pmatrix}X'\\ X''\end{pmatrix}$, the following (which proves (i) and its dual (i)^o):

$$\begin{pmatrix}1 & X'\\ 0 & X''\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & Y\end{pmatrix}=\begin{pmatrix}1 & X'Y\\ 0 & X''Y\end{pmatrix}=\begin{pmatrix}1 & \\ & XY\\ 0 & \end{pmatrix}=(XY)^{\Leftarrow}.\ ◻$$

Proposition 8.20 [Pr87; 2.3] *Let* K *and* H *be matrices. Then* $K\in\langle H\rangle$ *implies* $H^{\Leftarrow}\in\langle K^{\Leftarrow}\rangle$.

Proof By 8.10 it is enough to check the following two cases.

(a) $K=HX$ for some matrix X.

Then one has $K^{\Leftarrow}=(HX)^{\Leftarrow}=H^{\Leftarrow}.\begin{pmatrix}1 & 0\\ 0 & X\end{pmatrix}$ (by (i) of 8.19). So, by definition of closed set of (left!) matrices, $H^{\Leftarrow}\in\langle K^{\Leftarrow}\rangle$.

(b) H has the form $\begin{pmatrix}1 & \\ & X\\ 0 & \end{pmatrix}.K$

For this, consider: $H^{\Leftarrow}=\left(\begin{pmatrix}1 & \\ & X\\ 0 & \end{pmatrix}.K\right)^{\Leftarrow}=\left(\begin{pmatrix}1 & X'\\ 0 & X''\end{pmatrix}.\begin{pmatrix}K'\\ K''\end{pmatrix}\right)^{\Leftarrow}=\begin{pmatrix}K'+X'K''\\ X''K''\end{pmatrix}^{\Leftarrow}=$

$\begin{pmatrix}1 & K'+X'K''\\ 0 & X''K''\end{pmatrix}=\begin{pmatrix}1 & X'\\ 0 & X''\end{pmatrix}.\begin{pmatrix}1 & K'\\ 0 & K''\end{pmatrix}=\begin{pmatrix}1 & \\ & X\\ 0 & \end{pmatrix}.K^{\Leftarrow}$. Thus $H^{\Leftarrow}\in\langle K^{\Leftarrow}\rangle$, as required. ◻

The above proposition, together with **8.18**, shows that the operators "$\Leftarrow$" and "$\Rightarrow$" do induce a duality between $P^f(R)$ and $P^f({}_RR)$ (the same notation will be used).

Theorem 8.21 *The operators "$\Leftarrow$" and "$\Rightarrow$" induce a duality between* $P^f(R)$ *and* $P^f({}_RR)$. *In the case that* R *is an artin algebra, this extends to a duality between* $P(R)$ *and* $P({}_RR)$. *In particular* $P^f({}_RR) \simeq P^f(R)^{op}$ *and, if* R *is an artin algebra,* $P({}_RR) \simeq P(R)^{op}$. □

For the stronger conclusion (for artin algebras) see the discussion at the end of §12.1. That this duality does not operate at the level of modules is illustrated by the next example.

Example 2 I illustrate this process with an example and, at the same time, show that the duality is definitely at the level of pp-types rather than modules (i.e., the fact that p and q have the same hull does not imply that their duals have the same hull).

First consider the type of the element "2" in $\mathbb{Z}_8$: the corresponding matrix H is $\begin{pmatrix} 2 & 1 \\ 0 & -4 \end{pmatrix}$ (see Ex 8.3/3). Then $H^{\Leftarrow}$ is $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & -4 \end{pmatrix}$. We compute the left module that this presents. By using its transpose $\begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 1 & -4 \end{pmatrix}$, we may work on the right. This matrix defines a morphism from R^2 to R^3: let a, b and c be the images in the cokernel of the canonical generators of R^3. The relations between these are generated by $a+2b+c=0$ and $c4=0$. One may then see (exercise) that the resulting module decomposes as $\langle b\rangle \oplus \langle a-b2\rangle$ - a copy of $\mathbb{Z} \oplus \mathbb{Z}_4$. In particular, the type of a is reducible.

Now take the element "1" of $\mathbb{Z}_8$: its pp-type is generated by the (formula with) matrix (8). The transpose of $(8)^{\Leftarrow}$ is $\begin{pmatrix} 1 \\ 8 \end{pmatrix}$ and this presents a module with generators a and b related by $a=b8$. That is, we obtain the pp-type of the element "8" in $\mathbb{Z}$.

Before going on to the main application, I mention various other consequences.

Corollary 8.22 [Pr87; 2.5] *Let* R *be a commutative ring. Then the lattice of finitely generated pp-types is symmetric, in the sense of being isomorphic to its own dual.* □

The next result will be used in the main application: it is immediate from **8.21** and **11.6**.

Corollary 8.23 [Pr87; 3.1] *Suppose that* R *is right pure-semisimple. Then* R^{op} *has the descending chain condition on finitely generated pp-types.* □

Corollary 8.24 [DR72; 1.1] also see [Aus74a; 3.6] *If* R *is of finite representation type on the right then it is so on the left also.* □

Corollary 8.25 *If* R *is right and left pure semisimple then* R *is of finite representation type.* □

Both of these follow immediately from **8.23** and **11.38**. The second result was found independently by a number of authors (see [Ful76; p433] for some references).

It is interesting to compare the proof of the above in [FR75; Thm] with the one given here. Their proof establishes the equivalence of the acc for monomorphisms between finitely generated right modules and the dcc for epimorphisms between finitely generated left modules (and then they feed this into [Aus74a; 3.1]). Their proof uses the Auslander-Bridger transpose ([AB69]) which is defined on the level of modules, in contrast to the duality here which lives at the level of pp-types.

Turning again towards our main goal, we recall lemma **7.21**: the use of that lemma here is similar to its use in proving that a theory of U-rank n has no more than n dimensions (**7.23**), but our hypothesis will concern modules of a given length and this, together with the Harada-Sai lemma **11.20**, will replace the assumption of finite U-rank.

<u>Theorem 8.26</u> [Pr87; 3.6] *If the ring R is right pure-semisimple then, for every positive integer n, there are only finitely many indecomposable finitely presented <u>left</u> modules of length n.*

Proof Within this proof "module" will mean left module. Let N_λ $(\lambda \in \Lambda)$ be the set of all indecomposable left modules of length n (one of each isomorphism type). For each $\lambda \in \Lambda$, set Y_λ to be the set of all pp-1-types realised in N_λ. Observe that the Y_λ are pair-wise disjoint. Put $X_0 = \bigcup_\Lambda Y_\lambda$.

By the Harada-Sai lemma (and **8.5**) there is no strictly increasing chain in X_0, so certainly X_0 has a maximal element: let p_0 be such an element. Let us suppose that the indecomposable N_0 realises p_0. Put $X_1 = X_0 \setminus Y_0$.

Take a maximal element p_1 in X_1, ... Continue in this way.

This process will come to a halt at a finite stage exactly if Λ is finite - which is what is want. So suppose for a contradiction that the process continues without end, and let $p_0, p_1, \ldots, p_k, \ldots$ be the "maximal" pp-types obtained.

Consider the descending chain $p_0 \geqslant p_0 \cap p_1 \geqslant \ldots \geqslant p_0 \cap p_1 \cap \ldots \cap p_k \geqslant \ldots$ of pp-types of left R-modules. By **8.23** this chain stabilises at some finite stage and so there exists k such that $p_{k+1} \geqslant p_1 \cap \ldots \cap p_k$. By **7.21** there exists a pp-type q realised in N_{k+1} and there exists $i \in \{0, \ldots, k\}$ such that $q \geqslant p_i$: note that $q > p_i$. But since p_i is maximal in X_i and since q is in X_i (since p_{k+1} is), this is impossible. Thus we have our contradiction, and the result follows. □

This is all very well, but since it is even open whether a right pure-semisimple ring is left artinian it would be much more satisfactory to transfer this to the right modules. After seeing the above result, D. Simson pointed out to me that a result of his allows one to accomplish this transfer. His result is the following.

<u>Theorem 8.A</u> [Sim77a; "Note added in proof" and Corrigendum p.256] *Suppose that R is right pure-semisimple, and let I be the minimal injective cogenerator (direct sum of the injective hulls of the simple modules). Set $T = \mathrm{End}\, I$. Then T is right pure-semisimple and left artinian; moreover there is a duality T-mod $\simeq$ (mod-R)$^{\mathrm{op}}$ between the category of finitely presented left T-modules and the category of finitely presented right R-modules (in the direction "$\Leftarrow$" it is given by the functor $(-, I)$).* □

Putting this together with **8.26** one obtains the following.

<u>Theorem 8.27</u> [Pr87; 3.8] *Suppose that R is right pure-semisimple. Then for each positive integer n there are, up to isomorphism, only finitely many (indecomposable) modules of length n.*

Proof Apply **8.26** to T as described in **8.A**. Note that the dual of a simple module is simple. Hence if the module N has length k, so does its dual (apply induction to the exact sequence $0 \longrightarrow S \longrightarrow N \longrightarrow N/S \longrightarrow 0$ where S is a simple submodule of N). Since this duality preserves length, the result follows. □

CHAPTER 9 TYPES AND THE STRUCTURE OF PURE-INJECTIVE MODULES

This chapter is concerned with the relation between types or, more properly, pp-types, and the structure of their hulls. Some of the results proved in this chapter will be used in the more global considerations of Chapter 10 but, in the main, we concentrate here on local structure.

Replacement of minimal pp-definable subgroups by minimal pairs is a key step in going from the totally transcendental to the general case. Irreducible types which share a minimal pair need not be equal, but they do have isomorphic hulls (§1). In consequence, given a complete theory, each unlimited indecomposable pure-injective has the same multiplicity in every pure-injective model. Another result which was seen in the totally transcendental case in §4.6 and is now proved in full generality, is that if an indecomposable pure-injective has a minimal pair, then the corresponding quotient of subgroups has the structure of a 1-dimensional vectorspace over the division ring associated to the indecomposable (**9.6**).

If two types have linked realisations then they have isomorphic parts: specifically, between the positive and negative parts of each type there is an interval, defined in terms of the linking formula, and these intervals are isomorphic (with positive and negative parts corresponding). This means that irreducible types which have isomorphic hulls are syntactically similar. For instance, if one of them has one of the (syntactically defined) dimensions of Chapter 10, then so does the other (and the values are equal). Finally in §2, the relation between the hull of a type and its direct summands is explicated (**9.16**) and there is a syntactic criterion on the type for there to be an indecomposable direct summand of its hull.

By this stage, the reader will have encountered a number of notions of "isolation". The relationship between these is clarified in the third section. It is shown that a type is isolated iff it is finitely generated and neg-isolated. Then there is the following chain of implications for an irreducible type p: p isolated $\Rightarrow$ p contains a minimal pair $\Rightarrow$ the hull of p is isolated in the space of indecomposables $\Rightarrow$ p is neg-isolated. At the beginning of the section it is shown that the (algebraic) weight of a neg-isolated type is bounded by the number of pp formulas required to isolate its negative part.

We say that a pure-injective module is an elementary cogenerator if every model of its theory purely embeds in some direct power of it. We see (§4) that every complete theory has an elementary cogenerator, and that every totally transcendental module is an elementary cogenerator. This notion is tied in with neg-isolation of pp-types. In particular, if p is irreducible and neg-isolated and if its hull is a direct summand of some product of indecomposable pure-injectives, then one of the factors must already be isomorphic to $N(p)$.

9.1 Minimal pairs

Recall: that $\varphi/\psi \in p$ means $\varphi \in p^+$ and $\psi \in p^-$ (and $\varphi > \psi$); that φ/ψ is a minimal pair if $\varphi > \psi$ and there is no pp formula strictly between φ and ψ. Observe that the property of having a minimal pair depends only on the lattice of pp-types. In particular, if $p \in S^T(0)$ and if $\varphi > \psi$, then φ/ψ is a minimal pair for p iff φ/ψ is a minimal pair for $jp \in S^{T^{\aleph_0}}(0)$. Therefore, when it is convenient to do so, we may make the simplifying assumption $T = T^{\aleph_0}$. In this chapter, when not otherwise specified, we are working modulo some complete theory T of modules. The first lemma states, for easy reference, a consequence of modularity.

<u>Lemma 9.1</u> *Let T be arbitrary and suppose that p is a type or pp-type. If $\varphi/\psi \in p$ is a minimal pair and if $\varphi' \in p^+$, then $\varphi \wedge \varphi' / \psi \wedge \varphi'$ is a minimal pair in p.*

Proof One has, by modularity, that either $\varphi \wedge \varphi' / \psi \wedge \varphi'$ is a minimal pair or that $\varphi \wedge \varphi' = \psi \wedge \varphi'$. Since $\varphi \wedge \varphi' \in p^+$ and since $\psi \wedge \varphi' \leqslant \psi \in p^-$, the first case holds. □

The first main result, due to Ziegler, is that minimal pairs determine indecomposable pure-injectives uniquely. This was shown in §4.6 for totally transcendental theories, but in the general case one does not have a "least" pp formula in a type, so the argument has to be more complicated. The following "separation" lemma is needed.

Lemma 9.2 [Zg84; 7.10] *Let p, q be irreducible types over 0, and let φ, ψ be pp formulas with φ/ψ in both p and q. If the hulls of p and q are not isomorphic, then there is some pp formula θ with $\varphi > \theta > \psi$ and either: $\varphi/\theta \in p$ and $\theta/\psi \in q$; or $\varphi/\theta \in q$ and $\theta/\psi \in p$.*

Proof First it is shown that the set of formulas $p(\bar{v}) \wedge q(\bar{w}) \wedge \psi(\bar{v}-\bar{w})$ is inconsistent. If it were not, then there would be $\bar{a}$ realising p and $\bar{b}$ realising q, with $\psi(\bar{a}-\bar{b})$. It may be assumed that $T = T^{\aleph_0}$. Therefore 6.20 applies to yield from $N(p) \not\simeq N(q)$ that p is orthogonal to q. Hence $\bar{a}$ and $\bar{b}$ are independent over 0. Then from $\psi(\bar{a}-\bar{b})$ one deduces (by 5.26) that $\psi(\bar{a})$ holds - contradicting $\psi \notin p$.

Therefore there are $\varphi_0 \in p^+$ and $\psi = \psi_1, \psi_2, \ldots, \psi_n$ in p^- such that $\varphi_0(\bar{v}) \wedge q(\bar{w}) \wedge \psi(\bar{v}-\bar{w}) \rightarrow \bigvee_1^n \psi_i(\bar{v})$. By 4.29 there is $\varphi' \in p^+$, which may be taken with $\varphi' \leqslant \varphi \wedge \varphi_0$, such that $\sum_1^n \varphi' \wedge \psi_i \in p^-$. Adding φ' as a conjunct to each side of the above implication, one obtains $\varphi'(\bar{v}) \wedge q(\bar{w}) \wedge \psi(\bar{v}-\bar{w}) \rightarrow \bigvee_1^n \varphi' \wedge \psi_i(\bar{v}) \rightarrow \sum_1^n \varphi' \wedge \psi_i \in p^-$ $(*)$: let ψ' be the formula $\sum_1^n \varphi' \wedge \psi_i$. Observe that $\varphi \geqslant \varphi' > \psi' \geqslant \psi \wedge \varphi'$ $(**)$ (this is why ψ was required to be among the ψ_i). There are two cases to consider.

Suppose that $\psi' + \psi \in q^+$. Then set θ to be $\psi' + \psi$. Certainly one has $\psi \leqslant \theta \leqslant \varphi$ and $\theta/\psi \in q$. It must be shown that $\theta \in p^-$. Now, $\varphi' \in p^+$, but $\theta \wedge \varphi' = (\psi' + \psi) \wedge \varphi' = \psi' + \psi \wedge \varphi'$ (by modularity) and this, by $(**)$, equals ψ', which is in p^-. So $\varphi/\theta \in p$.

On the other hand, if $\psi' + \psi \in q^-$, set θ to be $\varphi' + \psi$. Clearly $\psi \leqslant \theta \leqslant \varphi$ and $\theta/\psi \in p$ (for $\varphi' \in p^+$). So it remains to show that φ/θ is in q, i.e., that $\theta \notin q$. If this were not so, then we would have the following formula in $q(\bar{w})$: $\exists \bar{v}\,(\varphi'(\bar{v}) \wedge \psi(\bar{v}-\bar{w}))$. Together with $(*)$, this (quickly) gives that $\exists \bar{v}\,(\psi'(\bar{v}) \wedge \psi(\bar{v}-\bar{w}))$ is in $q(\bar{w})$ - that is, $\psi' + \psi \in q$ - contradiction, as required. □

Corollary 9.3 [Zg84; §7] *Suppose that p and q are irreducible types which contain the same minimal pair. Then $N(q) \simeq N(p)$.* □

This corollary says that irreducible types which share a minimal pair have "the same" hull. Such types need not, however, be equal. For an example, take R to be the ring $K[X,Y]/\langle X,Y\rangle^2$, and consider R as a module over itself (cf. Ex 2.1/6(vi)). Let $\varphi(v)$ say "$vJ = 0$" and let $\psi(v)$ say "$v \in xR$". Then φ/ψ is a minimal pair, but there are many irreducible types which contain this minimal pair: for each $\alpha \in K$, the type of a non-zero element of $(\alpha x + y)R$ does so.

Corollary 9.4 [Zg84; 8.12] *Suppose that the interval $[\varphi:\psi]$ has finite length. Then the open set (φ/ψ) of $\mathcal{I}(T)$ has only finitely many points (no more than the length of $[\varphi:\psi]$).* □

The second corollary is generalised in 10.20.

Thus, given any complete theory of modules, T, if φ/ψ is a T-minimal pair then the open set (φ/ψ) of $\mathcal{I}(T)$ contains just one, isolated, point of $\mathcal{I}(T)$ (by 4.35 this open set is non-empty). Of course, if $T \leqslant T'$ then there is no reason in general to expect that (φ/ψ) defines a single point in $\mathcal{I}(T')$.

The remaining results of the section generalise some seen in §4.6.

Theorem 9.5 *Suppose that the type p is irreducible and limited. Then there is an integer $\mathrm{n}(p)$ such that, if M is any pure-injective model then the multiplicity of $N(p)$ in M is exactly $\mathrm{n}(p)$.*

Proof By **4.44** there is a minimal pair $\varphi/\psi \in p$ with $\mathrm{Inv}(T,\varphi,\psi)$ a finite integer $k>1$. Let M be any pure-injective model of T and set $M = N(p)^{(n)} \oplus M' \oplus M_C$ where M' is a discrete pure-injective, M_C is continuous, and $N(p)$ does not divide M (and, of course, n is finite).

Then $\mathrm{Inv}(T,\varphi,\psi) = \mathrm{Inv}(N(p),\varphi,\psi)^n.\mathrm{Inv}(M',\varphi,\psi).\mathrm{Inv}(M_C,\varphi,\psi)$. Next, we appeal to **10.1**, the proof of which may be read now. Since M_C has no indecomposable direct summand, **10.1** yields $\mathrm{Inv}(M_C,\varphi,\psi)=1$. Since M' has no factor isomorphic to $N(p)$, **10.1** and **9.3** yield $\mathrm{Inv}(M',\varphi,\psi)=1$. Thus $n = \mathrm{n}(p) = \mathrm{Inv}(T,\varphi,\psi)/\mathrm{Inv}(N(p),\varphi,\psi)$, as required. □

In §4.6 an algebraic significance was attached to $\mathrm{Inv}(N(p),\varphi,\psi)$ when the overlying theory is t.t. and $\varphi/\psi \in p$ is a minimal pair. This invariant is now shown to have the same meaning in the general case.

<u>Proposition 9.6</u> *Suppose that the indecomposable pure-injective N has a minimal pair, φ/ψ. Then the quotient $\varphi(N)/\psi(N)$ has the structure of a 1-dimensional vectorspace over the division ring $D_N = \mathrm{End}N/J\mathrm{End}N$. Setting $d_N = |D_N|$, we have:*

(a) if p is such that $N(p) \simeq N$ then there is a minimal pair $\varphi'/\psi' \in p$ with $[\varphi'(N):\psi'(N)] = d_N$;

(b) if $\varphi'(N) > \psi'(N)$ then there are φ'', ψ'' pp with $\varphi' \geqslant \varphi'' > \psi'' \geqslant \psi'$ and with $[\varphi''(N):\psi''(N)] = d_N$.

Proof Suppose that p is any type with $N(p) \simeq N$. By **10.1** and 9.12 below, p contains a minimal pair, say φ'/ψ'. Let $\bar{a}$ realise p in N.

By **2.8**, $S\bar{a} = p^+(N)$, where $S = \mathrm{End}N$. By **4.30**, $B = \sum\{S\bar{b} : \bar{b}$ is in N and $\mathrm{pp}(\bar{b}) > p^+\}$ is a proper S-submodule of $S\bar{a}$. Define the map $\theta : S \longrightarrow S\bar{a}/B$ by $\theta f = f\bar{a} + B$. This is a well-defined left S-linear map. Moreover, by 4.27 (and 2.8), θ induces an isomorphism (of left S-modules) between S/JS and $S\bar{a}/B$. In particular, $|D_N| = |S\bar{a}/B|$.

Next, I claim that B is just $p^+(N) \cap \psi'(N)$: this is immediate from the fact that ψ' neg-isolates p. That is proved below (see 9.26), but I give a direct proof now. Let θ be a pp formula not in p^+. Since p is irreducible, there is φ_0, which may be taken to be below φ', with $\varphi_0 \wedge \theta + \varphi_0 \wedge \psi'$ in p^-. By modularity, the pair $\varphi_0 \wedge \varphi'/\varphi_0 \wedge \psi'$ is minimal (it does not collapse, since ψ' is in p^-). So it must be that $\varphi_0 \wedge \theta$ is below $\varphi_0 \wedge \psi'$ (so, in particular, below ψ'). Thus p is indeed equivalent to $p^+ \wedge \neg\psi'$.

So B is just $p^+(N) \cap \psi'(N)$. Again by modularity, $S\bar{a}/B = p^+(N)/p^+(N) \cap \psi'(N) \simeq \varphi'(N)/\psi'(N)$.

For (b), suppose that $\varphi'(N) > \psi'(N)$. Choose an element a in $\varphi'(N) \setminus \psi'(N)$ and let p be its type. By (a), p has a minimal pair, φ_0/ψ_0 say, with $|\varphi_0(N)/\psi_0(N)| = d_N$ where, by **9.1**, it may be supposed that $\varphi_0 \leqslant \varphi'$. Since ψ' and ψ_0 are in p^-, there is (**4.29**) some $\varphi_1 \in p^+$ with $\varphi_1 \wedge (\psi' + \varphi_1 \wedge \psi_0) = \varphi_1 \wedge \psi' + \varphi_1 \wedge \psi_0$ not in p^+, so $\psi' + \varphi_1 \wedge \psi_0 \notin p^+$. Now, by 9.1, $(\varphi_1 \wedge \varphi_0)/(\varphi_1 \wedge \psi_0)$ is a minimal pair in p. Consider the pair $(\varphi_1 \wedge \varphi_0 + \psi')/(\varphi_1 \wedge \psi_0 + \psi')$. Clearly $\varphi_1 \wedge \varphi_0 + \psi' \in p^+$, and it was noted that $\varphi_1 \wedge \psi_0 + \psi'$ is not in p^+. By modularity
$|(\varphi_1 \wedge \varphi_0 + \psi')(N)/(\varphi_1 \wedge \psi_0 + \psi')(N)| = |(\varphi_1 \wedge \varphi_0)(N)/(\varphi_1 \wedge \psi_0)(N)| = d_N$. Also, $\varphi' \geqslant \varphi_1 \wedge \varphi_0 + \psi' > \varphi_1 \wedge \psi_0 + \psi' \geqslant \psi'$, and so (b) is proved. □

Therefore, if N is an indecomposable pure-injective, either with no minimal pair, or with a minimal pair φ/ψ where $[\varphi(N):\psi(N)]$ is infinite, then N cannot be limited in any theory.

Example 1 Even if D_N is finite, it need not be the case that N contains a minimal pair. Let R be a complete valuation domain with maximal ideal infinitely generated and finite residue field. Then, by **9.12** below, R has no minimal pairs. On the other hand, R is pure-injective and $D_R = R/J$ is finite.

9.2 Associated types

Suppose that p and q are two types such that there is a realisation $\bar{a}$ of p, and a realisation $\bar{b}$ of q, with $\bar{a}$ and $\bar{b}$ linked. This section is devoted to exploring the relationship between types thus connected. Particular questions addressed are: if q is realised in $N(p)$, how is this fact reflected in the syntactic structure of the types p and q? if the type p has a certain (syntactically expressed) finiteness property and if $N(p)\simeq N(q)$, then does q share this property? One may say that we are looking in more detail at the relationship of non-orthogonality (by **6.20** this is almost literally true, since in this section one usually may replace T by $T^{\aleph_0}$).

We know, by **4.31**, that linked types are structurally connected: here we examine this relation in more detail. The first lemma is a general result about additive relations (i.e., subgroups of a product $A\times B$ of groups). For readability, I denote $(\exists w\,\varphi(v,w))(M)$ by $\exists w\,\varphi(M,w)$. The following results would be most succinctly expressed by using T^{eq}.

Lemma 9.7 *Let $\varphi(v,w)$ be a pp formula and let M be any module. Then the quotients of pp-definable subgroups, $\exists w\,\varphi(M,w)/\varphi(M,0)$ and $\exists v\,\varphi(v,M)/\varphi(0,M)$ are naturally isomorphic. Under this isomorphism, the coset $c+\varphi(M,0)$ is mapped to $d+\varphi(0,M)$, where d is such that $\varphi(c,d)$ holds: in particular, the isomorphism is definable.*

Proof First, the alleged map is well-defined. For if $\varphi(c,d)$ and $\varphi(c,d')$ hold, then $d-d'$ lies in $\varphi(0,M)$: also, if $\varphi(c,d)$ and $\varphi(c-c',0)$ hold, then so does $\varphi(c',d)$. Second, the kernel of the map is precisely $\varphi(M,0)$: so it is 1-1. Clearly it is onto. □

Corollary 9.8 (see [Gar80]) *Let a and b be elements of a module M and suppose that they are linked by the pp formula $\varphi(v,w)$. Then the quotients of pp-definable subgroups, $\exists w\,\varphi(M,w)/\varphi(M,0)$ and $\exists v\,\varphi(v,M)/\varphi(0,M)$ are definably isomorphic, by an isomorphism which takes a to b.* □

This was already noted in the context of t.t. theories in §4.6. Garavaglia used this result heavily in [Gar80] and [Gar80a; §5], though most of those uses are now redundant. But, extracting more information from this isomorphism, we obtain the next result, which is quite central.

What we have to note is that this isomorphism of groups "respects pp formulas" and so induces an isomorphism between the relevant intervals in the lattice of pp-definable subgroups.

More precisely, let $\varphi(v,w)$ be pp and let $\psi(v)$ be a pp formula which lies between $\exists w\,\varphi(v,w)$ and $\varphi(v,0)$ in the lattice of pp-definable subgroups of M. Then $\psi(v)$ is equivalent to $\psi(v)+\varphi(v,0)$. Therefore the isomorphism between $\exists w\,\varphi(M,w)/\varphi(M,0)$ and $\exists v\,\varphi(v,M)/\varphi(0,M)$ which is defined by φ induces a map of such pp-definable subgroups $\psi(M)$. The image of $\psi(M)$ lies between $\exists v\,\varphi(v,M)$ and $\varphi(0,M)$ and is itself pp-definable by the formula $\exists v(\varphi(v,w)\wedge\psi(v))$. Therefore one obtains the following result.

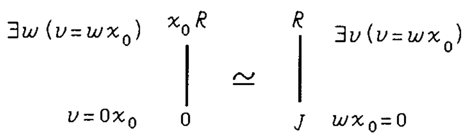
Proposition 9.9 [Zg84; 8.9] also see [Gar80a; §6] *Suppose that φ is a pp formula, and that $\bar{a}$ and $\bar{b}$ are such that $\varphi(\bar{a},\bar{b})\wedge\neg\varphi(\bar{a},\bar{0})$ holds. Then, in the lattice of pp-definable subgroups, the intervals $[\exists\bar{w}\,\varphi(\bar{v},\bar{w}),\ \varphi(\bar{v},\bar{0})]$ and $[\exists\bar{v}\,\varphi(\bar{v},\bar{w}),\ \varphi(\bar{0},\bar{w})]$ are naturally isomorphic. The isomorphism f is such that, for $\psi(\bar{v})\in[\exists\bar{w}\,\varphi(\bar{v},\bar{w}),\ \varphi(\bar{v},\bar{0})]$, one has $\psi(\bar{a})$ iff $(f\psi)(\bar{b})$.*

Proof The map f, and its purported inverse, g, are defined as follows, on $\psi(\bar{v})\in[\exists\bar{w}\,\varphi(\bar{v},\bar{w}),\ \varphi(\bar{v},\bar{0})]$ and on $\theta(\bar{w})\in[\exists\bar{v}\,\varphi(\bar{v},\bar{w}),\ \varphi(\bar{0},\bar{w})]$ respectively: $(f\psi)(\bar{w})\equiv\exists\bar{v}\,(\varphi(\bar{v},\bar{w})\wedge\psi(\bar{v}))$; $(g\theta)(\bar{v})\equiv\exists\bar{w}\,(\varphi(\bar{v},\bar{w})\wedge\theta(\bar{w}))$.

Clearly, by the discussion above, these are order-preserving, so the only point to be checked is that $gf\psi$ is equivalent to ψ (and by symmetry, $fg\theta$ is equivalent to θ). Suppose first that $\psi(\overline{c})$ holds. Since $\psi(\overline{v})$ implies $\exists\overline{w}\,\varphi(\overline{v},\overline{w})$, there is $\overline{d}$ such that $\varphi(\overline{c},\overline{d})$ holds. Thus one has $\varphi(\overline{c},\overline{d})\wedge\psi(\overline{c})$ so, by definition, $f\psi(\overline{d})$ holds. Thus one has $\varphi(\overline{c},\overline{d})\wedge f\psi(\overline{d})$ and so, by definition, $gf\psi(\overline{c})$ holds. Conversely, suppose that $gf\psi(\overline{c})$ holds. So, by definition of g, there is $\overline{d}$ with $\varphi(\overline{c},\overline{d})\wedge f\psi(\overline{d})$. Since $f\psi(\overline{d})$ holds, there is some $\overline{e}$ with $\varphi(\overline{e},\overline{d})\wedge\psi(\overline{e})$. One deduces $\varphi(\overline{c}-\overline{e},0)$. But $\varphi(\overline{v},\overline{0})$ implies $\psi(\overline{v})$: hence $\psi(\overline{c}-\overline{e})$ holds. Together with $\psi(\overline{e})$, this yields $\psi(\overline{c})$, as required.

Therefore f and g are indeed isomorphisms. □

Example 1 Let R be the ring $K[x_i\ (i\in\omega): x_ix_j=0\ (i,j\in\omega)]$, where K is a field. Since R is local, it is indecomposable and, since it has dcc on pp-definable subgroups ([Z-HZ78; Thm 5]), it is t.t., so pure-injective. Therefore every two non-zero elements are related.



Take, for illustration, $x_0\ (\in J(R))$ and $1\in R$ to be the elements: a linking formula is $\varphi(v,w)\equiv(v=wx_0)$ (clearly $R\vDash\varphi(x_0,1)\wedge\neg\varphi(x_0,0)$).

<u>Corollary 9.10</u> *Suppose that* $\overline{a}$ *and* $\overline{b}$ *are linked. If there is* $\varphi\in\mathrm{pp}(\overline{b})$ *such that the interval* $[\varphi,0]$ *in the lattice of pp-definable subgroups has the dcc, then* $\mathrm{tp}(\overline{a})$ *has a minimal pair.*

Proof If $\theta(\overline{a},\overline{b})$ is a pp formula linking $\overline{a}$ and $\overline{b}$, then it may be assumed that $\theta(\overline{v},\overline{w})$ implies $\varphi(\overline{w})$, so the result follows quickly from **9.9**. □

We may say that types p and q (over 0) are said to be **linked** if there are $\overline{a}$ and $\overline{b}$ realising p and q respectively, such that $\overline{a}$ and $\overline{b}$ are linked - that is, such that there is a pp formula φ with $\varphi(\overline{a},\overline{b})\wedge\neg\varphi(\overline{a},\overline{0})$. This is essentially, and if $T=T^{\aleph_0}$, is literally, non-orthogonality of p and q (by **6.20**).

<u>Proposition 9.11</u> (essentially [Zg84; 8.10]) *Suppose that* p *and* q *are types over* 0 *such that* p *is irreducible and such that* p *and* q *are linked - hence, by* **4.31**, $N(p)$ *is a factor of* $N(q)$. *If* p *contains a minimal pair, then so does* q.

Proof Suppose that φ'/ψ' is a minimal pair in p. Let $\overline{a},\overline{b}$ be realisations of p and q respectively, and let θ be pp such that $\theta(\overline{a},\overline{b})\wedge\neg\theta(\overline{a},\overline{0})$ holds. Neither $\exists\overline{w}\,\theta(\overline{v},\overline{w})\wedge\psi'(\overline{v})$ nor $\theta(\overline{v},\overline{0})$ is in p^+ so, by **4.29**, there is $\varphi\leqslant\varphi'$ in p^+ such that $\chi(\overline{v})$, being $\varphi(\overline{v})\wedge\exists\overline{w}\,\theta(\overline{v},\overline{w})\wedge\psi'(\overline{v})+\varphi(\overline{v})\wedge\theta(\overline{v},\overline{0})$, is in p^-.

Let $\theta'(\overline{v},\overline{w})$ be $\varphi(\overline{v})\wedge\theta(\overline{v},\overline{w})$; so we have $\theta'(\overline{a},\overline{b})\wedge\neg\theta'(\overline{a},\overline{0})$.

Consider **9.9** applied to $[\exists\overline{w}\,\theta'(\overline{v},\overline{w}),\theta'(\overline{v},\overline{0})]$ and $[\exists\overline{v}\,\theta'(\overline{v},\overline{w}),\theta'(\overline{0},\overline{w})]$; by that result, it will be enough to produce a minimal pair in the first interval with $\overline{a}$ satisfying the upper, but not the lower, member of the pair. Since $\exists\overline{w}\,\theta'(\overline{v},\overline{w})\leqslant\varphi'(\overline{v})$ one has that $\exists\overline{w}\,\theta'(\overline{v},\overline{w})/\exists\overline{w}\,\theta'(\overline{v},\overline{w})\wedge\psi'(\overline{v})$ is a minimal pair (by **9.1**). The lower member need not be in the required interval, but the sum $\exists\overline{w}\,\theta'(\overline{v},\overline{w})\wedge\psi'(\overline{v})+\theta'(\overline{v},\overline{0})$ is precisely $\chi(\overline{v})$ above and so is in p^-, so $\exists\overline{w}\,\theta'(\overline{v},\overline{w})/\chi(\overline{v})$ is a minimal pair, as required. □

This result is, in essense, generalised by **9.16** below. Most of the results of this section which deal with minimal pairs are generalised (to higher dimensions) in Chapter 10.

<u>Corollary 9.12</u> [Zg84; 8.10] *Suppose that* p *and* q *are related irreducible types. If one of them contains a minimal pair, then so does the other.* □

Corollary 9.13 *Suppose that N is an indecomposable pure-injective. If there is some N-minimal pair, then every interval $\varphi(N) > \psi(N)$ of N contains an N-minimal pair.*

Proof The hypothesis implies that there is some type p, over 0, which contains a minimal pair and is realised in N (cf. proof of 10.1).

Let $\varphi(N) > \psi(N)$. Choose $\bar{a}$ in $\varphi(N) \setminus \psi(N)$ and let q be its type. Then, by 9.12, q contains a minimal pair - φ'/ψ' say. By 9.1, $\varphi\wedge\varphi'/\varphi\wedge\psi'$ is a minimal pair in q. Then, by modularity, $\varphi\wedge\varphi'+\psi / \varphi\wedge\psi'+\psi$ is a minimal pair which, clearly, is between φ and ψ. □

Further corollaries of 9.9 will be seen in Chapter 10.

Next, consider the problem of describing the indecomposable direct summands of $N(p)$. Proposition 4.31 gives some information on this, but here we look for a "syntactic" criterion.

Let p be a type over 0. Say that the pp formula ψ is **large in** p if $\psi \in p^-$ and if, for all $\psi_1, \psi_2 \in p^-$ with $\psi \leqslant \psi_1, \psi_2$, there is some φ in p^+ with $\psi < \varphi$ and $(\psi_1\wedge\varphi) + (\psi_2\wedge\varphi) \in p^-$: that is (compare 4.29), "p looks irreducible above ψ".

Lemma 9.14 [Zg84; 7.5] *Let p be a type over 0.*

(*a*) *If $\psi \leqslant \psi' \in p^-$ and if ψ is large in p, then so is ψ'.*

(*b*) *p is irreducible iff every formula in p^- is large in p.*

(*c*) *If φ/ψ is a minimal pair in p, then ψ is large in p.*

Proof (a) Given $\psi_1, \psi_2 \in p^-$ with $\psi_i \geqslant \psi' (\geqslant \psi)$, there is, since ψ is large in p, some φ in p^+ with $\varphi > \psi$ and $(\psi_1\wedge\varphi) + (\psi_2\wedge\varphi) \in p^-$. We need such a formula φ of p^+ with $\varphi > \psi'$. So try $\varphi' = \varphi + \psi$.

It is an immediate application of modularity that $\psi_1\wedge\varphi' + \psi_2\wedge\varphi' = \psi_1\wedge\varphi + \psi_2\wedge\varphi + \psi' = \psi'' + \psi'$ (say). This formula is in p^- (so we finish) since, otherwise, $\varphi\wedge(\psi' + \psi'') = \varphi\wedge\psi' + \psi''$ (for $\varphi \geqslant \psi''$) would be a formula in p^+ which was contained in $\varphi\wedge\psi_1 + \psi'' = \psi''$ - contradiction.

(b) and (c) follow from the definitions, 4.29 and (a). □

The type $q \in S^T(0)$ is **associated to** $p \in S^T(0)$ ***via*** ψ if ψ is in p^- and if, for every pp formula $\varphi \geqslant \psi$, one has $\varphi \in p^+$ iff $\varphi \in q^+$ ("above ψ, p looks like q").

Example 2 Take T to be theory of the abelian group $\mathbb{Z}_6^{\aleph_0}$ and, for p, q, take the types of an element of order 6, respectively of order 3. Then p is associated to q *via* the formula $v2 = 0$.

Lemma 9.15 [Zg84; 7.5] *Let p and q be types over 0.*

(*a*) *If q is irreducible and associated to p via ψ, then ψ is large in p.*

(*b*) *If φ/ψ is a minimal pair common to p and q, then p and q are associated via ψ.*

Proof (a) This is immediate from the definition and 9.14(b).

(b) This also is immediate from the definition since, with notation as there, if $\theta \geqslant \psi$ then $\theta \in p^+$ iff $\theta \geqslant \varphi$ iff $\theta \in q^+$. □

Theorem 9.16 [Zg84; 7.6] *Let p be any type over 0.*

(*a*) *If ψ is large in p, then there is an irreducible type q associated to p via ψ.*
The hull of q is a direct summand of the hull of p: moreover, the isomorphism type of $N(q)$ is determined by ψ.

(*b*) *If N is any direct summand of $N(p)$, then there is q with $N \simeq N(q)$, and there is ψ such that q is associated to p via ψ; ψ will be large in p iff N is indecomposable.*

Proof (a) Suppose that ψ is large in p. Let $q^+ \geq p^+$ be a pp-type maximal with respect to having the property that for every $\varphi \in q^+$ one has $\psi + \varphi \in p$.

Let $\bar{a}$ realise p, and note that for each $\varphi \in q^+$, $(\varphi + \psi)(\bar{a})$ holds. Hence $q^+(\bar{w}) \wedge \psi(\bar{a}-\bar{w})$ is finitely satisfied in $N(\bar{a})$: therefore it is satisfied, by $\bar{b}$ say, in $N(\bar{a})$. It is claimed that $\mathrm{pp}(\bar{b}) = q^+$.

Let $\psi_1 \notin q^+$ be pp. By maximality of q^+, there is $\varphi \in q^+$ with $\varphi \wedge \psi_1 + \psi \in p^-$. On the other hand, if one had $\psi_1(\bar{b})$ then one would have $(\varphi \wedge \psi_1)(\bar{b})$ and $\psi(\bar{a}-\bar{b})$; so $\varphi \wedge \psi_1 + \psi$ would be in p^+ - contradiction. Thus the pp-type of $\bar{b}$ is q^+. Set q to be $(q^+)^{\sim}$: it is claimed that q is irreducible.

So let ψ_1, ψ_2 be in q^-. By maximality of q^+ there is (clearly) some φ in it with $\varphi \wedge \psi_i + \psi \in p^-$ for $i = 1, 2$. Then, since ψ is large in p, there is φ' in $p^+ (\leq q^+)$ with $\varphi' > \psi$ and such that $\varphi' \wedge (\varphi \wedge \psi_1 + \psi) + \varphi' \wedge (\varphi \wedge \psi_2 + \psi)$ is in p^-. That is (by modularity) $\varphi' \wedge \varphi \wedge \psi_1 + \varphi' \wedge \varphi \wedge \psi_2 + \psi$ is in p^-. Hence, by definition of q, $\varphi' \wedge \varphi \wedge \psi_1 + \varphi' \wedge \varphi \wedge \psi_2$ is in q^-. So, noting that $\varphi' \wedge \varphi$ is in q^+, irreducibility of q follows by **4.29**.

If $\varphi \in q^+$ and $\varphi \geqslant \psi$ then, by definition of q, $\varphi \in p^+$. Since also $q^+ \geq p^+$, q is associated to p *via* φ. Thus the first statement is proved.

Now we show that ψ determines the isomorphism type of $N(q)$. Let q_1 be any irreducible type associated to p *via* ψ. Let $\bar{b}_1$ in $N(\bar{a})$ realise $q_1^+(\bar{w}) \wedge \psi(\bar{a}-\bar{w})$ as above. It will be shown that the type of $\bar{b}_1$ is q_1.

So let ψ_1 be in q_1^-. Since q_1 is irreducible, there is $\varphi \in q_1^+$ with $\psi \wedge \varphi + \psi_1 \wedge \varphi$ in q_1^-. If $\psi + \psi_1 \wedge \varphi$ were in q_1^+, then so would be $\varphi \wedge (\psi + \psi_1 \wedge \varphi) = \varphi \wedge \psi + \varphi \wedge \psi_1$ - contradiction. So $\psi + \psi_1 \wedge \varphi$ is in q_1^-. By definition of "associated *via* ψ", it follows that $\psi + \psi_1 \wedge \varphi$ is in p^-. Therefore $\psi(\bar{a}-\bar{b})$ and $\varphi(\bar{b})$ yield $\neg\psi_1(\bar{b})$. Thus the type of $\bar{b}_1$ is q_1, and so q_1 is realised in $N(p) = N(\bar{a})$.

Finally note that if q and q_1 both are irreducible and associated to p *via* ψ then they are associated to each other *via* ψ (by the definition) and so, by what has just been shown, applied to q_1 and q in place of q_1 and p, their hulls are isomorphic, as required.

(b) Let N be a direct summand of $N(\bar{a})$, where $\bar{a}$ realises p. Set $N(\bar{a}) = N \oplus N'$, and write $\bar{a} = (\bar{a}_0, \bar{a}')$ accordingly; so N is the hull of $\bar{a}_0$ (cf. Exercise 4.1/10). Let q be the type of $\bar{a}_0$.

By **4.28**, the pp-type of $\bar{a}'$ strictly contains p^+. So choose any pp formula ψ with $\psi(\bar{a}') \wedge \neg\psi(\bar{a})$; one has $\neg\psi(\bar{a}_0)$. It is claimed that q is associated to p *via* ψ.

Certainly, if φ is in p^+ then φ also is in q^+. Conversely, if $\varphi \in q^+$ and $\varphi \geqslant \psi$ then, since $\psi(\bar{a}')$ holds and so $\varphi(\bar{a}')$ holds, from $\varphi(\bar{a}_0)$ one deduces $\varphi(\bar{a})$ - that is, φ is in p^+, as required for the first statement.

If N is indecomposable, then by 9.14(b), ψ will be large in q, hence large in p by 9.15(a). Conversely, if ψ is large in p then, by part (a), q must be irreducible. □

9.3 Notions of isolation

A number of notions of isolation have appeared in these notes. In this section, I connect one of them - neg-isolation - with the weight of a type, and then compare all of them.

Recall (§8.1) that the type p is neg-isolated (by $\psi_1, \ldots, \psi_n$) if there are $\psi_1, \ldots, \psi_n$ in p^- such that p is equivalent to $p^+ \wedge \bigwedge_1^n \neg\psi_i$ (my apologies for this ugly word, but I haven't come up with a better one). This notion is, of course, relative to the theory being considered; in particular, it does not depend only on p^+. By **9.20** below, such a type is isolated iff its pp-part is finitely generated. This notion is implicit (at least for irreducible types) in **[Gar80a]** (definition 3 and following) and is more or less explicitly used in **[Zg84]** (it was explicitly known to Ziegler (personal communication)). It has already been met in §8.1. Notice that the definition makes good sense only for types in finitely many free variables.

Suppose that p is a type over 0. In §6.4, we defined the weight, wt(p), of p to be the weight of (i.e., number of indecomposable factors (counting multiplicity) of) the hull, $N(p_*)$, of the free part, p_*, of p, provided this is discrete. Also the algebraic weight, algwt(p), of the pp-type p, was defined to be the weight of $N(p)$. Because I am mainly interested in the algebraic meaning of neg-isolation I will usually assume, when dealing with this notion, that the over-theory is closed under products, so that the algebraic and model-theoretic versions of weight coincide. Recall that if N is not discrete then we set wt$(p)=\infty$.

__Lemma 9.17__ [Pr82a] *Suppose that* p *is a type over* 0, *with* p *equivalent to* $p^+ \wedge \bigwedge_1^n \neg\psi_i$. *Then* algwt$(p) \leqslant n$.

Proof Observe first that if $p^+ \wedge \bigwedge_1^n \neg\psi_i$ defines a complete T-type, then it also defines a complete $T^{\aleph_0}$-type, since T and $T^{\aleph_0}$ have the "same" lattice of pp-definable subgroups. So it may be supposed that $T = T^{\aleph_0}$

For each $i = 1,\dots,n$, let q_i be a type maximal with respect to containing p^+ and not containing ψ_i. By **4.33**, q_i is irreducible.

Let $\bar{a}_i$ in $N_i = N(\bar{a}_i)$ realise q_i. Then the element $\bar{a} = (\bar{a}_1,\dots,\bar{a}_n)$ in $N_1 \oplus \dots \oplus N_n$ satisfies p^+ and also $\neg\psi_i(\bar{a})$ holds for each i (since $\neg\psi_i(\bar{a}_i)$ holds). Hence $\bar{a}$ realises p. Then, since $N(p)$ is therefore a factor of $N_1 \oplus \dots \oplus N_n$ and since the N_i are indecomposable, it follows that wt$(p) \leqslant n$, as required. ◻

__Corollary 9.18__ *Every neg-isolated type has finite weight. In particular, every isolated type has finite weight.* ◻

__Corollary 9.19__ [Pr82a; 1.1] *Suppose that* p *is a neg-isolated type over* 0: *say* p *is equivalent to* $p^+ \wedge \bigwedge_1^n \neg\psi_i$, *where the* ψ_i *have been chosen to minimise* n. *Then*:

(*a*) p *is irreducible iff* $n = 1$;

(*b*) *if* $n = 2$, *then* algwt$(p) = 2$.

Proof (a) ⇒ Suppose that $n \geqslant 2$. Then, by **4.28**, there is some φ in p^+ with $\varphi\wedge\psi_1 + \varphi\wedge\psi_2 \in p^-$. One has that $p^+ \wedge \bigwedge_1^n \neg\psi_i$ is equivalent to $p^+ \wedge \neg(\varphi\wedge\psi_1 + \varphi\wedge\psi_2) \wedge \bigwedge_3^n \neg\psi_i$. For the first is p, so certainly proves the second: on the other hand, since $\neg(\varphi\wedge\psi_1 + \varphi\wedge\psi_2)$ implies $\neg(\varphi\wedge\psi_1) \wedge \neg(\varphi\wedge\psi_2)$, the other direction is clear. This contradicts minimality of n, as required.

The other direction is immediate from **9.17**.

(b) This is immediate by part (a) and **9.17**. ◻

The following example shows that, in general, the inequality, wt$(p) \leqslant n$, in **9.17** cannot be improved beyond **9.19**, even when n is minimised.

Example 1 Let R be the ring $K[x,y : x^2 = y^2 = xy = yx = 0]$, where K is a finite field with p^m elements. The pp-definable subgroups of R are just the ideals, and they have been described already in Ex 2.1/6(vi). In particular, below the radical $J = xR + yR$ there are $p^m + 1$ non-zero (1-dimensional) ideals $I_k = (x + yk)R$, where $k \in \mathrm{P}(K)$ (and $I_\infty = yR$).

Let T be the theory of $R^{\aleph_0}$: then T is totally transcendental. There is, moreover, a consistent type "at J" which is isolated by the formula $\exists v_1, v_2\, (v = v_1 x + v_2 y) \wedge \bigwedge \{\neg\exists w\, (v = w(x + yk)) : k \in \mathrm{P}(K)\}$. Clearly, the conjunction defining p^- cannot be shortened, since I_k is not contained in the union $\bigcup_{l \neq k} I_l$. Thus the minimum value for "n" as in **9.17** is $p^m + 1$.

It follows by **9.19**(a) that p is not irreducible. In fact wt$(p) = 2$.

Consider $R\oplus R$ and its element $a=(x,y)$. One has $a=(1,0)x+(0,1)y$, so $p^+(a)$ holds. On the other hand, it is clear that, for every $k\in P(K)$ and for every $w\in R\oplus R$, $a\neq w(x+yk)$. Therefore a realises p. But R is indecomposable (being a local ring) and pure-injective (it is even finite). So $N(p)=N(a)=R\oplus R$ and so, indeed, the weight of p is 2.

It is worthwhile noting that if K in the above example is taken to be infinite, then the only point which changes if that p is no longer neg-isolated, since there are infinitely many pp formulas "just below" p. In particular, one still has $\mathrm{wt}(p)=2$.

Example 2 A finitely generated irreducible type which is not neg-isolated is the type of any non-zero element of $\mathbb{Q}$ in the theory of the abelian group $(\mathbb{Z}_{2^\infty})^{\aleph_0}$. This type is not neg-isolated since, although p^+ is equivalent to the formula "$v=v$", p^- is equivalent to the set of formulas $\{v2^n\neq 0 : n\in\omega\}$ but to no finite subset.

We have now come across four different senses in which a type p over the empty set may be isolated. They are as follows:

(i) the type p is isolated (in the space of types);
(ii) the type p contains a minimal pair;
(iii) the hull of p is isolated in the space, $\mathcal{I}(T)$, of indecomposables;
(iv) the type p is neg-isolated.

The remainder of this section is devoted to examining the inter-relationships between these. It is the irreducible types in which we are interested, but I bring in irreducibility only where needed.

<u>Proposition 9.20</u> [Pr81b; Lemma1], [PP87; 6.5] *The type p over 0 is isolated iff it is finitely generated and neg-isolated.*

Proof It is immediate from the definitions that if p is finitely generated and neg-isolated then it is isolated. For the converse, suppose that $\varphi\wedge\bigwedge_1^n\neg\psi_i$ isolates p, where φ and the ψ_i are pp. If p were not finitely generated, there would be (by 3.1) an infinite descending chain $\varphi=\varphi_0>\varphi_1>\dots$ of pp formulas in p. Then $\varphi\wedge\bigwedge_1^n\neg\psi_i$ would imply each φ_k: that is, $\varphi\wedge\bigwedge_1^n\neg\psi_i\leqslant\varphi_k$ for each k. Hence, one would have $\varphi=\bigcup_1^n\psi_i\cup\varphi_k$ for each k (and note that φ_k cannot be omitted!). By the strong statement of Neumann's Lemma (2.12) it would follow that the index of φ_k in φ were no more that $n!$ - a contradiction for k sufficiently large.

Then, if φ' is a pp formula equivalent to p^+, one has $\varphi'\wedge\bigwedge_1^n\neg\psi_i$ equivalent to p: so p is also neg-isolated. □

<u>Corollary 9.21</u> *If the type p over 0 is isolated then it has a minimal pair.* □

The converse is false. Moreover, being isolated is not relatedness-invariant, even for irreducible types: that is, if p is isolated and $N(q)\simeq N(p)$ it need not be that q is isolated, although under many conditions, such as total transcendality, it will be. The two examples below illustrate these points. One does at least have the following.

<u>Corollary 9.22</u> *Suppose that p and q have linked realisations and that q is isolated. Then p contains a minimal pair.*

Proof By the proof of 9.17, the hull of q is a direct sum of indecomposables, each being the hull of a type with a minimal pair. By **4.31**, $N(p)$ and $N(q)$ have a non-zero direct summand in common so p is linked to an irreducible type with a minimal pair. Therefore, by 9.12, p contains a minimal pair, as required. □

Example 2a Let p be the (irreducible) type of the element $(0,1)$ in the theory of $\mathbb{Z}_{(2)}\oplus\mathbb{Z}_{(3)}$. Then p contains the minimal pair $(2|v)/(6|v)$ - clearly minimal since

$\mathrm{Inv}(\mathbb{Z}_6, 2|v, 6|v)=3$. Yet (see Ex 2.1/7(ii)) p^+ is not finitely generated, since it is equivalent to the infinite set of formulas $\{2^n|v : n\in\omega\}$.

Example 3 Let the ring and module be as described in Ex 7.2/6. The prime-pure-injective model is indecomposable and is the hull of both an element with isolated type and one with non-isolated type. So isolation is not relatedness-invariant.

Proposition 9.23 [Zg84; 4.9, 7.10] *If the irreducible type p over 0 has a minimal pair then the hull, $N(p)$, of p is an isolated point in $\mathcal{I}(T)$.* □

This is immediate from **9.3** and the definition of the topology on $\mathcal{I}(T)$ (the minimal pair defines an isolating neighbourhood). The result does use irreducibility of p. For consider the type of an element a in $\prod_p \mathbb{Z}_{(p)}$ which has infinitely many components equal to the "1" of the relevant factor and the other components all 0. This type contains a minimal pair but there is no sense in which the points comprising the hull of this element are collectively isolated.

It is unknown whether or not the converse is invariably true. It is shown in §10.4 that if R is countable, or if T has zero continuous part, then the converse does hold - that is, if N is isolated in $\mathcal{I}(T)$ then some, and hence (**9.12**) every, non-zero type realised in $N(p)$ has a minimal pair.

Proposition 9.24 (Ziegler; private communication) *Suppose that the type p is neg-isolated, and let q be such that $N(q)$ is isomorphic to $N(p)$. Then q is neg-isolated.*

Proof Suppose, for a contradiction, that q were not neg-isolated. For each $\psi\in q^-$, let b_ψ, in the pure-injective N_ψ, have pp-type generated by $q^+\cup\{\psi\}$ (it may be assumed that $T=T^{\aleph_0}$). Consider the element $b=(b_\psi)_\psi$ in $N=\prod\{N_\psi : \psi\in q^-\}$. Certainly b satisfies q^+: I claim that it realises q. If it also realised θ in q^-, then each of its components would do so hence, by choice of the b_ψ, every ψ in q^- would, in conjunction with q^+, imply θ - so θ would neg-isolate q - contrary to our assumption.

Since N is pure-injective, there is a realisation $a=(a_\psi)_\psi$ of p in N. The pp-type of a is the intersection of the pp-types of its components, a_ψ, so, since p is neg-isolated, there is θ with $\mathrm{pp}(a_\theta)=\mathrm{pp}(a)$. Therefore the projection, π, to the θ-component strictly preserves the pp-type of a, hence must strictly preserve the pp-type of b - contradiction, as required. □

Corollary 9.25 *If the indecomposable pure-injective N is isolated in $\mathcal{I}(T)$ then every non-zero type realised in N is neg-isolated.*

Proof By **9.24** it will be enough to show that N realises *some* neg-isolated type. Suppose that (φ/ψ) is an isolating neighbourhood of N. Let p be a type which contains φ/ψ and is maximal pp with respect to omitting ψ. Then p is irreducible and neg-isolated by ψ (**4.33**). Since (φ/ψ) isolates N, the hull of p is N. Thus the result follows. □

As a counterexample to the converse of **9.25**, one has the following.

Example 4 Let T be the theory of the abelian group $\mathbb{Z}_{(2)}$. Then $\mathcal{I}(T)$ has two points: $\overline{\mathbb{Z}_{(2)}}$ and $\mathbb{Q}$: $\overline{\mathbb{Z}_{(2)}}$ is isolated; $\mathbb{Q}$ is not. Nevertheless, the type of any non-zero element of $\mathbb{Q}$ is neg-isolated - by the formula "$v=0$".

On the other hand, the theory of the abelian group $\mathbb{Z}_{2^\infty}\oplus\mathbb{Z}_{(2)}$ is an example where the neg-isolated types do happen to coincide with the types realised in isolated points of $\mathcal{I}(T)$.

The type of the element $(0,1)$ in the theory of $\mathbb{Z}_{(2)}\oplus\mathbb{Z}_{(3)}$ gives an example of a neg-isolated, but not isolated, type, the hull of which is isolated in $\mathcal{I}(T)$.

By **9.23** and **9.25**, if an indecomposable pure-injective has a T-minimal pair then every type realised in it is neg-isolated. Recall that, for any t.t. theory, the types realised in the prime model are precisely the isolated types. So one may ask if there is a generalisation involving the prime pure-injective model (the fact just deduced perhaps suggests this). It will be shown in Chapter 10 (**10.24**) that, if the lattice of pp-definable subgroups has no densely ordered subset, then there is a prime pure-injective model which does indeed satisfy the condition that every type realised has a minimal pair (and it realises all the irreducible pp-types with a minimal pair).

In summary, we have the following chain of implications; also, we see that, for a finitely generated type, the various notions of isolation all coincide (most of that already is in §8.2).

<u>Corollary 9.26</u> *Suppose that* p *is an irreducible type over* 0. *Then* $(i)\Rightarrow(ii)\Rightarrow(iii)\Rightarrow(iv)$.
If p *is finitely generated then the conditions* (*i*)-(*iv*) *are equivalent.*
(*i*) p *is isolated.*
(*ii*) p *contains a minimal pair.*
(*iii*) *The hull,* $N(p)$, *of* p *is isolated in the space,* $\mathcal{I}(T)$, *of indecomposables.*
(*iv*) p *is neg-isolated.*

Proof The first statement summarises **9.21**, **9.23** and **9.25**. The second statement then follows by **9.20**. □

Suppose that T is totally transcendental of finite Morley rank. Then (exercise) every irreducible type is isolated. So, by the above, it follows that every point of $\mathcal{I}(T)$ is isolated and hence $\mathcal{I}(T)$ is finite (cf. §11.4).

<u>Corollary 9.27</u> *Suppose that the theory* T *is totally transcendental and let* p *be any irreducible type over* 0. *Then the following conditions are equivalent:*
(*i*) p *is isolated;*
(*ii*) p *contains a minimal pair;*
(*iii*) *The hull,* $N(p)$, *of* p *is isolated in the space,* $\mathcal{I}(T)$, *of indecomposables;*
(*iv*) p *is neg-isolated.* □

9.4 Neg-isolated types and elementary cogenerators

In this section the algebraic significance of neg-isolation is examined. We will, in particular, see another implication of the distinction, made in **8.3**, between the two ways in which a type may be irreducible. This distinction is rather important in the context of the representation theory of finite-dimensional algebras, since it relates to finite presentation of certain simple functors (§12.2) and hence to the existence of almost split sequences ([Aus74a; 2.7]).

The material of this section comes mostly from [Pr80e] and [Pr82a].

Let N be a pure-injective module. Say that N is an **elementary cogenerator** if every pure-injective summand of a model of the theory of $N^{\aleph_0}$ is a direct summand of some power of N (of course any such member of $\mathcal{P}(N^{\aleph_0})$ is a direct summand of some ultrapower of N). That is, N is an elementary cogenerator iff it is a cogenerator in the category whose objects are the pure submodules of models of $\mathrm{Th}(N^{\aleph_0})$ and whose morphisms are the pure embeddings. This notion generalises that of an injective cogenerator for a torsionfree class (see Chpt. 15). I take this point farther in §15.3.

Example 1

(a) Let N be the abelian group $\mathbb{Z}_{2^\infty}$. So $\mathcal{I}(\mathrm{Th}(N^{\aleph_0})) = \{\mathbb{Z}_{2^\infty}, \mathbb{Q}\}$. In order to show that N is an elementary cogenerator, it must be established that $\mathbb{Q}$ (purely) embeds in some power of $\mathbb{Z}_{2^\infty}$. But, since $\mathbb{Z}_{2^\infty}$ contains elements of arbitrary high order, $(\mathbb{Z}_{2^\infty})^{\aleph_0}$ contains elements of infinite order: the hull of any such element is a copy of $\mathbb{Q}$.

(b) Take N to be the abelian group $\overline{\mathbb{Z}_{(2)}}$; so $\mathcal{I}(\mathrm{Th}(N^{\aleph_0})) = \{\overline{\mathbb{Z}_{(2)}}, \mathbb{Q}\}$. I claim that N is not an elementary cogenerator. For suppose that the type of a non-zero element of $\mathbb{Q}$ were realised by an element $a=(a_i)_i$ in some power $\overline{\mathbb{Z}_{(2)}}^{\aleph_0}$ of $\overline{\mathbb{Z}_{(2)}}$. Then for each $i \in I$ and each $n \in \omega$, one would have $2^n | a_i$, since $2^n | a$. Therefore each a_i would be zero and so a would be zero - contradiction.

__Proposition 9.28__ *The pure-injective module N is an elementary cogenerator iff every indecomposable direct summand of a model of the theory of N embeds as a direct summand of some power of N.*

Proof This is immediate from **4.38** and **4.39**. □

__Proposition 9.29__ *Let T be any complete theory of modules. Let N, $\{N_\lambda\}_\lambda$ be in $\mathcal{I}(T)$, and suppose that N is the hull of an irreducible type which is neg-isolated (with respect to T). Suppose that N is a direct summand of the product $\prod_\lambda N_\lambda$. Then N is a direct summand of one of the N_λ.*

Proof Let ψ be pp such that p is equivalent to $p^+ \wedge \neg\psi$ (using **9.19**). Suppose that $\bar{a} = (\bar{a}_\lambda)_\lambda \in \prod N_\lambda$ realises p: then each $\bar{a}_\lambda$ satisfies p^+. Moreover, it must be that, for some μ, $\neg\psi(\bar{a}_\mu)$ holds. Hence $\bar{a}_\mu$ realises p, and the conclusion follows. □

__Corollary 9.30__ [Pr82a; 1.3] *Let N be any module and let p be a type over* 0 *which is irreducible and neg-isolated in the theory of N. Let I be any set. Then p is realised in N^I iff p is realised in N.*

Proof The direction "⇒" is as in the proof above. The other direction is clear since "p" is irreducible and neg-isolated equally in the theory of N and in that of $N^{\aleph_0}$ and so we may regard p in either context. □

__Proposition 9.31__ [Pr82a; 1.7] *Suppose that N is pure-injective and let T be the theory of N. If every irreducible neg-isolated type in $S^T(0)$ is realised in (some power of) N, then N is an elementary cogenerator.*

Proof By **9.29** the parenthesised hypothesis is no weaker.

To see that N is an elementary cogenerator, let $N_0 \in \mathcal{P}(N^{\aleph_0})$. Let X be the set of morphisms from N_0 to N. I claim that the morphism from N_0 to N^X given by $a \mapsto (fa)_{f\in X}$ is a pure embedding.

Let $\bar{a}$ be in N_0 and let p be its pp-type: then certainly $\mathrm{pp}((f\bar{a})_f) \geq p$. So let $\psi \in p^-$: then $p^+ \wedge \neg\psi$ extends (by **4.33**) to a neg-isolated irreducible type q which, by **4.39**, may be taken to be a type for T (rather than $T^{\aleph_0}$). Our hypothesis implies that q is realised in N, say by $\bar{c}$. By **2.8** there is a morphism $f \in X$ taking $\bar{a}$ to $\bar{c}$. Since $\neg\psi(\bar{c})$ holds it follows that $\neg\psi((fa)_f)$ holds also. Thus the image of $\bar{a}$ in N^X has pp-type exactly p^+. This is so for every tuple in N_0 so, by definition, N_0 is purely embedded in N^X, as required. □

__Theorem 9.32__ [Pr82a; 1.6] *Let N be pure-injective. Then N is an elementary cogenerator iff N realises every neg-isolated irreducible type in $D^N(0)$.*

Proof This is immediate from **9.30** and **9.31**. □

__Corollary 9.33__ [Pr84; 3.4] *Suppose that the module N is totally transcendental. Then N is an elementary cogenerator.*

Proof Since N is totally transcendental, isolation and neg-isolation coincide; also, every model of Th(N) realises every isolated type. Therefore N realises every neg-isolated type and so the result follows by **9.32**. ▫

Corollary 9.34 *Let N be totally transcendental and let M be elementarily equivalent to N. Then M is a direct summand of some power of N.* ▫

Exercise 1 [Gar80a; Lemma17] Let M be a module such that every power, M^{λ}, of M is a direct sum, $M^{(\kappa)}$, of copies of M. Show that M is t.t. and weakly saturated. [Hint: see the proof of **2.11**.] Deduce that if M is indecomposable then M has finite pp-rank iff every power of M is isomorphic to a direct sum of copies of M [Gar80a; Thm 13].

Exercise 2 Garavaglia [Gar80a; Thm10] actually proves the following. Suppose that M is a direct sum of finitely many indecomposable modules. Then the following are equivalent: (i) every power of M is a direct sum of copies of M; (ii) M is totally transcendental, $M^{(\aleph_0)}$ is weakly saturated and, for every M-minimal pair φ/ψ, one has $|\varphi(M)/\psi(M)|^{\aleph_0} = |M|^{\aleph_0}$. Prove this (Garavaglia's original proof is rather long, but the structure theory of §4.6 helps to cut down the work).

We saw above that $\overline{\mathbb{Z}_{(p)}}$ is not an elementary cogenerator. We may extract the key point from that example. Let N be pure-injective. Suppose that there are pp formulas $\varphi_0, \varphi_1, \ldots$ and ψ, such that $\bigcap\{\varphi_i(N) : i \in \omega\} = \psi(N)$ but also such that, for every $n \in \omega$, one has $\bigcap\{\varphi_i(N) : i \leqslant n\} > \psi(N)$. Then N is not an elementary cogenerator. For any complete type which is maximal with respect to containing $\{\varphi_i : i \in \omega\}$ and not containing ψ (this is consistent by hypothesis) is, by **4.33**, neg-isolated, but is, by hypothesis, not realised in N. So, by **9.32**, N is not an elementary cogenerator. The next result is immediate from **9.32**.

Corollary 9.35 [Pr80e] *If N is weakly saturated and pure-injective, then N is an elementary cogenerator.* ▫

Corollary 9.36 *Every complete theory T has a model which is an elementary cogenerator.* ▫

Corollary 9.37 [Pr82a; 1.12] *Let $\mathcal{K}$ be any universal Horn class of modules. Then there is some $N \in \mathcal{K}$ such that $\mathcal{K}$ may be described in each of the following ways.*

(a) $\{M : M$ embeds in some power of $N\}$

(b) $\{M : M$ purely embeds in some power of $N\}$.

Proof Set $T' = \mathrm{Th}(\oplus\{M_T : T$ is a complete extension of $\mathrm{Th}(\mathcal{K})$ and M_T is an arbitrarily chosen model of $T\})$. Take N to be any elementary cogenerator for T' (such exists by 9.36). ▫

Existence of elementary cogenerators is an important ingredient in Facchini's treatment [Fac85] of decomposition of pure-injective modules (his Theorem 1 is just existence of an elementary cogenerator for T^*).

CHAPTER 10 DIMENSION AND DECOMPOSITION

"Finiteness" conditions on the lattice of right ideals of a ring correspond to structure theorems for the injective modules (see §1.1). In this chapter, we look at finiteness conditions on the lattice of pp-types which correspond to structure theorems for pure-injective modules. We have already seen one example in Chapter 3: the descending chain condition on pp-definable subgroups is equivalent to every pure-injective module being a direct sum of indecomposable submodules. In §4.6 we were able to develop a structure theory for such modules.

One of the conditions that we consider turns out to be equivalent, at least over countable rings, to there being no continuous pure-injectives (an example of a "good structure" theorem). Also considered is a stronger condition, under which there is a good structure theory for the pure-injective models, analogous to that in §4.6.

If one considers the proof of **3.14**, then one sees that the point is that, under the assumed finiteness condition, every pure-injective model has an indecomposable direct summand. In the first section, we see that a very much weaker hypothesis suffices for that conclusion.

The finiteness conditions of §1, though sufficient to imply zero continuous part, are far from necessary. In fact, each is just the basis for an inductively defined sequence of finiteness conditions. For example, one may, in some sense, factor out that part of a theory which has the dcc. It is possible that, in what remains, there is some non-trivial part which has the dcc - in which case factor it out; and so on. For many theories this process will not terminate until the trivial theory is reached, in which case one deduces that the theory has continuous part zero.

Such inductive extension of finiteness conditions is the topic of the second section. Although our interest is in two particular examples of the process, the first part of the discussion is carried out in a more general framework: this saves effort and emphases an underlying idea. We end up with two dimensions, m-dimension and breadth, which generalise the conditions considered in the first section. We say that a theory "has a given dimension" if the dimension of the lattice of pp-definable subgroups of a model has dimension less than "∞". Having m-dimension is a strictly stronger condition than having breadth. I also discuss Ziegler's definition of "width" (breadth is just a smoothed out version of this). Any theory with m-dimension also has breadth (equivalently, width).

Any theory with width, in particular any theory with m-dimension, has continuous part zero. For countable theories, the converse also holds, This is shown in the third section. It is not known whether an uncountable theory with continuous part zero necessarily has width.

Any theory with width has continuous part zero but, if one assumes the stronger condition that the theory has m-dimension, one may develop a structure theory for the pure-injective models. Theories with m-dimension, as well as countable theories, satisfy the condition that a point of the space of indecomposable pure-injectives is isolated iff it contains a minimal pair. Under this condition, the topological (Cantor-Bendixson rank) and lattice-theoretic (m-dimension) analyses of a theory fit together, and one is able to lift the structure theory of §4.6 well beyond the totally transcendental case. All this is in the fourth section.

In the fifth section, we look at a dimension which is co-extensive with m-dimension but which grows faster and has a "direction": the Krull dimension of the lattice of pp-definable subgroups (this is what Garavaglia originally considered).

Then there is a supplementary section on T^{eq}: an environment in which, for example, cosets of one pp-definable subgroup in another are elements of an appropriate sort. We see that if a theory has m-dimension then, in T^{eq}, every type is non-orthogonal to a regular type: this need not be so if we restrict to T.

In the sixth section, we relate foundation rank to Krull dimension. Then we go on consider a 2-valued rank which, generalising U-rank, is applicable to theories of modules which have m-dimension. Finally, we describe a rank which Pillay introduced for any ω-stable non-multidimensional theory and it is seen that, provided we work in T^{eq}, this rank corresponds to the analysis of §4.

There is a supplementary section on valuation domains. This contains background material, and the classification of the pure-injective modules.

10.1 Existence of indecomposable direct summands

The main point of the proof of 3.14 is that, under the assumed finiteness condition (dcc on pp-definable subgroups), one may split off an indecomposable factor. In fact, nowhere near the full strength of the dcc is required.

<u>Proposition 10.1</u> [Gar80a; Lemma14,Thm1] *Let N be a pure-injective module and let φ/ψ be an N-minimal pair. Then N realises an irreducible pp-type p which contains φ/ψ, (p may be taken to be neg-isolated by ψ).*
In particular, N has an indecomposable direct summand.

Proof Let p be maximal pp containing φ and omitting ψ. By 4.33 p is irreducible and is neg-isolated by ψ. It will be shown that p is realised in N.

Let a be any element in $\varphi(N)\setminus\psi(N)$ and consider the set of pp formulas $\Phi(v,w)=p^+(v)\wedge\psi(w)\wedge(a=v+w)$. This set is finitely satisfied in N. For, given any $\varphi'\in p^+$, one has $\varphi\wedge\varphi'+\psi=\varphi$ because φ/ψ is a minimal pair. Therefore, since $a\in\varphi(N)$, it follows that there are b and c in N with $\varphi\wedge\varphi'(b)$, $\psi(c)$ and with $a=b+c$.

Since N is pure-injective, Φ is realised in it (2.8), say by b, c.

So one has $p^+(b)$. If $\psi(b)$ also held then, from $\psi(c)$ and $a=b+c$, one would deduce $\psi(a)$ - which is not so. Therefore b satisfies $p^+(v)\wedge\neg\psi(v)$. Since p is neg-isolated by ψ this means that b is a realisation of p in N - as required. □

So the presence of a minimal pair in a pure-injective guarantees the existence of an indecomposable factor. The next example shows that this condition is certainly not necessary.

Example 1 [Zg84; Example before 9.4] Let R be a complete valuation domain with infinitely generated maximal ideal (see §10.V). Since (10.V1) the pp-definable subgroups of R are just the principal ideals, certainly $R=\overline{R}$ has no minimal pair. Yet by 10.V3 every pure-injective elementarily equivalent to R has an indecomposable factor. The next result covers this example.

<u>Theorem 10.2</u> [Zg84; §7] *Let N be pure-injective. Suppose that the interval $[\varphi(N),\psi(N)]$ in the lattice of pp-definable subgroups of N is a* **chain** *(every two elements are comparable). Then N realises an irreducible type which contains φ/ψ.*
In particular, N has an indecomposable direct summand.

Proof (Compare with the proof of 10.1). Let a be any element in $\varphi(N)\setminus\psi(N)$. The type of a splits the chain $[\varphi,\psi]$ into an "upper cut" $U=\mathrm{pp}(a)\cap[\varphi,\psi]$ and "lower cut" $L=[\varphi,\psi]\setminus U$. Of course, every formula in U is above every formula in L.

Let p be a maximal pp-type containing U and missing L. Then 4.33 applies, to give that p is irreducible and is equivalent to $p^+\wedge\{\neg\psi':\psi'\in L\}$.

Consider the set $\Phi(v,w)=p^+(v)\wedge\{\psi(w)\}\wedge\{a=v+w\}$. It is claimed that this set is finitely satisfied in N. Since (clearly) Φ is $\wedge$-closed, it will be enough to take any $\varphi'\in p^+$ and show that $a\in\varphi'(N)+\psi(N)$. One has $\varphi\geqslant\varphi'\wedge\varphi+\psi\geqslant\psi$; since $\varphi'\wedge\varphi\in p^+$, one has $\varphi'\wedge\varphi+\psi\in p^+$. By construction, $p^+\cap[\varphi,\psi]=\mathrm{pp}(a)\cap[\varphi,\psi]$. Hence a satisfies $\varphi'\wedge\varphi+\psi$; so certainly $a\in\varphi'(N)+\psi(N)$.

Since N is pure-injective it follows that there is (b,c) in N satisfying $\Phi(v,w)$. One has $p^+(b)$. If one had $\psi(b)$, then one could combine this with $\psi(c)$ to conclude $\psi(b+c)$ - that is $\psi(a)$ - contradiction. Hence b is a realisation of p in N, and the theorem is proved. □

10.2 Dimensions defined on lattices

In the introduction to this chapter I mentioned the possibility of recursively stripping away the "dcc part" of a complete theory so as to get a decomposition theory. Precisely that idea will be followed up in §10.4 below. In this section we look rather at the corresponding process carried out on the lattice of pp-definable subgroups. In §10.1 two conditions were given under which one has indecomposable factors. Since I wish to consider the effect of inductively extending each of them, I give a more abstract treatment which covers them both.

The context for this treatment is that of modular lattices. Of course the example we are interested in is the lattice of pp-definable subgroups of a module. Readers familiar with the Gabriel-Rentschler definition of Krull dimension will note that, unlike that dimension, the dimensions defined here do not have a "direction". For the "directional" versions of these dimensions the reader should consult §10.5.

This section is based on [Pr82a]. Similar ideas have been developed independently by Simmons [Si8?], [Si86].

Let C be a class of modular lattices which is closed under sublattices and quotients. The two examples to have in mind are: C consists of the (isomorphism class of the) two-point lattice $\mathbf{2}$; C is the class $\mathcal{CH}$ of all chains (= linear orders).

Let L be any modular lattice. Define $\approx_C(L)$, or just $\approx$, to be the congruence generated by the set of all the sub-intervals of L which belong to C. Thus $\approx_C(L)$ is the smallest relation, $\approx$, on L such that the quotient $L/\approx$ is a (modular) lattice in which all pairs $(\varphi,\psi)\in L\times L$ with $[\varphi,\psi]\in C$ become identified under the canonical projection $\pi: L\longrightarrow L/\approx$. Let us write $\varphi\leqslant_C\psi$ if $\pi\varphi\leqslant\pi\psi$.

The first result gives a useful description of the way in which those sub-intervals of L which lie in C generate this congruence.

Let us agree to the convention that, in this section, L is a modular lattice and C is a class as above. I have in mind lattices with 1 (top) and 0 (bottom), but this affects only a few points.

<u>Lemma 10.3</u> *Let* $\varphi,\psi\in L$.

(*a*) $\varphi\approx\varphi\wedge\psi$ *iff* $\varphi+\psi\approx\psi$.

(*b*) $\varphi\approx\psi \iff \varphi+\psi\approx\varphi\wedge\psi \iff \varphi\approx\varphi\wedge\psi$ *and* $\psi\approx\varphi\wedge\psi$.

(*c*) $\varphi\approx\psi$ *iff there is a finite sequence* $\varphi+\psi=\varphi_0\geqslant\varphi_1\geqslant\ldots\geqslant\varphi_n=\varphi\wedge\psi$ *such that, for each* i, *the interval* $[\varphi_i,\varphi_{i+1}]$ *is in* C.

(*d*) $\varphi\geqslant_C\psi$ *iff there is a finite sequence* $\varphi+\psi=\varphi_0\geqslant\varphi_1\geqslant\ldots\geqslant\varphi_n=\varphi$ *with* $[\varphi_i,\varphi_{i+1}]\in C$ *for each* i.

Proof Let $\pi: L\longrightarrow\!\!\!\!\rightarrow \pi L = L/\approx_C$ be the canonical projection.

(a) This is immediate since πL is a modular lattice and so $\pi\varphi=\pi(\varphi\wedge\psi)=\pi\varphi\wedge\pi\psi$ iff $\pi\psi\geqslant\pi\varphi$, and this is so iff $\pi(\varphi+\psi)=\pi\varphi+\pi\psi=\pi\psi$.

(b) This follows in the same way (using (a)).

(c) Let us set $\varphi\sim\psi$ if there is a finite sequence between $\varphi+\psi$ and $\varphi\wedge\psi$ of the form described. It will be sufficient to show that "$\sim$" is a congruence, since it certainly collapses all intervals of L which are in C and any congruence which does this must contain "$\sim$".

For convenience, say that there is a "C-chain from φ to ψ" if there is a sequence $\varphi=\varphi_0\geqslant\varphi_1\geqslant\ldots\geqslant\varphi_n=\psi$ with $[\varphi_i,\varphi_{i+1}]\in C$ for each i. Observe that our assumptions on C imply that, if $\varphi\geqslant\varphi'\geqslant\psi'\geqslant\psi$ and if there is a C-chain from φ to ψ, then there is a C-chain from φ' to ψ'. Also, if there is a C-chain from φ to ψ then, for any φ', there is a C-chain from $\varphi\wedge\varphi'$ to $\psi\wedge\varphi'$. Moreover the existence of a C-chain from φ to ψ and of one from ψ to θ implies the existence of a C-chain from φ to θ.

To see that "$\sim$" is an equivalence relation is easy; only transitivity needs a little checking. So suppose that $\varphi\sim\psi\sim\theta$. Then there is a C-chain from $\varphi+\psi$ to $\varphi\wedge\psi$ and hence there is one from $\varphi+\psi$ to ψ. Similarly, there is a C-chain from $\psi+\theta$ to ψ. Therefore, by modularity, there is a C-chain from $\varphi+\psi+\theta$ to ψ. Analogously, there is a C-chain from ψ to $\varphi\wedge\psi\wedge\theta$. So, in particular, there is a C-chain from $\varphi+\theta$ to $\varphi\wedge\theta$. Hence $\varphi\sim\psi$.

Next it is shown that "$\sim$" is a lattice congruence. So suppose $\varphi\sim\psi$ and let $\theta\in L$. There is a C-chain from $\varphi+\psi$ to $\varphi\wedge\psi$ and hence there is one from $\varphi+\psi+\theta$ to $(\varphi\wedge\psi)+\theta$. Now, $(\varphi+\theta)+(\psi+\theta)=\varphi+\psi+\theta$ and also $(\varphi+\theta)\wedge(\psi+\theta)\geqslant\varphi\wedge\psi+\theta$. Hence there is a C-chain from $(\varphi+\theta)+(\psi+\theta)$ to $(\varphi+\theta)\wedge(\psi+\theta)$ - so $\varphi+\theta\sim\psi+\theta$. The proof that $\varphi\wedge\theta\sim\psi\wedge\theta$ is entirely analogous.

Modularity of $L/\sim$ follows, since any quotient lattice of a modular lattice is modular (the identity for modularity is $(c\wedge(a\vee b))\vee b=(c\vee b)\wedge(a\vee b)$ - see [Co81; Exercise II.4.1]).

(d) This follows immediately from (c) since $\varphi\geqslant_C\psi$ iff $\varphi+\psi\sim\varphi$. ◻

The two examples which are of interest to us arise from 10.1 and 10.2, being given by taking $\mathfrak{X}$, respectively $C\mathcal{K}$, for C. By 10.3 one has, using the obvious notation, $\varphi\approx_{\mathfrak{X}}\psi$ iff the interval $[\varphi+\psi,\varphi\wedge\psi]$ has finite length. The description of the points which are identified under $\approx_{C\mathcal{K}}$ is also given by 10.3(c).

The process which takes L to $L/\approx$ may be applied to any modular lattice. In particular it may be applied to $L/\approx$. Let us therefore define the following sequence of congruences and corresponding quotients on L: $L^0=L$, $L^1=L'=L/\approx$, $\approx^1=\approx$. Having defined the congruence $\approx^\alpha$ on L and the quotient $L^\alpha=L/\approx^\alpha$, set $L^{\alpha+1}=(L^\alpha)'$ and let $\approx^{\alpha+1}$ be the congruence on L such that $L/\approx^{\alpha+1}=L^{\alpha+1}$. At a limit ordinal λ we let $\approx^\lambda$ be $\bigcup\{\approx^\alpha:\alpha<\lambda\}$ and set $L^\lambda=L/\approx^\lambda$. Thus one has the sequence of surjective morphisms $L\twoheadrightarrow L^1\twoheadrightarrow L^2\twoheadrightarrow\ldots\twoheadrightarrow L^\alpha\twoheadrightarrow\ldots$. In all this notation we may, if necessary, introduce "C".

Define $\dim_C L$ to be the least ordinal α such that $L^{\alpha+1}$ is the trivial lattice: if there is no such ordinal we set $\dim_C L=\infty$, saying that $\dim_C L$ is undefined or that L does not have C-dimension. Observe that if λ is a limit and if $1_L\approx^\lambda 0_L$ then, for some $\alpha<\lambda$, one must have $1_L\approx^\alpha 0_L$ - so $\dim L$ is well-defined.

For example, $\dim_{\mathfrak{X}}L=0$ iff L has finite length; $\dim_{\mathfrak{X}}(\omega,<)=1$, $\dim_{\mathfrak{X}}(\omega^2,<)=2=\dim_{\mathfrak{X}}(\omega^2+\omega,<)$ (exercise, also cf. 10.41 below).

Finally, let L^∞ be the colimit of the system of morphisms shown above: $L^\infty=L/(\bigcup\{\approx^\alpha:\alpha \text{ is an ordinal}\})$.

<u>Lemma 10.4</u> $\dim_C L\leqslant\alpha$ *iff for all quotients* $\pi:L\twoheadrightarrow L_1$ *of* L *one has* $\dim_C L_1\leqslant\alpha$.

Proof One direction is trivial. For the other it is enough (by induction) to show that $\approx_C(L_1)\supseteq\pi(\approx_C(L))$, where by the latter I mean $\{(\pi\varphi,\pi\psi):(\varphi,\psi)\in\approx_C(L)\}$. But this is clear, since if $\varphi,\psi\in L$ with $[\varphi,\psi]$ in C then the assumptions on C entail $[\pi\varphi,\pi\psi]\in C$. ◻

The next result is a characterisation of those lattices L with $\dim_C L<\infty$. Say that the modular lattice is C-dense if for all $\varphi>\psi$ in L there are φ' and ψ' in L with $\varphi\geqslant\varphi'>\psi'\geqslant\psi$ and $[\varphi',\psi']\in C$. So "$C_{\mathfrak{X}}$-dense" means "contains no densely ordered interval".

<u>Proposition 10.5</u> *The following conditions on the lattice* L *are equivalent:*

(i) $\dim_C L<\infty$;

(ii) *every quotient of* L *is* C*-dense.*

Proof (i)$\Rightarrow$(ii) Suppose that there is some quotient $\pi:L\twoheadrightarrow L_1$ of L which contains an interval $[\pi\varphi,\pi\psi]$ with no non-trivial sub-interval in C. If there were distinct points $\theta>\theta'$ between $\pi\varphi$ and $\pi\psi$ which became identified under the congruence $\approx_C$ on L_1 then, by 10.3, there would be a non-trivial interval in C between θ and θ' and hence between $\pi\varphi$ and $\pi\psi$ - contrary to assumption. That is, $[\pi\varphi,\pi\psi]/\approx$ "equals" $[\pi\varphi,\pi\psi]$. So, clearly, $\dim_C L_1=\infty$. So, by 10.4, $\dim_C L=\infty$.

(ii)⇒(i) The assumption implies that on every quotient, L_1, of L, the congruence $\approx_{\mathcal{C}}$ is non-trivial unless L_1 is itself trivial. In particular this applies to L^∞. Hence $L^\infty = 0$, as required. □

Even if L is $\mathcal{C}$-dense it may be that $\dim_{\mathcal{C}} L = \infty$. For example, it follows from **10.5** that $\dim_{\mathbf{2}}(\mathbb{Q}, <) = \infty$. Consider the ordered set $\mathbb{Q}\times\mathbf{2}$ with the lexicographic ordering ($\mathbf{2}$ being the two-point lattice) - thus every point of $\mathbb{Q}$ is replaced by a two-point interval. Then clearly $\mathbb{Q}\times\mathbf{2}$ is $\mathcal{C}_{\mathbf{2}}$-dense but, by **10.5**, $\dim_{\mathbf{2}}\mathbb{Q}\times\mathbf{2} = \infty$.

From now on we will be concerned only with $\dim_{\mathbf{2}}$ which will be called **m-dimension** and denoted "m-dim" ("m" for minimal congruence). Ziegler says just "dimension", but the term is already over-worked in these notes. Also, $\dim_{\mathcal{CK}}$ which will be called "**breadth**" and denoted "br".

Exercise 1 Let B be a boolean algebra. Show that $B/\approx_{\mathbf{2}} = B$ iff B is atomless and this will be so iff $B/\approx_{\mathcal{CK}} = B$. Deduce that $\dim B < \infty$ iff $\operatorname{br} B < \infty$ iff every quotient of B is atomic (in fact $\dim B = \operatorname{br} B$). Boolean algebras B with $\dim B < \infty$ are termed **superatomic**.

The following result appears to be "folklore": it is stated explicitly in [Gar80a; Lemma 10].

Corollary 10.6 *The following conditions on L are equivalent:*

(i) $\text{m-dim}\, L < \infty$;

(ii) the ordered set $(\mathbb{Q}, <)$ *is not a subquotient (i.e. sublattice of a quotient) of* L.

Proof (i)⇒(ii) It follows from **10.5** (and **10.4**) that $(\mathbb{Q}, <)$ is not a quotient of L. More generally, suppose that the quotient $L \twoheadrightarrow L_1$ has a densely ordered subset. Then by **10.3**(c) it is clear that no two distinct points of this sublattice may become identified under the projection $L_1 \twoheadrightarrow L_1/\approx_1$. So, by induction, one has $\dim L_1 = \infty$ and hence $\dim L = \infty$.

(ii)⇒(i) Let $\pi: L \twoheadrightarrow L_1$ be a quotient of L which is not trivial. Since $(\mathbb{Q}, <)$ does not embed in L_1, there must be points $\pi\varphi > \pi\psi$ of L_1 such that the interval $[\pi\varphi, \pi\psi]$ is just the two-point lattice (exercise). Therefore the congruence $\approx_1$ is non-trivial on L_1. In particular, it follows that $L^\infty = 0$, as required. □

Before going on to the equivalent result for breadth, I present Ziegler's definition of "width" of a lattice (on which I have based the definition of breadth). I present this here, rather than in §5, where the various ranks are compared, since at one point (**10.12**) it seems that width is the appropriate tool.

So let L be a modular lattice. Define, by induction, the **width**, $\mathrm{w}[\varphi,\psi]$ or just $\mathrm{w}(\varphi/\psi)$, of an interval $[\varphi,\psi]$ of L as follows (by the way, a rather different measure of the "width", and indeed "breadth", of a poset is also in use):

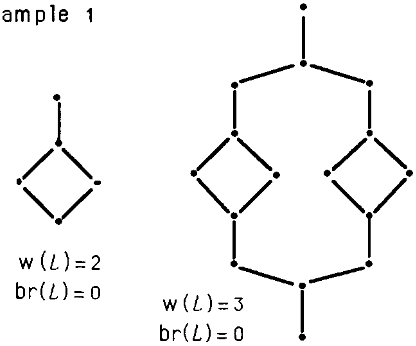
$\mathrm{w}(\varphi/\psi) = 0$ iff $\varphi = \psi$;

$\mathrm{w}(\varphi/\psi) = \alpha$ iff $\mathrm{w}(\varphi/\psi)$ is not less than α and if, for all θ_1, θ_2 with $\varphi \geqslant \theta_i \geqslant \psi$ $(i=1,2)$, one has $\mathrm{w}(\theta_1/\theta_1\wedge\theta_2) < \alpha$ or $\mathrm{w}(\theta_2/\theta_1\wedge\theta_2) < \alpha$.

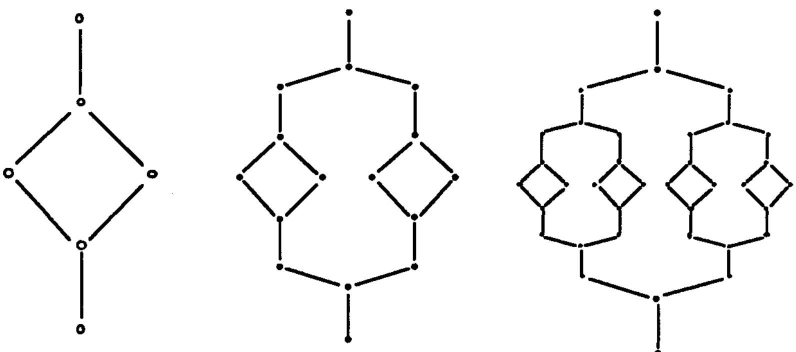
Set $\mathrm{w}(\varphi/\psi) = \infty$ if $\mathrm{w}(\varphi/\psi) \neq \alpha$ for each ordinal α. Define $\mathrm{w}(L) = \mathrm{w}(1_L, 0_L)$.

Observe that $\mathrm{w}(\varphi/\psi) = 1$ iff the interval $[\varphi,\psi]$ is a chain: $\mathcal{CK} = \{L : \mathrm{w}(L) = 1\}$.

Example 1



An example with breadth 1 can be got by placing lattices as opposite, of increasing width, on top of each other. To illustrate breadth "∞", I picture a sequence of nested sub-lattices of a lattice (which can be taken to be the union of these).



beginning of refining sequence for breadth $=\infty$

Exercise 2 Show that if $L \twoheadrightarrow L_1$ is an epimorphism of modular lattices and if $w(L) < \infty$ then $w(L_1) < \infty$.

Next we see that a lattice has **width** (i.e. has width $< \infty$) iff it has breadth. The difference between these two measures of complexity lies only in their rates of growth - and these are not entirely unrelated.

<u>Lemma 10.7</u> *Let* L *be any modular lattice. Then*:

(a) $\operatorname{br} L \leq w(L)$;

(b) *if* $\operatorname{br} L = \alpha$ *then* $w(L) < \omega^{\alpha+1}$, *hence*:

(c) $\operatorname{br} L < \infty$ *iff* $w(L) < \infty$.

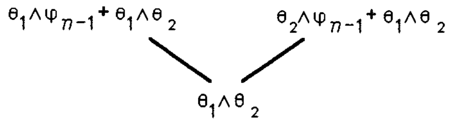
Proof (a) The proof is by induction on $w(L) = \alpha$, where the case $\alpha = 1$ is clear since any chain $(w(L)=1)$ has breadth 0.

So suppose that we have the result for all ordinals $\beta < \alpha$ and suppose that $w(L) = \alpha$. By definition of width, for each pair θ_1, θ_2 in L, at least one of the intervals $[\theta_1, \theta_1 \wedge \theta_2]$, $[\theta_2, \theta_1 \wedge \theta_2]$ has width strictly less than α - say $w([\theta_1, \theta_1 \wedge \theta_2]) = \beta < \alpha$. Let $\sim$ be the congruence $\approx_{\mathcal{C}\mathcal{K}^\beta(L)}$. By the induction hypothesis, $\operatorname{br}[\theta_1, \theta_1 \wedge \theta_2] \leq \beta$ and so $\theta_1 \sim \theta_1 \wedge \theta_2$. Thus, in $L/\sim$, one has $(\theta_1/\sim) \leq (\theta_2/\sim)$. Therefore, if $\approx$ is $\bigcup\{\approx_{\mathcal{C}\mathcal{K}^\beta(L)} : \beta < \alpha\}$, then $L/\approx$ is a chain (every two points are comparable). Then, momentarily separating the cases α a limit or not a limit, one checks that $\operatorname{br} L \leq \alpha$, as required.

(b) This is proved by induction on $\alpha = \operatorname{br} L$.

So suppose that $\mathrm{br}\,L = 0$. By 10.3 there is a finite sequence in L, $1_L = \varphi_0 > \varphi_1 > \dots > \varphi_n = 0_L$ with each $[\varphi_i, \varphi_{i+1}]$ a chain. It is claimed that $w(L) \leqslant n$ (in fact this is a rather lax bound). The proof of the claim is by induction on n, where the base case is clear.

Take any incomparable pair θ_1, θ_2 in L and consider $\theta_1 = \theta_1 \wedge \varphi_0 + \theta_1 \wedge \theta_2 \geqslant \theta_1 \wedge \varphi_1 + \theta_1 \wedge \theta_2 \geqslant \dots \geqslant \theta_1 \wedge \varphi_n + \theta_1 \wedge \theta_2 = \theta_1 \wedge \theta_2$. If not all these inclusions are strict then one has, by induction (for clearly each gap is a chain), that $w(\theta_1/\theta_1 \wedge \theta_2) < n$, as required. If, on the other hand, these inclusions are all strict, and if the same were true of the chain with θ_1 and θ_2 interchanged, then one would have, towards the bottom of each chain, the diagram shown below.



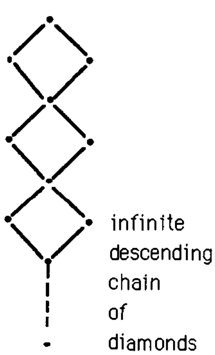
But, since $[\varphi_{n-1}, \varphi_n = 0_L]$ is a chain one has, say, $\theta_1 \wedge \varphi_{n-1} \geqslant \theta_2 \wedge \varphi_{n-1}$ and hence $\theta_1 \wedge \varphi_{n-1} + \theta_1 \wedge \theta_2 \geqslant \theta_2 \wedge \varphi_{n-1} + \theta_1 \wedge \theta_2 > \theta_1 \wedge \theta_2$, which of course contradicts the definition of $\theta_1 \wedge \theta_2$!

Thus the case $\alpha = 0$ has been dealt with. The general case is proved in just the same way. Suppose $\mathrm{br}\,L = \alpha$. Then there is (by 10.3) a sequence $1_L = \varphi_0 > \varphi_1 > \dots > \varphi_n = 0_L$ in L such that, if $\pi : L \twoheadrightarrow L/\approx^{\alpha}$ is the canonical projection, then each interval $[\pi\varphi_i, \pi\varphi_{i+1}]$ is a chain. The claim, inductive on n, is that $w(L) \leqslant \omega^{\alpha}.n$. For the base case we need a separate argument and have to use the fact that if $\theta_1 \leqslant \theta_2 \leqslant \theta_3$ then $w(\theta_3/\theta_1) \leqslant w(\theta_3/\theta_2) + w(\theta_2/\theta_1)$ (cf. [Zg84; 7.4]) (the first part of the argument was a special case of this and the details are left as an exercise).

Take any incomparable pair θ_1, θ_2 in L and consider $\pi\theta_1 = \pi\theta_1 \wedge \pi\varphi_0 + \pi\theta_1 \wedge \pi\theta_2 \geqslant \dots \geqslant \pi\theta_1 \wedge \pi\varphi_n + \pi\theta_1 \wedge \pi\theta_2 = \pi\theta_1 \wedge \pi\theta_2$. If not all the inclusions are strict then one has, on applying the induction hypothesis to the lattice $[\theta_1, \theta_1 \wedge \theta_2]$, the image of which under π can be spanned by $n-1$ chains, that $w[\theta_1, \theta_1 \wedge \theta_2] \leqslant \omega^{\alpha}(n-1)$, as required. Otherwise, as in the case $\alpha = 0$, one derives, say, $\pi\theta_1 \wedge \pi\varphi_{n-1} \geqslant \pi\theta_2 \wedge \pi\varphi_{n-1} > \pi\theta_1 \wedge \pi\theta_2$ – contradiction as before.

Thus the result is established.

(c) This is immediate from (a) and (b).



One should note that the connection between $\mathrm{br}\,L$ and $w(L)$ is not very tight. For example the lattice shown has $\mathrm{br} = 1$ and has width 2 ($<< \omega^2$).

Having established the connection between breadth and width, one has the equivalents to $\mathrm{br} < \infty$ given by the next result, which is immediate from 10.5 and 10.7.

<u>Corollary 10.8</u> [Pr82a; 1.20] *The following conditions are equivalent:*

(*i*) $\mathrm{br}\,L < \infty$;

(ii) $w(L) < \infty$;
(iii) no non-trivial quotient of L *satisfies*

$$\forall\varphi,\psi\,(\varphi>\psi \to \exists\theta_1,\theta_2\,(\varphi \geqslant \theta_1+\theta_2 > \theta_1 > \theta_1\wedge\theta_2 \geqslant \psi);$$

(iv) every quotient of L *"has dense chains" - i.e., is* $\mathcal{CK}$*-dense.* ▫

Of course $\text{m-dim}\,L \geqslant \text{br}\,L$ for all L, since $\mathfrak{2} \subseteq \mathcal{CK}$, and the example $(\mathbb{Q}, <)$ shows that it is possible to have m-dimension "∞" but breadth zero.

If M is a module then $\dim_{\mathcal{C}} M$ means $\dim_{\mathcal{C}} L$, where L is the lattice, $\text{Latt}^f(T)$, of pp-definable subgroups of M. For T a complete theory of modules $\dim_{\mathcal{C}} T$ means $\dim_{\mathcal{C}} M$, where M is any model of T. Also, I sometimes use $\text{m-dim}(\varphi/\psi)$ and $\text{br}(\varphi/\psi)$ rather than $\dim[\varphi,\psi]$ and $\text{br}[\varphi,\psi]$.

10.3 Modules with width

This section presents results of Ziegler which relate the condition $w(T) < \infty$ to that of T having continuous part zero. In the countable case, the conditions are equivalent: in the uncountable case, the former implies the latter.

Theorem 10.9 [Zg84; 7.8(1)] *Suppose that* N *is pure-injective and that* $w(N) < \infty$. *Then* N *has continuous part zero.*

Proof If N' is any direct summand of N then its lattice of pp-definable subgroups is a quotient of that of N so, by **10.4**, $w(N') < \infty$ also. Therefore **10.8** implies that every non-zero direct summand of N has a non-trivial interval which is a chain. So, by **10.2**, every direct summand of N has an indecomposable factor. ▫

Corollary 10.10 [Gar80a; Thm1] *Suppose that* N *is pure-injective with* $\text{m-dim}\,N < \infty$. *Then* N *has continuous part zero.* ▫

Corollary 10.11 [Zg84; 7.1(1)] *If* T *is a complete theory with* $w(T) < \infty$ *then* $T_C = 0$; *so every pure-injective model of* T *is the pure-injective hull of a direct sum of indecomposables.* ▫

Within various classes of modules, ring-theoretic finiteness conditions may be given which ensure that the situation of **10.11** obtains. Examples of such conditions are provided for: injective modules **1.12**; projective modules **14.14**, **14.15**; modules over commutative regular rings **16.26**; modules over valuation domains and Dedekind domains §2.$\mathbb{Z}$, §10.V; and modules over path algebras of quivers §13.3.

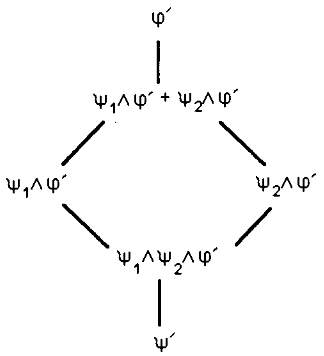
Let me remark at this stage that whenever these dimensions are being discussed, it may be assumed, if convenient, that our theories are closed under product: for T and $T^{\aleph_0}$ have the same lattice of pp-definable subgroups.

Before embarking on the proof of the converse to **10.11** in the countable case, it is worthwhile noting the following local version of **10.9**.

Proposition 10.12 [Zg84; 7.8(1)] *Let* T *be a complete theory and let* $\varphi>\psi$ *be pp formulas with* $w[\varphi,\psi] < \infty$. *Then every type* p *with* $\varphi/\psi \in p$ *has a large formula. In particular,* $N(p)$ *has an indecomposable direct summand.*

Proof I reproduce Ziegler's proof and actually work with the equivalent (10.7) assumption that $w(\varphi/\psi) < \infty$, since breadth seems to be too coarse for this proof.

So suppose that $\varphi/\psi \in p$. Choose $\varphi'/\psi' \in p$ with $w(\varphi'/\psi') = \alpha$ minimal (and $<\infty$ by the hypothesis). It is claimed that ψ' is large in p.



To see this, take $\psi_1,\psi_2 \in p^-$ with $\psi_1,\psi_2 \geqslant \psi'$. Then, for $i=1,2$, one has $\psi' \leqslant \psi_1\wedge\psi_2\wedge\varphi' \leqslant \psi_i\wedge\varphi' \leqslant \varphi'$. If, for some $i\in\{1,2\}$, one has $\psi_1\wedge\psi_2\wedge\varphi' = \psi_i\wedge\varphi'$ then $\psi_1\wedge\varphi' + \psi_2\wedge\varphi' = \psi_{3-i}\wedge\varphi' \in p^-$, as required. So it may be supposed that the inclusions $(\psi_1\wedge\varphi')\wedge(\psi_2\wedge\varphi') < \psi_i\wedge\varphi'$ are strict. Thus, one has the diagram shown. Since $w(\varphi'/\psi') = \alpha$ one has, from the definition of width, that $w(\psi_1\wedge\varphi' + \psi_2\wedge\varphi' / \psi_i\wedge\varphi') < \alpha$ for $i=1$ or $i=2$.

Now, $\psi_i\wedge\varphi' \in p^-$, so minimality of α yields $\psi_1\wedge\varphi' + \psi_2\wedge\varphi' \notin p^+$, as required (for the definition of "large").

The last statement now follows by **9.16**. ◻

<u>Theorem 10.13</u> [Zg84; 7.8(2)] *Suppose that* T *is a complete theory of modules such that, with respect to the* $\mathcal{CK}$ *congruence,* $\mathrm{Latt}^{\mathrm{f}}(T)^{\infty}$ *is non-trivial (i.e.,* $\mathrm{br}\,T=\infty$*) but countable. Then* $T_{\mathrm{C}} \neq 0$.

Proof A type with no large formula is to be constructed: by **9.16**, this will be enough. The construction is performed in $L^{\infty} = \mathrm{Latt}^{\mathrm{f}}(T)^{\infty}$ and is then pulled back to $L=\mathrm{Latt}^{\mathrm{f}}(T)$ *via* the canonical projection $\pi: L \twoheadrightarrow L^{\infty}$.

Since L^{∞} is assumed to be countable, let us suppose that it is enumerated as $\{\sigma_n : n\in\omega\}$ say, in such a way that each element occurs infinitely often.

Very roughly, one may view the construction as a process of continual refinement between what is to be p^+ and what is to be p^-.

At the end of the n-th stage of the construction, one will have decided for each $m\leqslant n$ whether $\sigma_m \in p^+$ or $\sigma_m\in p^-$ and one will set ρ_n to be the intersection of those σ_m in p^+ with $m\leqslant n$. Moreover, one will have constructed a finite set I_n of intervals of the form φ/ψ (i.e. $[\varphi,\psi]$) such that: (i) each such "top point" φ satisfies $\varphi\leqslant\rho_n$; (ii) for each interval φ/ψ in I_n there is an interval φ_0/ψ_0 of I_{n-1} with $\varphi_0/\psi_0 \geqslant \varphi/\psi$ (in the sense of Exercise 2.4/1) and with $\varphi_0\geqslant\varphi$.

Set $I_{-1} = [1_L, 0_L]$.

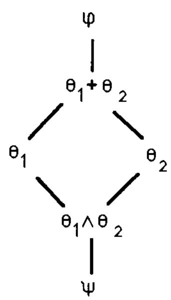
The $(n+1)$-st stage of the construction is as follows. Consider $\sigma=\sigma_{n+1}$: there are two cases.

<u>Case (a)</u> $\sigma\wedge\varphi > \sigma\wedge\psi$ for each $\varphi/\psi\in I_n$.

In this case put $\sigma=\sigma_{n+1}$ into p^+, set $\rho_{n+1}=\rho_n\wedge\sigma_n$ and form I_{n+1} by replacing each $\varphi/\psi\in I_n$ by $\sigma\wedge\varphi/\sigma\wedge\psi$ $(=\rho_{n+1}\wedge\varphi/\rho_{n+1}\wedge\psi$ since, by induction, $\varphi\leqslant\rho_n)$.

<u>Case (b)</u> $\sigma\wedge\varphi = \sigma\wedge\psi$ for some $\varphi/\psi\in I_n$.

In this case put σ into p^- and choose, for each such "type (b)" pair $\varphi/\psi\in I_n$, a pair of incomparable formulas θ_1,θ_2 as shown - there is such a pair, since $\mathrm{br}(\varphi/\psi)=\infty$ - and put both intervals $\theta_1/\theta_1\wedge\theta_2$ and $\theta_2/\theta_1\wedge\theta_2$ into I_{n+1}.



Those intervals ("type (a)") φ/ψ of I_n for which $\sigma\wedge\varphi$ strictly contains $\sigma\wedge\psi$ are replaced by $\sigma\wedge\varphi/\sigma\wedge\psi$ in I_{n+1}. Set $\rho_{n+1}=\rho_n$.

Let us note immediately that: (ii) if $\varphi'/\psi'\in I_{n+1}$, then there is $\varphi/\psi\in I_n$ with $\varphi/\psi\geqslant\varphi'/\psi'$ and $\varphi\geqslant\varphi'$; (i) each "top point", φ', of an interval $\varphi'/\psi'\in I_{n+1}$ is below ρ_{n+1} (by induction).

Moreover, if $k\geqslant m$ is such that $\sigma_k=\Sigma\varphi_i'$ - the sum of all "top elements" of members of I_{n+1} then, since $\sigma_k\geqslant\varphi'$ for each such φ', the statement (ii) above (inductively extended) implies that case (a) will obtain at the k-th stage and so this sum σ_k will be put into p^+. Also, if $\varphi'/\psi'\in I_{n+1}$ and if $l>n+1$ is such that $\sigma_l=\psi'$, then, from $\sigma_l\wedge\varphi'=\sigma_l\wedge\psi'$ (and (the inductively extended) (ii)), it follows that case (b) will obtain at the l-th stage and hence ψ will be put into p^-. Note also that the same argument shows that if $\sigma_t=\sigma_{t+r}$ then σ_{t+r} will be put into p^+ or p^- according as σ_t was; so our construction does hang together.

Each point eventually is assigned either to p^+ or to p^- (and p^+ will be given by the descending sequence $\rho_0\geqslant\rho_1\geqslant\ldots\geqslant\rho_n\geqslant\ldots$). The type q which we want has, of course, to lie in L, and it is simply defined to be the pullback along $\pi:L\twoheadrightarrow L^\infty$ of the "type" which has been constructed (so $\varphi\in q^+$ iff $\pi\varphi\in p^+$). "Consistency" must be checked. The first point to note is that p^+ (and so q^+) is a filter - this is clear on considering the ρ_n's - moreover, arguing as in the last paragraph, one sees that p^+ (so q^+) is closed upwards. Then the fact that $(L^\infty)'$ equals L^∞ is, together with Neumann's Lemma **2.12**, enough to ensure that q is indeed consistent.

The other point to be checked is that no formula is large in q. Looking at the definition of "large", one sees that it is simply a lattice property which may be tested equally (and more easily) on p as on q.

So let $\sigma=\sigma_{n+1}$ be any formula in p^-. Let $I_n=\{\varphi_1/\psi_1,\ldots,\varphi_t/\psi_t,\varphi_{t+1}/\psi_{t+1},\ldots,\varphi_{t+r}/\psi_{t+r}\}$, where the first t are (in relation to σ) of "type (b)" and the rest are of "type (a)". Since $\sigma\in p^-$, $t\geqslant1$. For $j=1,\ldots,t$ let $\theta_{ji}/\theta_{j1}\wedge\theta_{j2}$ $(i=1,2)$ be the two corresponding intervals in I_{n+1}. It is claimed that for each $j=1,\ldots,t$ and $i=1,2$, one has $\sigma+\theta_{ji}\in p^-$, but that for every $\varphi\in p^+$ one has $\Sigma_{j=1}^{t}\Sigma_{i=1}^{2}(\sigma+\theta_{ji})\wedge\varphi\in p^+$ - which will be as required; for this yields that σ is not large in p (cf. before **9.14**).

First, for each j, $\sigma+\theta_{ji}$ is in p^- (say $i=1$). For one has $(\sigma+\theta_{j1})\wedge\theta_{j2}=(\sigma+\theta_{j1})\wedge\varphi_j\wedge\theta_{j2}=(\theta_{j1}+\sigma\wedge\varphi_j)\wedge\theta_{j2}$ (by modularity) $=(\theta_{j1}+\sigma\wedge\theta_{j1})\wedge\theta_{j2}$ (since $\sigma\wedge\varphi_j=\sigma\wedge\psi_j$ and $\varphi_j\geqslant\theta_{j1}\geqslant\psi_j$) $=\theta_{j1}\wedge\theta_{j2}=(\sigma+\theta_{j1})\wedge\theta_{j1}\wedge\theta_{j2}$. But $\theta_{j2}/\theta_{j1}\wedge\theta_{j2}\in I_{n+1}$ so, at the appropriate stage (since case (b) will obtain), $\sigma+\theta_{j1}$ will be put into p^-, as claimed.

Now let $\tau=\sigma_{m+1}$ (m large enough) be an arbitrary member of p^+. Fix $\varphi'/\psi'\in I_{m+1}$: then $\varphi'/\psi'\leqslant\varphi_j/\psi_j$ for some φ_j/ψ_j in I_{n+1} and, since τ is put into p^+ (case (a)), one has $\tau\geqslant\varphi'$.

If $j>t$ then (by construction at the $(n+1)$-st stage), one has $\sigma\wedge\varphi_j\geqslant\varphi'$, so one concludes that $\varphi'\leqslant\sigma\wedge\tau\leqslant(\Sigma_{ji}(\sigma+\theta_{ji}))\wedge\tau$.

If $j\leqslant t$ then one has for, $i=1$ or $i=2$, that $\varphi'\leqslant\theta_{ji}$. Since also $\varphi'\leqslant\tau$, one again concludes that $\varphi'\leqslant\theta_{ji}\wedge\tau\leqslant(\Sigma_{ji}(\sigma+\theta_{ji}))\wedge\tau$.

Thus one has that $(\Sigma_{ji}(\sigma+\theta_{ji}))\wedge\tau$ is above each "top" element in I_{n+1}. So (as argued before), at an appropriate stage $(\Sigma_{ji}(\sigma+\theta_{ji}))\wedge\tau$ will be put into p^+, as required. □

Corollary 10.14 [Zg84; 7.1(2)] *Suppose that T is a complete theory of modules over a countable ring. Then T has continuous part zero iff* $\mathrm{br}\,T<0$ *iff* $\mathrm{w}(T)<\infty$. □

10.4 Classification for theories with dimension

In this section a structure theory is developed for the pure-injective models of those theories which have m-dimension in the sense of §2. By 10.10, we know that every pure-injective model of such a theory is discrete. The structure theory, which is due to Ziegler, is analogous to, indeed generalises, that of §4.6. In particular, it gives precise limitations on the number of times each member of $\mathcal{I}(T)$ can occur in the decomposition of a pure-injective model. I do not know to what extent a similar structure theory may be developed if T has width but not dimension.

Our analysis of a theory, T, with m-dimension proceeds by successively stripping away the simplest components of T. "Simplest" will be measured by CB-rank in $\mathcal{I}(T)$, but our first task is to tie this in with m-dimension. This dimension was defined for intervals in the lattice of pp formulas, but there is an natural way in which this induces a dimension on the points of $\mathcal{I}(T)$. It is shown that, at least if T is countable or $\dim T < \infty$, then this latter dimension coincides with CB-rank in $\mathcal{I}(T)$.

Recall (9.3) that for any complete theory T, if the interval φ/ψ is T-minimal then the open set (φ/ψ) in $\mathcal{I}(T)$ has just one point N (say) and so, in particular, N is isolated. For the theory of this section to apply to T it appears that we need the converse also (recall that when I say that a point is isolated, I mean that it is isolated as a point of $\mathcal{I}(T)$ rather than $\mathcal{I}(T)/\approx$):

Condition $(\wedge)$ if $T' \leqslant T$ then N is an isolated point of $\mathcal{I}(T')$ iff there is a T'-minimal pair φ/ψ with $(\varphi/\psi) = \{N\}$.

Thus, in a theory T satisfying $(\wedge)$, the isolated points of $\mathcal{I}(T)$ are precisely those which realise a type with a T-minimal pair or, to put it another way, an open set (φ/ψ) contains an isolated point iff there is a minimal pair between φ and ψ. This condition on T and its components implies that the topological and lattice-theoretic measures of the complexity of pure-injectives will coincide, not just at the base level, but also through the inductive extensions - CB-rank and m-dimension. Observe that if $\text{m-dim}\, T < \infty$ then any isolated neighbourhood in $\mathcal{I}(T)$ contains just one point. Indeed, if $N \equiv N'$ are indecomposable pure-injectives and $\text{m-dim}\, N < \infty$ then, by 10.5, N has an N-minimal pair φ/ψ say. Since $N' \equiv N$, φ/ψ also is a minimal pair for N'. So by 9.3 and 10.1, $N \simeq N'$: this result is due to Garavaglia [Gar80a; Cor 6].

Before examining the consequences of condition $(\wedge)$, let us establish that the condition is a reasonable one. We will see that if R is countable then $(\wedge)$ holds for any complete theory of modules over R. Then, more to the point as regards a structure theory, we see that if T has breadth (in particular if T has m-dimension) then T satisfies $(\wedge)$ (in fact this last may be proved locally).

<u>Theorem 10.15</u> [Zg84; 8.3] *Let T be a complete theory over a countable ring and let $\varphi > \psi$ be pp formulas. If there exists a densely ordered subset of the interval $[\varphi, \psi]^T$, then the open set (φ/ψ) of $\mathcal{I}(T)$ contains $2^{\aleph_0}$ points. In particular, if $[\varphi, \psi]$ does not contain a minimal pair then $|(\varphi/\psi)| > 1$; so T satisfies condition$(\wedge)$.*

Proof First of all, $2^{\aleph_0}$ types are produced. Then it is shown that no point of $\mathcal{I}(T)$ can realise more than countably many of them.

It may be supposed that the densely ordered subset is indexed by the rationals so that $\varphi_r < \varphi_s$ iff $r < s$ $(r, s \in \mathbb{Q})$. Define the partial types corresponding to real cuts: $p_\alpha = \{\varphi_r : r \geqslant \alpha\} \cup \{\neg\varphi_s : s < \alpha\}$ $(\alpha \in \mathbb{R})$.

Let $\{\theta_i\}_{i \in \omega}$ be an enumeration of all pp-formulas; we use this enumeration to give sufficient coherence to our extensions of the p_α to complete types. For each $\alpha \in \mathbb{R}$, define a sequence of extensions of p_α by putting $p(\alpha,0)=p_\alpha$ and, having defined $p(\alpha,i)$, setting $p(\alpha,i+1)=p(\alpha,i)\wedge\theta_i$ if this is consistent and $p(\alpha,i+1)=p(\alpha,i)\wedge\neg\theta_i$ if not. Then let q_α be the limit of these: $q_\alpha=\bigcup_i p(\alpha,i)$. Set $X_\alpha=\{i\in\omega:\theta_i\in q_\alpha\}$. Each q_α is irreducible by **4.33** - since it has in effect been defined to be maximal with respect to missing the chain $p_\alpha{}^-$.

Next we show that these $2^{\aleph_0}$ irreducible types represent $2^{\aleph_0}$ indecomposable pure-injectives. It will be enough to show that if $N\in\mathcal{I}(T)$ then N realises at most $\aleph_0$ of the q_α. Suppose that, on the contrary, N realises uncountably many of the q_α - say $a_\beta\in N$ realises q_β for suitable β. Choose and fix some non-zero element $a\in N$. Since N is indecomposable there is, for each a_β, some pp formula ψ_β linking a and a_β: $\psi_\beta(a,a_\beta)\wedge\neg\psi_\beta(0,a_\beta)$.

Since there are only countably many pp formulas, there is an uncountable set, D, of reals with $\psi_\alpha(w,v)=\psi_\beta(w,v)$ for all α,β in D. Let $n\in\omega$ be such that $\psi_\alpha(0,v)=\theta_n$ for each $\alpha\in D$. For each such α the formula θ_n does not go into q_α, so by construction it must be that $p(\alpha,n)$ proves $\neg\theta_n$. Choose $\beta<\alpha$ both in D such that $X_\beta\cap n=X_\alpha\cap n$ (there are only finitely many subsets of n). So, for $i<n$, $\theta_i(a_\alpha)$ holds iff $\theta_i(a_\beta)$ does and, in this case, $\theta_i(a_\alpha-a_\beta)$ holds. Since $p_\beta{}^+\geqslant p_\alpha{}^+$ we also have $p_\alpha(a_\alpha-a_\beta)$ (since there is $r\in\mathbb{Q}$ with $\alpha>r>\beta$!). Therefore $a_\alpha-a_\beta$ satisfies $p(\alpha,n)$ and hence $\neg\theta_n(a_\alpha-a_\beta)$, that is $\neg\psi_\alpha(0,a_\alpha-a_\beta)$, holds. On the other hand, from $\psi_\beta(a,a_\beta)\wedge\psi_\alpha(a,a_\alpha)$ we deduce (since $\psi_\alpha=\psi_\beta$) that $\psi_\alpha(0,a_\alpha-a_\beta)$ holds - contradiction as desired. □

Exercise 1 Suppose that $|R|=\kappa$. Suppose that there is a subset of $\mathrm{Latt}^f(T)$ of cardinality κ and with λ cuts. Find (set-theoretic) conditions on κ and λ which allow the proof of **10.15** to go through.

The generalisation of **10.15** to uncountable rings is not true without some set-theoretic assumptions: I discuss an example of Hodkinson after **10.22**.

Our main interest in this section is in developing a structure theory for theories with m-dimension. Fortunately such theories do have property ($\wedge$). Indeed, one has the following.

Theorem 10.16 cf. [Zg84; 8.11] *Suppose that* $T_C=0$ (*in particular suppose that* m-dim$T<\infty$). *Then* T *satisfies* ($\wedge$).

Proof By the remarks at the beginning of this section, it will be enough to show that if the interval $[\varphi,\psi]$ does not contain a minimal pair then the neighbourhood (φ/ψ) contains more than one point (that seems to be the best we can do under the hypothesis of the theorem). Our aim, then, is to produce two irreducible types which contain φ/ψ and which are so different that they cannot be realised in the same indecomposable pure-injective. Assume, then, that there is no minimal pair between φ and ψ.

One irreducible type is found easily. Let p be maximal with respect to containing φ and missing ψ; by **4.33**, p is irreducible and neg-isolated.

Now, recall that if q is related to p then q also must be neg-isolated (9.24). This suggests the following approach to finding a second point of (φ/ψ).

Let φ_r ($r\in\mathbb{Q}$) be a densely ordered subset of $[\varphi,\psi]$ such that $r<s$ iff $\varphi_r<\varphi_s$. Let α be any irrational and define q to be a maximal pp-extension of the cut, p_α, at α (as in the proof of **10.15**). Then q will be irreducible by **4.33**, and one is tempted to think that q cannot be neg-isolated - for the pp formulas which define its negative part form a chain with no maximal element. How, though, does one exclude the possibility that adding a single $\neg\varphi_r$ ($r<\alpha$) to q^+ will imply $\neg\varphi_s$ for all s with $r\leqslant s<\alpha$? I don't expect that this can happen for all α but, in the absense of a proof for that, we reverse our strategy and note that what we have to avoid is adding a pp formula whose intersections with the various φ_r collapse. Then we will invoke the assumption $T_C=0$ to get an *irreducible* type.

Therefore, let q be a type maximal with respect to containing φ, missing ψ and having the property that for every $\theta \in q^+$ and $r<s$ one has $\theta \wedge \varphi_r < \theta \wedge \varphi_s$. Note that no φ_r is in q^+. Observing that (using **4.40**) one may suppose that $T = T^{\aleph_0}$, one sees that q is consistent (this is also easy to see directly).

Since it has been assumed that T has zero continuous part, one deduces that $N(q)$ is discrete. So $N(q)$ realises an irreducible type containing φ/ψ. If (φ/ψ) contained just one point then the corresponding indecomposable would be the hull of p. We will see that this leads to a contradiction.

By **9.24**, q is neg-isolated (since p is), by ψ' say. Since $\psi' \in q^-$, there are $\tau \in q^+$ and $r<t$ in $\mathbb{Q}$ with $\tau \wedge \psi' \wedge \varphi_r = \tau \wedge \psi' \wedge \varphi_t$ (□). Choose $s \in \mathbb{Q}$ with $r<s<t$ and consider the formula $\psi'' \equiv \varphi_s \wedge \tau + \psi'$.

First we see that this formula is not in q^+. For
$\psi'' \wedge \tau \wedge \varphi_t = (\varphi_s \wedge \tau + \psi' \wedge \tau) \wedge \varphi_t = \varphi_s \wedge \tau + \psi' \wedge \tau \wedge \varphi_t = \varphi_s \wedge \tau + \psi' \wedge \tau \wedge \varphi_s$ (by (□))
$= (\varphi_s \wedge \tau + \psi') \wedge \tau \wedge \varphi_s = \psi'' \wedge \tau \wedge \varphi_s$. So $\psi'' \wedge \tau \notin q^+$; hence $\psi'' \notin q^+$.

So, since ψ' neg-isolates q, there is $\theta \in q^+$ with $\theta \wedge \psi'' \leq \psi'$. Then
$\theta \wedge \tau \wedge \varphi_s \leq \theta \wedge \tau \wedge \psi' \wedge \varphi_t = \theta \wedge \tau \wedge \psi' \wedge \varphi_r$ (by (□)) $= \theta \wedge (\tau \wedge \varphi_r) \wedge \psi' = \theta \wedge \tau \wedge \varphi_r$, the first inequality and last equality since $\psi' \geq \theta \wedge \psi'' \geq \theta \wedge \tau \wedge \varphi_s \geq \theta \wedge (\tau \wedge \varphi_r)$. But $\theta \wedge \tau \in q^+$ - contradiction. □

Exercise 2 The proof above shows that if $\varphi > \psi$ are such that for every continuous pure-injective $E \in \mathcal{P}(T)$ one has $\varphi(E) = \psi(E)$ (in the terminology of [Zg84], (φ/ψ) is small), and if (φ/ψ) contains just one point, then there is a minimal pair between φ and ψ.

Ziegler proves the results of this section under the assumption that R is countable or that we are working within an interval which is "small" in this sense. The condition ($\wedge$) that I work under (cf. [Zg84; 8.1]) includes the first case but does not entirely encompass the second; however, the reader should have no difficulty in seeing how to replace the global assumption "$T_C = 0$" by Ziegler's local one.

From now on in this section we assume that we are working with a theory T which satisfies condition ($\wedge$).

Supposing that (φ/ψ) isolates a single point N (say), let us consider the interval $[\varphi, \psi]$. Every non-trivial sub-interval also isolates N, so contains a minimal pair, but the example given after **10.5** warns us that we cannot immediately conclude that $\dim[\varphi,\psi] < \infty$. But, in fact, m-dim$(\varphi/\psi)$ is 0 - in other words the interval $[\varphi, \psi]$ has finite length.

Proposition 10.17 ($\wedge$) *Suppose that the open set (φ/ψ) contains just one point. Then the interval $[\varphi, \psi]$ has finite length.*

Proof Let Θ be the set of elements, θ, of $[\varphi, \psi]$ such that $[\theta, \psi]$ has finite length: clearly Θ is +-closed. Let p be maximal with respect to containing φ and avoiding the ideal generated by Θ; p will be consistent unless $[\varphi, \psi]$ has finite length and, by **4.33**, p is irreducible. Since $|(\varphi/\psi)| = 1$, it must be that $N(p) = N$ is the unique point in (φ/ψ). By condition ($\wedge$) there is some type q with a minimal pair realised in N. So, by **9.12**, p must contain a minimal pair. We show that this is impossible unless $\alpha = 0$.

Suppose then, that $\varphi'/\psi' \in p$ is a minimal pair. Since $\psi' \notin p$ there must be $\varphi'' \in p^+$ such that $\varphi'' \wedge \psi' \leq \theta$ for some $\theta \in \Theta$; in particular $\varphi'' \wedge \psi' + \psi \in \Theta$. Since $\varphi'' \wedge \varphi' + \psi$ is minimal above (or equal to) $\varphi'' \wedge \psi' + \psi$, it must also be in Θ. But $\varphi'' \wedge \varphi' + \psi \in p^+$ - contradicting the definition of p. □

Now define the **derivatives** of T as follows: $T^{(\alpha)} = \mathrm{Th}(\oplus\{N^{\aleph_0} : \text{CB-rk}\, N \geq \alpha\})$. Thus: $T^{(0)} = T$; $T' = T^{(1)}$ is obtained from T by removing all isolated points from $\mathcal{I}(T)$ and "taking the theory" of what remains; and so on. Thus (since the set of points of CB-rank $\geq \alpha$ is a closed set) $\mathcal{I}(T^{(\alpha)}) = (\mathcal{I}(T))^{(\alpha)}$, where the derivative on the right-hand side is the Cantor-Bendixson derivate.

In general, we have (9.4) that if the interval $[\varphi, \psi]$ is of finite length then the neighbourhood (φ/ψ) contains only finitely many, necessarily isolated, points (no more that its length). Therefore we obtain the following.

Proposition 10.18 ($\wedge$) *The derivative, T', of the theory T is axiomatised by $T \cup \{\varphi \leftrightarrow \psi : \varphi/\psi$ has finite length modulo $T\}$. Therefore, the lattice of pp-definable subgroups for T' is just that of T, modulo the congruence $\approx_{\mathfrak{L}}$.*

Proof Let T'' be $T \cup \{\varphi \leftrightarrow \psi : \varphi/\psi$ has finite length modulo $T\}$. If $N \in \mathcal{I}(T)$ has CB-rank at least 1 then it does not lie in any (φ/ψ) with φ/ψ of finite length, so N satisfies T''. On the other hand, by condition ($\wedge$), no isolated point satisfies T''. Hence $\mathcal{I}(T'') = \mathcal{I}(T')$, as required. ◻

If this is extended inductively, one obtains that the α-th derivative of a theory T satisfying ($\wedge$) is derived from T by declaring all intervals of m-dimension $< \alpha$ to be trivial (see below).

Let us first define "m-dimension" for the points N of $\mathcal{I}(T)$ by: $\text{m-dim}\,N = \min\{\text{m-dim}[\varphi, \psi] : \varphi(N) > \psi(N)\} = \min\{\text{m-dim}[\varphi, \psi] : N \in (\varphi/\psi)\}$. That is, m-dim N is the minimum dimension of a non-trivial pp-definable interval of N, where the dimension of an interval is measured relative to T. This turns out to be the same as the CB-rank of N, as measured in $\mathcal{I}(T)$.

Proposition 10.19 cf. [Zg84; 8.6] ($\wedge$) *If $N \in \mathcal{I}(T)$ then $\text{m-dim}\,N = \text{CB-rk}\,N$ and the α-th derivative $T^{(\alpha)}$, of the theory T is axiomatised by $T \cup \{\varphi \leftrightarrow \psi : \dim \varphi/\psi < \alpha\}$.*
In particular $\text{m-dim}\,T = \text{CB-rk}\,\mathcal{I}(T)$ and the lattice of pp-definable subgroups for $T^{(\alpha)}$ is just that for T, modulo the congruence $\approx_\alpha$ which collapses all sub-intervals of dimension strictly less than α. The identification is the natural one, which sends φ to $\varphi/\approx_\alpha$.

Proof The proof that $\text{m-dim}\,N = \text{CB-rk}\,N$ is by induction on α. The case $\alpha = 0$ is immediate by condition ($\wedge$) and the definition of the topology. So suppose that the equality holds for points of $\mathcal{I}(T)$ of dimension strictly less than α and also suppose inductively that the β-th derivatives of T are axiomatised as stated, for $\beta < \alpha$. Let $T^{(\alpha)}$ be the α-th derivative of T and suppose that $\text{CB-rk}\,N = \alpha$; so N is an isolated point of $\mathcal{I}(T^{(\alpha)})$ and hence its CB-rank in $\mathcal{I}(T^{(\alpha)})$ is zero. Therefore, by condition ($\wedge$) and the case $\alpha = 0$, N contains a non-trivial interval of finite length with respect to $T^{(\alpha)}$. Also, the assumption on axiomatisations implies (consider the congruence $\varphi \sim \psi \Leftrightarrow T^{(\alpha)} \vdash \varphi \leftrightarrow \psi$) that $\varphi(N) = \psi(N)$ modulo $T^{(\alpha)}$ iff $\dim[\varphi, \psi] < \alpha$ modulo T. It follows from the definition that $\text{m-dim}\,N = \alpha$. The correctness of the axiomatisation of $T^{(\alpha)}$ now follows (by 10.18) ◻

It is not known whether the first statement follows without hypothesis ($\wedge$) (cf. [Zg84; after 8.6]).

Exercise 3 ([Zg84; 8.7]) ($\wedge$) Show that if B is a basis of basic open neighbourhoods of $N \in \mathcal{I}(T)$ then $\text{m-dim}\,N = \min\{\text{m-dim}[\varphi, \psi] : (\varphi/\psi) \in B\}$.

The first corollary follows from, and generalises, 9.4.

Corollary 10.20 [Zg84; 8.12] ($\wedge$) *Let φ and ψ be pp formulas such that $\text{m-dim}(\varphi/\psi) = \alpha$ (modulo T). Let n be the length of the interval $[\varphi, \psi]/\approx_\alpha$ (this is what Ziegler calls the "multiplicity" of (φ/ψ)). Then (φ/ψ) contains no more than n points of $\mathcal{I}(T)$ of m-dimension α.* ◻

The next corollary follows from the description of the topology on the space of indecomposables over a Dedekind domain (cf. §2.$\mathbb{Z}$, Ex4.7/1; alternatively, see the proof of 10.28 below).

Corollary 10.21 [Gar80a; Thm3] *If* R *is a Dedekind domain and if* T^* *is the largest complete theory of modules over* R *then* $\text{m-dim}\,T^* = 2$. ◻

Corollary 10.22 [Zg84; 8.1,8.4] *Let* T *be a complete theory of modules over a countable ring. Then the following conditions are equivalent:*
(*i*) $\text{m-dim}\,T < \infty$;
(*ii*) $\text{CB-rk}\,\mathcal{I}(T) < \infty$;
(*iii*) $\mathcal{I}(T)$ *is countable*;
(*iv*) $|\mathcal{I}(T)| < 2^{\aleph_0}$.

Proof Since there are only countably many pp formulas, (ii)⇒(iii) (each point uniquely determines a pair of formulas which isolates it in the appropriate derivative). Certainly, (iii)⇒(iv) and, by 10.15, (iv)⇒(i). Since R is countable, 10.15 yields that T satisfies ($\wedge$), so 10.19 applies. In particular, $\text{CB-rk}\,\mathcal{I}(T) = \text{m-dim}\,T$ and so (i)⇒(ii) follows. ◻

It was left open in [Zg84] whether 10.22(i)⇔(iv) remains true over an uncountable ring. Hodkinson investigated this problem and, as a by-product of a general construction that he introduced, he proved a result which shows that the answer depends on one's choice of set theory. He produced a commutative von Neumann regular ring of cardinality $\aleph_1$ whose largest theory of modules has m-dimension ∞ but with only $\aleph_1$ indecomposable (pure-)injectives (it is perhaps worth observing that also T^* satisfies ($\wedge$)). Thus, whether or not $\text{m-dim} = \infty$ implies $2^{\aleph_0}$ indecomposable pure-injectives, depends on whether or not one takes on board the continuum hypothesis. Hodkinson's proof is complex and the methods are very different in character from those used here. So I refer the reader to [Hod85] and [Hod85a] for details.

We have defined the m-dimension of an indecomposable pure-injective to be the minimum value of $\text{m-dim}[\varphi,\psi]$ where $N \in (\varphi/\psi)$, and we saw that this coincides with the Cantor-Bendixson rank of N in $\mathcal{I}(T)$. Similarly, we may define the dimension of a type p by: $\text{m-dim}\,p = \min\{\text{m-dim}[\varphi,\psi] : \varphi/\psi \in p\}$. We show that this value coincides with $\text{m-dim}\,N(p)$.

Lemma 10.23 [Zg84; 8.7,8.10] *If* $p \in S^T$ *is any irreducible type, then* $\text{m-dim}\,N(p) = \text{m-dim}\,p$. *In particular, if* p *and* q *are irreducible types with* $N(p) \simeq N(q)$ *and if* $\text{m-dim}\,p < \infty$, *then* $\text{m-dim}\,p = \text{m-dim}\,q$.

Proof Let $a \in N(p)$ realise p. Since (4.66) the set of (φ/ψ) with $\varphi/\psi \in p$ provide a neighbourhood basis for $N(p)$, this follows by the exercise after 10.19 (it also follows from 9.9). ◻

Now we are in a position to prove the promised structure theorem. Let T be a complete theory of modules with $\dim T < \infty$. Let N be any point of $\mathcal{I}(T)$. As in 4.53 and 9.6, let $d(N)$ be the cardinality of the division ring $\text{End}N/\text{JEnd}N$. We assign the members of $\mathcal{I}(T)$ to four sets, according to essentially the same criteria used in §4.6:

$N \in$ IL iff $\text{CB-rk}(N) = 0$ and for any (equivalently, by 9.6, every) isolating neighbourhood (φ/ψ) of N, one has $\text{Inv}(T,\varphi,\psi)$ finite - set $\kappa(N) = \text{Inv}(T,\varphi,\psi)/d(N)$;

$N \in$ IUF iff $\text{CB-rk}(N) = 0$, $d(N)$ is finite but for any (equivalently every) isolating neighbourhood (φ/ψ) of N, one has $\text{Inv}(T,\varphi,\psi)$ infinite - set $\kappa(N) = \aleph_0$;

$N \in$ IUI iff $\text{CB-rk}(N) = 0$ and $d(N)$ is infinite - set $\kappa(N) = 1$;

$N \in$ U iff $\text{CB-rk}(N) > 0$ - set $\kappa(N) = 0$.

Then we obtain the following generalisation of 4.62/4.63.

Theorem 10.24 [Zg84; 9.1, 9.2] *Let* T *be any theory of modules with* $\dim T < \infty$ *and let* $\mathcal{I}(T)$ *be partitioned as above. Then the pure-injective models of* T *are precisely the modules of the form* $\mathrm{pi}(\oplus_i N_i^{(\kappa_i)})$ *where:*

$\kappa_i = \kappa(N_i)$ *if* $N \in \mathrm{IL}$;

$\kappa_i \geq \aleph_0$ *if* $N \in \mathrm{IUF}$;

$\kappa_i \geq 1$ *if* $N \in \mathrm{IUI}$;

$\kappa_i \geq 0$ *if* $N \in \mathrm{U}$.

In particular, there is a prime-pure-injective module M_0 *which is given by setting the* κ_i *to have their minimal admissible values,* $\kappa(N_i)$.

Proof The proof is directly analogous to that of **4.61**. First, it is clear, using **9.3**, that M_0 is a direct summand of a model.

Suppose that $\mathrm{Inv}(T,\varphi,\psi)$ is finite. Then the interval $[\varphi,\psi]$ is of finite length, so we may reduce to the case of a minimal pair φ/ψ. Then the open set (φ/ψ) contains a single point N and, by definition of $\kappa(N)$, one has $\mathrm{Inv}(N^{\kappa(N)},\varphi,\psi) = \mathrm{Inv}(T,\varphi,\psi)$ (it was noted above that the definition was independent of the minimal pair chosen). Then, since (φ/ψ) isolates N, it follows that $\mathrm{Inv}(M_0,\varphi,\psi) = \mathrm{Inv}(T,\varphi,\psi)$.

Suppose, on the other hand, that $\mathrm{Inv}(T,\varphi,\psi)$ is infinite. If the interval $[\varphi,\psi]$ is of infinite length then it contains an infinite sub-chain, each gap of which must contain a minimal pair. Either (φ/ψ) contains infinitely many points or else contains an indecomposable which itself has its lattice of pp-definable subgroups of infinite length - in either case $\mathrm{Inv}(M_0,\varphi,\psi)$ is infinite. If, on the other hand, the interval $[\varphi,\psi]$ is of finite length, then every point in (φ/ψ) must be isolated and so the fact that $\mathrm{Inv}(T,\varphi,\psi)$ is infinite means, by definition of the $\kappa(N)$, that also $\mathrm{Inv}(M_0,\varphi,\psi)$ is infinite, as required. □

The prime model of an arbitrary theory (if it exists) realises exactly the isolated types. One sees that the prime pure-injective model above realises precisely the irreducible types with minimal pairs. Also, by **9.11**, every type realised in M_0 has a minimal pair.

Even without assuming $(\wedge)$, one may say the following ([Zg84; 9.3]).

(i) The indecomposable pure-injective N is a summand of every *discrete* pure-injective model of T iff $\text{CB-rk}\, N = 0$.

(ii) The indecomposable pure-injective N is a summand of *every* pure-injective model of T iff N contains a T-minimal pair.

The first equivalence is immediate by **4.68**: and the direction "$\Leftarrow$" of the second follows from **10.1** and **9.3**. As for (ii)$\Rightarrow$; clearly N must be isolated so, by the exercise after **10.16**, it will be enough to show that N has an isolating neighbourhood of the form (φ/ψ) where φ/ψ is "small" in the sense discussed after **10.16**. But if there were no such neighbourhood, then clearly N could be replaced by a direct sum of continuous pure-injectives in $\mathcal{P}(T)$ - contrary to hypothesis.

Exercise 4 Use the above to prove that if the countable theory of modules T is not $\aleph_0$-categorical, then it has infinitely many non-isomorphic countable models (**6.32**).

10.5 Krull dimension

Analysing the lattice of pp-definable subgroups in terms of m-dimension proved to be of value to us in developing a structure theory for discrete pure-injectives. We also found that the property of having breadth was relevant to the non-existence of continuous pure-injectives. There are a couple of other measures of the complexity of a poset or modular lattice which are probably more familiar than m-dimension and breadth. These other measures share a feature:

they have a "direction" (so their values on a lattice do not, in general, coincide with their values on the opposite lattice).

The first such measure which I consider is Krull dimension in the sense of Gabriel and Rentschler. The second measure, considered in §6, is the "height" of a lattice. I will, also in §6, define a 2-valued rank (following Garavaglia) which will give us the Krull dimension, but which is finer and will allow us to encompass classical Krull dimension and a rank introduced by Pillay.

Krull dimension The Krull dimension of a modular lattice (or even of a poset) was introduced in [RG67] as a means of generalising the classical notion of Krull dimension for commutative noetherian rings. Rather than being defined in terms of the prime ideals of the ring, the definition of their dimension makes sense for any module (indeed, for any modular lattice) and is in terms of the lattice of all submodules. It turns out that, for commutative noetherian rings, this dimension coincides with the "classical Krull dimension" of the poset of prime ideals (see [GR73; §9]). This and related notions have become important tools in non-commutative ring theory.

Garavaglia [Gar80a] had the idea of using the Krull dimension of the lattice of pp-definable subgroups as a measure of the complexity of a theory of modules. He showed that when this dimension is defined (less than "∞") every pure-injective model is discrete. This work of Garavaglia was a major inspiration for the work of Ziegler described in this chapter.

Now, we will see (after **10.27**) that a modular lattice has Krull dimension iff it has m-dimension in the sense of §10.2: there are, however, a number of places where Krull dimension is a more natural measure than m-dimension. The way in which I define Krull dimension is not the usual one as seen, for example, in the monograph of Gordon and Robson [GR73] but rather is more in the spirit of §10.2. The idea is that a lattice with the dcc has Krull dimension 0; that a lattice has Krull dimension 1 if it does not have the dcc but, "when intervals with dcc are ignored" does have the dcc; and so on.

Therefore, let D be the class of all modular lattices with the dcc and let Kdim (for **Krull dimension**) be the corresponding notion of dimension as defined in §10.2. Thus, to measure the Krull dimension of a lattice L one forms the congruence "$\sim$" generated by all those sub-intervals of L which have the dcc - observe that, by **10.3**, $a > b$ in L are identified under this congruence iff $[a \vee b, a \wedge b]$ has the dcc. If $L' = L/\sim$ is trivial then $\mathrm{Kdim}\, L = 0$. If not, then apply the same process to L'; and so on. Recall what one does at limit ordinals λ: one defines L^{λ} to be L modulo the union of the $\sim^{\alpha}$ for $\alpha < \lambda$: then $L^{\lambda} = L/\sim^{\lambda}$ (in particular, L^{λ} cannot be trivial without L^{α} being so for some $\alpha < \lambda$, assuming that L has a 1 and 0). Then one sets $\mathrm{Kdim}\, L = \alpha$ if α is the least ordinal such that $L^{\alpha+1}$ is trivial. One may check that this does give the same notion as in [GR73] and [RG67] and gives the same value, modulo quibbles at limit ordinals (cf. [GR74]): the details of this are left as an exercise for the reader. The quibbles at limit ordinals are there because there is a choice of what to do at a limit stage: one may merely take the union of the congruences generated so far, or one may do this and then apply the derivative immediately.

If T is any complete theory of modules then by $\mathrm{EKdim}\, T$ is meant the Krull dimension of the lattice of pp-definable subgroups of any model of T. Ziegler uses "Kdim" but, since on occasion I will want to refer to the Krull dimension, $\mathrm{Kdim}\, M$, of the lattice of all submodules of some module M, I will follow Garavaglia and use $\mathrm{EKdim}\, M$ instead (for **elementary Krull dimension**).

<u>Proposition 10.25</u> [Gar80a; §2] *Let T be any complete theory of modules.*
(*a*) *T is totally transcendental iff* $\mathrm{EKdim}\, T = 0$.
(*b*) *If T is superstable then* $\mathrm{EKdim}\, T = 1$. ▫

Part (a) is obvious from 3.1, and part (b) also follows immediately from that result, on noting that a "finite index" interval certainly has the dcc. The converse of (b) is of course false since, for any theory T, T and $T^{\aleph_0}$ have isomorphic lattices of pp-definable subgroups but, if T is superstable and not totally transcendental, then $T^{\aleph_0}$ is not superstable.

Garavaglia also showed that if T^* is the largest theory of abelian groups then Kdim$T^* = 2$ (cf. 10.28 below). In fact the same is true for the largest theory of modules over the very similar ring $K[X]$ (K a field) and, indeed, over any commutative Dedekind domain. More generally, Ziegler showed that if R is commutative noetherian with every localisation a field or a discrete rank 1 valuation domain, then T^* has elementary Krull dimension ([Zg84; 8.2]).

There is little relation between the algebraic Krull dimension of a ring (which is the largest possible Krull dimension of a finitely generated module over that ring [GR73], so equals Kdim(R_R)) and the Krull dimension of the largest theory of modules. This may be seen by comparing these dimensions for $K[X]$ and $K[X,Y]$, where K is a field. The algebraic dimensions are respectively 1 and 2; the model-theoretic dimensions are respectively 2 and ∞. Of course we are not really comparing like with like: a better comparison would be the algebraic Krull dimension with the elementary Krull dimension of the largest theory of injective modules (to appreciate this, consider Gabriel dimension or quantifier-elimination - cf. 10.32 below) - and here we do find broad agreement. Another valid comparison would be between the algebraic Krull dimension of the functor category (mod-R,Ab) and the elementary Krull dimension of $T^*(R)$.

Let me also take the opportunity here to compare Krull dimension and m-dimension. The lemma which follows is obvious, since the congruence used to define Krull dimension is generated by the intervals with dcc, and these include the simple intervals (which generate the congruence used to define m-dimension).

<u>Lemma 10.26</u> *Let T be any complete theory of modules. Then* EKdim$T \leqslant$ m-dim$T$. □

One feature which separates Krull dimension from m-dimension is that the former has a "direction" in the sense that, in general, a lattice and its opposite do not have the same Krull dimension (consider any lattice with the acc but without the dcc); whereas it should be clear, from the definition, that a lattice and its opposite have the same m-dimension.

One may ask what relation there is between the dimension and the Krull dimension of a lattice. As already noted, one always has Kdim$L \leqslant$ m-dim$L$; in general one can say little else - for an ordinal, considered as a lattice, has Krull dimension zero but may have arbitrarily high m-dimension. On the other hand, one has the following (the proof is by induction since, if L has the acc, then a sub-interval has the dcc iff it has finite length).

<u>Lemma 10.27</u> *If the lattice L has the acc then* m-dim$L$ = Kdim$L$. □

Consider the following example. Let L be the chain consisting of elements a_{ij} $(i,j \in \omega)$ with the ordering given by: $a_{ij} < a_{i+1,k}$ and $a_{ij} > a_{i,j+1}$: then add a top and bottom element. Thus L consists of an infinite ascending chain where each "gap" is filled by an infinite descending chain; so L has neither the acc, nor the dcc. One easily checks that: Kdim$L = 1$; Kdim$L^{\mathrm{op}} = 1$ and m-dim$L = 2$.

The interested reader may work out the precise relationship between Kdim$L$, KdimL^{op} and dim$L$.

Although the dimensions grow at different rates, m-dimension and Krull dimension are co-extensive in the sense that, for any modular lattice L, one has Kdim$L < \infty$ iff m-dim$L < \infty$. The direction "$\Leftarrow$" is immediate, by 10.26). For the other direction, one need only note, for example, that if Kdim$L < \infty$ then L can contain no densely ordered subset (exercise) and so, by 10.6, m-dim$L < \infty$.

If R is a Dedekind domain (such as $\mathbb{Z}$ or $K[X]$) then the elementary Krull dimension of the theory of R-modules is 2. This was shown by Garavaglia. The proof is left as an exercise (one should find a suitable chain of pp formulas).

Theorem 10.28 [Gar80a; Thm 3], (also cf. [Zg84; 8.2]) *Suppose that* R *is a Dedekind domain which is not a field and let* T^* *be the largest theory of* R*-modules. Then* $\text{m-dim}\, T^* = \text{EKdim}\, T^* = 2$. □

The next result is immediate from the description of the pp-definable subgroups of an injective module given in 15.40 (for the definition of E-closed and the results used, see §15.1). Observe that we do not need to assume that the injective is coherent (has elimination of quantifiers) but only that the theory has an injective model - for each model displays the full complexity of the lattice of pp-definable subgroups, even if it does not realise every pp-type.

Proposition 10.29 *Let* E *be an injective module and let* T *be its complete theory. Then the elementary Krull dimension of* T *equals the Krull dimension of the opposite of the lattice consisting of the* E*-closures of those right ideals which are pp-definable as subgroups of* ${}_R R$. □

Corollary 10.30 *Let* E *be a coherent injective module (e.g., suppose that* R *is right coherent and let* E *be any injective) and let* T *be its complete theory. Then* $\text{EKdim}\, T = \text{Kdim}\,(\text{Latt}^{E\text{-fg}}(R))^{\text{op}}$ *where* $\text{Latt}^{E\text{-fg}}(R)$ *is the lattice of all* E*-finitely generated* E*-closed right ideals of* R.

In particular, if R *is right coherent, then the largest theory of injective* R*-modules* T_{inj} *has elementary Krull dimension equal to the Krull dimension of the opposite of the lattice of finitely generated right ideals of* R. □

Corollary 10.31 [Gar80a; Thm 5] *Suppose that the ring* R *has Krull dimension and let* E *be any injective* R*-module. Then* $\text{EKdim}\, E$ *exists and is bounded above by* $\text{Kdim}\,\text{Latt}^{\text{f}}(R)^{\text{op}}$. *Hence* E *has a decomposition as the injective hull of a direct sum of indecomposable injectives (a result due to Gabriel* [Gab62; p386, Thm 1]. *In fact* (1.12), *the same conclusion holds under weaker conditions).* □

If R is right noetherian, it follows that every injective module is t.t. (has EKdim 0) and it makes sense to consider the finer measure m-dimension. By 10.27 one obtains the following.

Corollary 10.32 *Let* E *be a* Σ*-injective* (= *t.t.* + *injective) module (for example, suppose that* R *is right noetherian and that* E *is any injective module). Then* $\text{m-dim}\, E = \text{Kdim}\,\text{Latt}^{E\text{-fg}}(R)$, *where* $\text{Latt}^{E\text{-fg}}(R)$ *is the lattice of* E*-closed right ideals of* R *(they are all* E*-finitely generated since* E *is* Σ*-injective).* □

Corollary 10.33 *If the ring* R *is noetherian then the largest theory* T_{inj} *of injective modules satisfies* $\text{m-dim}\, T_{\text{inj}} = \text{Kdim}\, R_R$. □

The **classical Krull dimension**, $\text{clKdim}\, P$, of a prime ideal P of the ring R is the foundation rank of P in the set, $\text{spec} R$, of all prime ideals of R, ordered by reverse inclusion. The **classical Krull dimension** of the ring is defined to be $\text{clKdim}\, R = \sup\{\text{clKdim}\, P : P \in \text{spec} R\}$. If R is commutative then its classical Krull dimension exists [GR73; §8]. This generalises the original definition of Krull dimension (as given in e.g. [Kap70]) and, insofar as the latter exists, coincides with it: Gordon and Robson show that for a commutative ring R one has $\text{clKdim}\, R = \text{Kdim}\, R$ [GR73; 8.12]. Furthermore, by 10.32, we have $\text{Kdim}\, T_{\text{inj}} = \text{Kdim}\, R$. Therefore we obtain the following corollary (if the prime P is non-minimal then work over R/P).

Corollary 10.34 *Suppose that R is a commutative noetherian ring and let P be a prime ideal of R. Then the (classical) Krull dimension of P is equal to the m-dimension of the theory of injective R/P-modules. As a theory of R-modules, this is obtained as* $\mathrm{Th}(\bigoplus\{E(R/Q)^{\aleph_0} : Q$ *is a prime with* $Q \geqslant P\})$.

It follows that $\mathrm{clKdim}\, R = \text{m-dim}\, T_{\mathrm{inj}}$, *where* T_{inj} *is the largest theory of injective R-modules.* □

Now we turn to commutative regular rings. This material overlaps with that in §16.2, but the approach here is different. A regular ring has Krull dimension iff it is semisimple artinian. It is, however, possible for the lattice of all (right) ideals to fail to have Krull dimension, while the lattice of all finitely generated right ideals has $\mathrm{Kdim} < \infty$. In fact, we will now see that this latter is the case for a commutative regular ring iff the CB-rank of the space of indecomposables (i.e. of $\mathrm{Spec}R$ - see §16.2) is defined. First, we need the fact that if R is a commutative regular ring then every complete theory of R-modules satisfies the condition ($\wedge$) (see 16.24). Therefore 10.19 applies and we deduce the following.

Theorem 10.35 [Gar80a; Thm 4] (cf. 16.25) *Suppose that R is a commutative regular ring and let T^* be the largest theory of R-modules. Then:*
$\text{CB-rk}\,\mathrm{Spec}R = \text{m-dim}\,T^* = \mathrm{EKdim}\,T^* = \mathrm{Kdim}(\mathrm{Latt}^f(R))^{\mathrm{op}}$.

Proof The first equality is 10.19, using condition ($\wedge$), and the third is by 10.30. So it remains to show that m-dimension and Krull dimension coincide. That is, it must be shown that an interval in the lattice of finitely generated right ideals which has the dcc must actually be of finite length. But that follows easily from 16.B. □

Here are some further examples, due to Garavaglia, of modules with elementary Krull dimension. The first result follows since, in a module over a commutative ring, every pp-definable subgroup is a submodule. The second then follows since every projective module is a direct summand of a free module (a module of the form $R^{(\kappa)}$).

Proposition 10.36 [Gar80a; 4.c] *Suppose that R is a commutative ring and let M be any R-module with Krull dimension. Then* $\mathrm{EKdim}\,M \leqslant \mathrm{Kdim}\,M$. □

Corollary 10.37 [Gar80a; 4.d] *Suppose that R is a commutative ring with Krull dimension. Let P be any projective module. Then* $\mathrm{EKdim}\,P \leqslant \mathrm{Kdim}\,R < \infty$. □

This also follows from 14.14.

Proposition 10.38 [Gar80a; Lemma7] *Let M be any module. Let I be any set and take F to be any filter on I. Then M has elementary Krull dimension iff M^I/F has, and then their dimensions are equal.* □

The last corollary follows since (Exercise 2.3/4) pp formulas commute with reduced products.

10.T T^{eq}

Now I say something about T^{eq}. The language for T^{eq} is that of T, expanded by new sorts, one for each definable equivalence relation on n-tuples. The elements of the sort corresponding to a given definable equivalence relation are to be thought of as equivalence classes for the relation. Also, to the language is added a function symbol, f_E, for each definable equivalence relation E: this function takes each n-tuple to its equivalence class mod E. For the complete definition, and discussion of the relation of T^{eq} to T, see, for example, [Mak84], [Poi85; §16.d]. The point is that properties such as stability, categoricity, complexity of models, ... are not changed in going from T to T^{eq}, but in T^{eq} discussion of many aspects becomes simplified. For instance, n-tuples of elements are just elements of a certain sort in T^{eq}. In the modules case, morphisms from a given finitely presented module to the model are just elements of one of the new sorts, as are the elements of quotients $\varphi(M)/\psi(M)$ of pp-definable subgroups (cf. end of §15.4; problem: clarify the relationship between T^{eq} and the classifying topos).

Here, we will be interested in sorts of the following kind. If φ is a pp formula, then there is a sort for cosets of φ (corresponding to the equivalence relation whose classes are the cosets of φ).

I will not give detailed proofs of the assertions that I make concerning this but, rather, indicate how this ties in with material that has been discussed elsewhere in the notes.

I will assume, for convenience, that $T = T^{\aleph_0}$.

Let ψ be a pp formula and let p be a type over 0 with $\psi \in p^-$. Then there is a corresponding complete type of cosets of ψ: I will denote this type by $[p/\psi](w)$. It says: "I am (a coset of ψ) not equal to ψ itself; for each $\varphi \in p^+$ I intersect the subgroup φ non-trivially and for each $\psi' > \psi$ with $\psi' \in p^-$, I am not contained in the subgroup ψ' (equivalently, I have trivial intersection with ψ' - this can be said by "there is no element a with $\psi'(a)$ and "$f_\psi a \in w$")". This does specify a complete type which (see below) is non-orthogonal to p.

We will see that types with minimal pairs thus "become" regular types in T^{eq}. Perhaps this gives some explanation of the importance of the property of having a minimal pair. It also means that §4.6 and §10.4 are even closer than one might have imagined.

I give a couple of examples to indicate what can happen in T^{eq}.

Example 1 $T = \mathrm{Th}(\mathbb{Z}_4{}^{\aleph_0})$. The possible U-ranks of elements are 0, 1 and 2. Let ψ be the pp formula which defines the group of elements of order 2 and let p be the type of an element of order 4. So $\mathrm{UR}(p) = 2$. Recall why this is so. Fix a model M: then p has a forking extension to M which is the type of an element of the form $m + a$ where m is an element of M of order 4 and a is an element of order 2 which is not in M. Consider the type of $[p/\psi]$: this is the type of a coset of the set of elements of order 2 and, from this point of view, the element $m + a$ is algebraic, since it is in $M + \tilde{M}2$. Thus one sees that $\mathrm{UR}([p/\psi]) = 1$. Note that p is non-orthogonal to $[p/\psi]$: clearly one increases in a model iff the other does.

In the same way, one sees that every type of the theory of $\mathbb{Z}_2{}^{\aleph_0} \oplus \mathbb{Z}_4{}^{\aleph_0}$ is non-orthogonal to a type of U-rank 1 (this illustrates a result of Lascar [Las84]), *provided* one works in T^{eq} (it is simply not true if one confines oneself to "elements" in the usual sense).

Example 2 $T = \mathrm{Th}(\mathbb{Z}_{(p)}{}^{\aleph_0})$. The same kind of argument applies. One may consider, for example, the type of a coset of $\tilde{M}p^3$ which "lies in" $\tilde{M}p^2$. Any forking extension of this type is algebraic, so it has U-rank 1. Thus we see that, although this theory is not superstable, nevertheless, every type is non-orthogonal, in T^{eq}, to a type with U-rank (=1).

Example 3 $T = \mathrm{Th}(\mathbb{Z}_{p^\infty})$. Consider the type, p, of a non-zero element of $\mathbb{Q}$. Since this type does not contain a minimal pair, fixing a coset of some formula in p^- does not allow us to drop the U-rank to 1. One sees, therefore, that p is not RK-equivalent to a type of U-rank 1.

A type $[p/\psi](w)$ forks if a formula is added which makes the difference between "w" and some named coset of ψ have non-empty intersection with some "ψ'/ψ" in $[p/\psi]^-$.

Using this, one sees that $[p/\psi]$ is RK-equivalent (in particular, is non-orthogonal) to p (consider $M \oplus N(p)$). The argument for regularity of types (6.23) applies as well in this more general situation, and we deduce that $[p/\psi]$ is regular iff there is no type q realised in $N(p)$ with $q^+ \supset p^+$ and $\psi \in q^-$: that is, $[p/\psi]$ is regular iff p is maximal in $N(p)$ such that $\psi \notin p$.

<u>Proposition 10.39</u> *If p is an irreducible type over 0 and $\psi \in p^-$ then $[p/\psi]$ is regular iff p is maximal among those types realised in $N(p)$ which omit ψ;*

this is the case iff p, *(or, rather, its pre-image in* $\mathrm{Th}(N(p))$*) is neg-isolated by* ψ *in* $\mathrm{Th}(N(p))$. ▫

Hence:

Theorem 10.40 *An irreducible type* p *is* RK-*equivalent to a regular type in* T^{eq} *iff* p *is related to a neg-isolated type hence, by* 9.24, *iff* p *is neg-isolated, in* $\mathrm{Th}(N(p))$. ▫

As a corollary (by 9.25 and 9.3):

Corollary 10.40a *If* $\text{m-dim}\, T < \infty$ *then every irreducible type is* RK-*equivalent to a regular type in* T^{eq}. ▫

Therefore, the analysis of §10.4 was based on removing successive layers of orthogonality classes of T-critical types.

Of course, if there are continuous pure-injectives (for example, if $w(T)=\infty$ and R is countable) then it is not even true that types of weight one suffice to classify the pure-injective models.

In the gap between $\text{m-dim}\, T=\infty$ and $w(T)=\infty$ it appears to be that it may or may not be the case, for a particular example, that regular types in T^{eq} suffice. For instance, it appears that certain algebras of tame but non-domestic representation type (see Chapter 13) provide examples where every indecomposable pure-injective realises a neg-isolated type and so there are enough regular types in T^{eq} to classify the models, but $\text{m-dim}\, T^* = \infty$.

For an example where there are not enough regular types in T^{eq}, take R to be a complete valuation domain with infinitely generated maximal ideal. Then the finitely generated ideals form a densely ordered chain and R is an indecomposable pure-injective (see §10.V). If a is a non-zero element of R, then the type of a is not neg-isolated (by 9.12, since this is true of the element 1). So regular types in T^* do not suffice, yet (10.V3) $w(T) < \infty$.

10.6 Dimension and height

height Given a lattice L and an element a of it, the **height** of a is defined to be its foundation rank. Dually, the depth of a, $\mathrm{dp}(a)$, is the height of $a^{(\mathrm{op})}$ in L^{op}.

For example, if L is an ordinal considered as a poset, then the height of an element is just itself. If R is a commutative noetherian ring and if spec is its set of prime ideals, ordered by inclusion, then the depth of a prime is just its Krull dimension in the classical sense. Chapter 5 contains many special cases of height (U-rank, pp-rank,..).

The relation between Krull dimension and height Suppose that the lattice L has the acc: so each point a has associated to it the Krull dimension, $\mathrm{Kdim}[1_L, a] = \text{m-dim}[1_L, a]$, and its depth. The basic result is the following (also cf. [Bas71], [Gul73], [Kr73], [Rh74], [GZ86]).

Proposition 10.41 [GR73; 10.2] *Let* L *be a modular lattice with the ascending chain condition and let* a *be an element of* L. *Set* $\alpha = \mathrm{Kdim}[1_L, a] = \text{m-dim}[1_L, a]$. *Then:* $\omega^{\alpha} \leqslant \mathrm{dp}(a) < \omega^{\alpha+1}$.

Proof The proof is by induction on α. In the case $\alpha=0$, the interval $[1,\alpha]$ has the dcc and hence has finite length, as required.

Suppose that the result is true for intervals of m-dimension strictly less than α. Let $\approx$ be the congruence which collapses all intervals of m-dimension strictly less than α: so the assumption is that, in the quotient lattice $L/\approx$, $[1,a/\approx]$ is of finite length. Suppose that $1=b_0/\approx > b_1/\approx > \ldots > b_n/\approx = a/\approx$ with each gap simple, $b_i \in L$ and $b_i > b_{i+1}$ for each i.

Then, in the interval $[b_i, b_{i+1}]$ in L, each element is $\approx$-equivalent to either b_i or b_{i+1}. Take (by acc) c in $[b_i, b_{i+1}]$ maximal such that $c \approx b_{i+1}$. Then, if $b_i \geqslant d > c$, one has $b_i \approx d$ and so, by the induction hypothesis, the depth of d in this interval satisfies $\omega^\beta \leqslant d < \omega^{\beta+1}$ for some $\beta < \alpha$. So, by definition of depth, the depth of c in $[b_i, b_{i+1}]$ is $< \omega^{\alpha+1}$ (also, by an obvious induction, it is at least ω^α). Hence, using the induction hypothesis, the depth of b_{i+1} in $[b_i, b_{i+1}]$ satisfies the inequality in the statement of the result.

So, by additivity of depth (cf. **10.43**), the result follows. □

<u>Corollary 10.42</u> [Gar80a; Thm 7] *Let T be a totally transcendental theory of modules, and let φ be a pp formula. Then*:
$\omega^{\text{m-dim}(\varphi)} \leqslant \text{pp-rk}(\varphi) < \omega^{\text{m-dim}(\varphi)+1}$. □

Suppose that the lattice L has Krull dimension (the example that we have in mind is the lattice of pp-definable subgroups). To each member a of L we assign a pair of ordinals. The first is the Krull dimension of the interval $[a,0]$ and the second is the foundation rank of the equivalence class of a in the well-founded lattice $[a,0]/\approx$, where $\approx$ is the congruence which collapses intervals of Krull dimension strictly less than the Krull dimension of $[a,0]$. Notationally, we will simplify by writing $\text{Krk}(a) = (\alpha,\beta) = (\text{K}(a), \text{rk}_{\text{K}(a)}(a))$.

This definition has more than one source. It was defined by Garavaglia in the first version of his paper **[Gar80a]** on elementary Krull dimension. The specific result for which it was introduced was superceded in the second version, so this Ord×Ord-valued rank makes no appearance in the latter. Essentially the same idea is well-known in the study of Krull dimension as a ring-theoretic tool (see, e.g., **[GR73]**). There is a notion there, termed "α-critical", which is really just our "α-regular" below. For the special case of Krull dimension zero (i.e. totally transcendental theories) Pillay (**[Pi84]**) introduced a rank closely linked to rk_0. Work of Lascar (**[Las84]**) on the relation between the U-rank and regularity of a type is influential here. Finally there are analogous ideas of Berline and Lascar in **[Be82]**, **[BeLa86]**.

The definition is extended to types by setting $\text{Krk}(p) = \min\{\text{Krk}(\varphi) : \varphi \in p^+\}$ – since Krk takes values in the lexicograhic product of two ordinals, this minimum does exist. Note that $\text{Krk}(p) = \text{Krk}(\varphi)$ for some $\varphi \in p^+$.

The point of introducing this rank is that it allows a finer analysis of types – one which parallels the use of U-rank in theories which are superstable. It should be clear that if we are dealing with a theory T satisfying $T = T^{\aleph_0}$, then a type p satisfies $\text{Krk}(p) = (0,\beta)$ iff p is a type having U-rank and with U-rank equal to β. I make the following tentative definitions.

First we define Krk for sub-intervals of L in the obvious way; if $[a,b]$ is a sub-interval of L then we define $\text{Krk}(a/b)$ to be $\text{Krk}_{[a,b]}(a)$.

Suppose that q is an extension of the type p where $\text{K}(p) = \alpha$. Say that q is an **α-forking** extension of p if either $\text{K}(q) < \alpha$ or $\text{K}(q) = \text{K}(p)$ and $\text{rk}_\alpha(q) < \text{rk}_\alpha(p)$. In particular then, an α-forking extension of a type is a forking extension, but the converse is in general false. The idea is that we ignore "small amounts" of forking.

For example, let T be the theory of $\mathbb{Z}_{(p)}{}^{\aleph_0}$ and let p be a type with $G(p) = M$: so $\text{Krk}(p) = (1,1)$. Let q be any forking extension of p with $M/G(q)$ p-torsion. Then q is not a 1-forking extension of p, because q does not have U-rank (observe that Krk cannot take values of the form $(\alpha,0)$ with $\alpha \geqslant 1$). If q is a non-algebraic 1-forking extension of p then the hull of the unlimited part of q is $\mathbb{Q}$.

This definition having been made, one is tempted to make some others. In particular, we may say that a type p is **α-regular** if $K(p)=\alpha$ and if p is orthogonal to each of its α-forking extensions.

Again let T be the theory of $\mathbb{Z}_{(p)}{}^{\aleph_0}$ and let p be any type with hull $\overline{\mathbb{Z}(p)}$. Then p is not regular - indeed the hull of p has no minimal pp-definable subgroup, so p is not even related to a regular type. On the other hand p is 1-regular. An example of a type which is not regular in this sense is tp(1) in $\mathbb{Z}_4{}^{\aleph_0}$.

The next result is in the first version of [Gar80a] (the case $\alpha=0$ survives as Lemmas 12 and 13 in [Gar80a]) and follows since rk_α is foundation rank in the lattice of pp formulas modulo $\sim_\alpha$. The case $\alpha=0$ is just Lascar's inequalities [Las76; §5] (see [Pi83] or [Poi85]) for the case of modules (by **5.12**). To see that the left-hand inequality should not be equality, consider the lattice of pp-definable subgroups of $\mathbb{Z}_{2\infty}\oplus\mathbb{Z}_2$, take a to be $v=v$, b to be $v2=0$ and c to be $v=0$.

<u>Proposition 10.43</u> *For any* $a\geqslant b\geqslant c$ *in the modular lattice* L *such that* $K(a)=\alpha$ *one has:* $\mathrm{rk}_\alpha(b/c)+\mathrm{rk}_\alpha(a/b)\leqslant \mathrm{rk}_\alpha(a/c)\leqslant \mathrm{rk}_\alpha(b/c)\oplus\mathrm{rk}_\alpha(a/b)$ *(here, "+" denotes ordinal addition and "⊕" denotes natural sum of ordinals - see, e.g.,* [Poi85; §19.b]). □

For totally transcendental theories of modules the definition of Krk allows us to connect and unify various of these ranks, including that of Pillay defined below. But the main point of introducing this 2-rank is that it gives a good generalisation of U-rank to non-superstable theories (of modules, but the question is how far can one take this in the general case - is modularity of the fundamental order enough?). For, although outside the superstable case one cannot expect to find enough regular types (at least in T) to classify the sufficiently saturated modules, one may ask when there are enough weight one types to do the job. The effectiveness of concentrating on types of weight one rather than on regular types is well-illustrated in Ziegler's paper [Zg84].

Pillay's rank In [Pi84], Pillay introduced a rank which provides a "layering" of orthogonality classes of regular types in non-multidimensional ω-stable theories. In particular, his results apply to modules. I describe briefly what he does, and then compare this notion of "rank" to those which we have considered already.

The ubiquituous theory is supposed at this stage only to be complete and stable. Let Ψ be a set of formulas and let φ be a formula. Then φ is **Ψ-algebraic** if whenever $M\prec_\Psi N$ one has $M\prec_\varphi N$. The notation $M\prec_\Psi N$ means that no formula $\psi\in\Psi$ increases between M and N: that is, if $\psi\in\Psi$ then $\psi(M)=\psi(N)$. So φ is Ψ-algebraic iff it cannot increase between two models without some formula in Ψ increasing. For example, $\emptyset$-algebraic is just "algebraic" in the usual sense.

Say that φ is **Ψ-minimal** if it is not Ψ-algebraic and, whenever $M\prec_\Psi N$, if $a,b\in\varphi(N)\setminus M$ then $\mathrm{tp}(a/M)=\mathrm{tp}(b/M)$. Thus $\emptyset$-minimal is just strongly minimal.

Pillay shows that if such φ and Ψ are with parameters in A then there is some stationary type p over A such that, whenever $A\subseteq M\prec_\Psi N$ and $a\in\varphi(N)\setminus M$, one has that the type of a over M is the non-forking extension of p to M. Moreover, (p,φ) is strongly regular.

Specialising now to the case of T non-multidimensional ω-stable, he defines a process rather like that seen in §10.4. He works over the prime model, M_0, for convenience. Let Ψ_0 be a maximal set of inequivalent $\emptyset$-minimal formulas over M_0 (the equivalence relation is "φ_1 is $\Psi\cup\{\varphi_2\}$-algebraic" - it is shown that this is an equivalence). Then let Ψ_1 be a maximal set of inequivalent Ψ_0-minimal formulas over M_0. And so on (through the ordinals). To each formula in Ψ_α there corresponds a strongly regular type as described above. Pillay shows that all orthogonality classes of regular types are reached in this way (and the layering is independent of choices made). The rank of (the orthogonality class of) a regular type is defined to be the ordinal α such that the type is non-orthogonal to the type corresponding to a formula in Ψ_α. The rank of T is defined to be the maximum α such that Ψ_α is non-empty.

Specialising to modules, one may see that negations of pp formulas do not contribute to the property of Ψ-minimality, so the formulas involved in the above may be taken to be pp. Thus, the orthogonality classes of rank 0 are those containing what I called a T-critical type (in §4.6). The rank 1 classes are those which are T_1-critical, where T_1 is the "derivative" of T which is obtained by removing all the points of $\mathcal{I}(T)$ which realise a T-critical type (note that all these are isolated). And so on.

The example $\mathbb{Z}_2{}^{\aleph_0} \oplus \mathbb{Z}_4{}^{\aleph_0}$ shows that this does not coincide with the derivative T' of §10.4, for Ψ_0 contains only the orthogonality class of $\mathbb{Z}_4$; so the rank of T is 1. Pillay then observes that in T^{eq} the rank of this example is 0, which is in accord with the rank in §10.4. Indeed, it seems that, as with Lascar's results on regular types and U-rank, one should work in T^{eq} to obtain "nice" results.

Pillay defines the derivative implicit in his rank: the "CB-rank" on the (unlimited) indecomposables which I described above (i.e, remove only T-criticals at each stage, rather than removing all isolated points), and then shows that the rank of a type in his sense does correspond to the "CB-rank" (in the sense just described) of its hull.

It does seem more reasonable to work in T^{eq} where this derivative is precisely the CB-derivative on $\mathcal{I}(T)$, for then Pillay's rank exactly corresponds with the CB-rank. By the results of §10.4 and **10.41**, one sees that the rank, in Pillay's sense, of a regular type p is exactly that α such that $\omega^\alpha \leqslant \mathrm{UR}(p) < \omega^{\alpha+1}$ - i.e., that α such that p is non-orthogonal to a type in T^{eq} of U-rank ω^α.

In Pillay's paper the above equality was left as an open question: an answer was given by Baudisch [Bd8?], who gave a sufficient condition (one satisfied by modules) on a non-multidimensional ω-stable theory for Pillay's rank to equal Lascar's "α" above (the condition concerns the relation between Morley rank and U-rank in T^{eq}). Certainly some condition on the theory is necessary: Pillay provides [Pi84] an example of a non-multidimensional ω-stable theory for which, even in T^{eq}, his rank and the rank "α" as above, are different.

10.V Valuation rings

A **valuation ring** is a commutative ring whose ideals are linearly ordered: it follows that every finitely generated ideal is principal. Such a ring has a unique maximal ideal, which will always be denoted by P. We shall be concerned with **valuation domains** (i.e. valuation rings without zero-divisors). If R is a valuation domain then it has a field of fractions Q: the R-submodules of Q are linearly ordered and are called **fractional ideals** of R (so these include the ideals of R and, note, I include, for convenience, Q among the fractional ideals). The reader may consult [End72], for example, for more about valuation domains, including the way in which they arise: for valuation rings in general, consult [FS85] (especially see Chpt XI, on pure-injective modules). Ziegler [Zg84; §5] classified the indecomposable pure-injectives over a valuation domain and showed that there are no continuous pure-injectives (this extends results of Kaplansky and Warfield). The classification over discrete rank 1 valuation domains was given in §2.$\mathbb{Z}$. The general classification will be described in this section. These rings also provide us with useful (counter-)examples.

A valuation domain is **discrete rank 1** if its ideals are exactly $R > P > P^2 > \ldots > P^n > \ldots 0$. These are the only noetherian valuation domains. (Often these valuation domains are called just "discrete", but there is a more general notion of discrete valuation domain, see [FS85; §I.2]).

Given any field K and totally ordered abelian group Γ, there is a valuation domain R with **residue class field** (i.e., R/P) K and **value group** (Krull, see [FS85; I.3.4]) Γ: then the principal ideals of R correspond bijectively to the positive elements of Γ.

An embedding $R \longrightarrow S$ of valuation domains is an **immediate extension** if the obvious maps between the respective lattices of ideals ($I \mapsto IS$ and $I' \mapsto I' \cap R$) are inverse bijections (so the value groups are isomorphic) and if the inclusion induces an isomorphism between the respective residue class fields: $R/P \simeq S/PS$. If $R \longrightarrow S$ is an immediate extension and if I is an ideal of R then I is finitely generated iff IS is so ([FS85; Exercise I.1.5]). The valuation domain R is **maximal** if it has no proper immediate extension: this is so iff R is complete in the topology which has as a neighbourhood base at 0 all the ideals. In particular, a maximal valuation domain is pure-injective. The most familiar example is the p-adic integers $\overline{\mathbb{Z}(p)}$. Every valuation domain has an immediate extension which is maximal. Much of this can be extended to valuation rings (see [FS85]).

<u>Lemma V1</u> [Zg84; §5] *Suppose that R is a valuation domain. Then the pp-definable subgroups of R are just the principal (=finitely generated) ideals.*

Proof The direction "$\Leftarrow$" is clear. The other direction depends on Warfield's characterisation of the finitely presented modules as the direct summands of direct sums of cyclic modules [War70; Thm 3]. For then, any matrix over R is equivalent to a "diagonal" one (see [FS85; Exercise II.3.8]), and the proof of **2.Z1** applies (I leave the details, and the details of the corollary which follows, as an exercise). □

<u>Theorem V2</u> [Zg84; after 5.1] (also see [War69; Cor 5]) *Suppose that R is a valuation domain. Then every pp formula in one free variable is equivalent, in every R-module, to a conjunction of formulas of the kind: $\varphi_{r,s}(v) \equiv \exists w\,(vs = wr)$. Moreover, there are no implications between these formulas beyond the obvious ones. In particular, the largest theory of R-modules has m-dimension less than "∞" iff the lattice of finitely generated ideals contains no densely ordered subset.* □

For m-dimension, see §10.2. What Warfield shows is that if R is a Prüfer domain, then purity is equivalent to the (in general weaker) RD-purity ("relative divisibility"). A module M is **RD-pure** in the module N if $Mr = Nr \cap M$, for each element $r \in R$.

By **V2** the pp-type of an element says (i) what the annihilator is, and (ii) to what extent the element and its multiples are divisible. It follows (exercise: cf. proof of **2.Z3**) that the indecomposable pure-injectives are obtained from the pairs $I > I'$ of fractional ideals *via* the corresponding pp-type, $p(v)$, which says "ann$\,v = I'$; every element of I divides v; vs is divisible by r only if $r \in Is$". From this we get the following description of the indecomposable pure-injectives ([War69; §6] contains partial results).

<u>Theorem V3</u> [Zg84; Remark after 5.1, Example(i) after 7.3] *Let R be a valuation domain, with quotient field Q. Then the indecomposable pure-injectives are the modules of the form $\overline{A/B}$, where $A > B$ are fractional ideals (with the possibility that A is Q). One has $\overline{A/B} \simeq \overline{C/D}$ iff there is $x \in Q\setminus\{0\}$ with $C = xA$ and $D = xB$. Furthermore, there are no continuous pure-injectives (indeed, the width, in Ziegler's sense, of the largest theory of R-modules is 2).*

Proof The last statement follows (with a bit of work) from the description of the pp formulas in **V2** and the description of width (and breadth) in §10.2 (going from a discrete rank 1 to a general valuation domain replaces the congruence which collapses two-point intervals with that which collapses intervals which are chains). The details are left to the reader. ▫

Exercise 1 Describe the topology on the space of indecomposables.

Together with 2.$\mathbb{Z}$8, this gives the complete list of indecomposable pure-injectives over a Prüfer domain ([War69; §6] has partial results).

Facchini [Fac8?; 5.9, 6.9, 6.4] extends, by different techniques, Ziegler's result to Prüfer rings (i.e., the ring is not assumed to be a domain). Also, Monari-Martinez [M-M84] re-derives the first part of V3 using purely algebraic arguments.

There is a detailed discussion in [FS85; ChptXI] of pure-injective modules over valuation domains (and, more generally, over valuation rings), especially in terms of height functions.

Example 1 Suppose that R is any valuation domain (not a field) and consider the theory of R (or $R^{\aleph_0}$ if P is finitely generated and the residue field R/P of R is finite). By V1 and V3, the members of $\mathcal{I}(R)$ are the isomorphism classes of fractional ideals of R (for every model is torsionfree and $R^{\aleph_0} \equiv R^{\aleph_0} \oplus I$ for every fractional ideal I). We always have $\overline{R}$ in $\mathcal{I}(R)$; necessarily not isomorphic to $\overline{R}$ is its quotient field Q (the hull of the unique type of U-rank 1). If the maximal ideal P is infinitely generated then we obtain $\overline{P}$ as a third member of $\mathcal{I}(R)$. Since every fractional ideal I other than Q is R-isomorphic to an ideal of R (exercise: consider a non-zero element of $\{r \in R : r^{-1} \notin I\}$) the members of $\mathcal{I}(T)$ may be characterised as the isomorphism classes of ideals, together with Q.

Example 2 [Zg84; Example after 9.3] The valuation domain R is archimedean if P is the only non-zero prime ideal of R; equivalently, if $\bigcap_n P^n = 0$. An equivalent condition is that the poset of non-zero finitely generated ideals of R be order-isomorphic to the positive elements of an additive subgroup (the value group) of the reals (any additive subgroup of the reals can be realised). In such a ring, the maximal ideal is finitely generated iff the ring is discrete rank 1.

Suppose that R is archimedean. Then every non-zero ideal of R is elementarily equivalent to R. For if I is any non-zero ideal of R and if r is a non-zero element of R then $I > Ir$ (otherwise I would be idempotent, contradicting that the intersection of the powers of P is the zero ideal): then check the invariants, using V1. Indeed, by [FS85; Exercise 4.3], all non-zero ideals are elementarily equivalent to R iff R is archimedean.

If R is discrete rank 1, then $\mathcal{I}(R) = \{\overline{R}, Q\}$. If R has value group $\mathbb{R}$ then (see [FS85; Example I.4.3]) $\mathcal{I}(R) = \{\overline{R}, \overline{P}, Q\}$. If R has value group $\mathbb{Q}$ then (see [FS85; Example I.4.4]) there are $2^{\aleph_0}$ non-isomorphic ideals, so $\mathcal{I}(R)$ has $2^{\aleph_0}$ points, yet $|\mathcal{I}(R)/\approx| = 2$! In this last example, since the pp-definable subgroups are densely ordered, no indecomposable pure-injective contains a minimal pair.

In connection with valuation domains, Shelah has shown [She86; §5] that there is a valuation domain which has a uniserial module (i.e., its submodules are linearly ordered) which is not simply the image of a fractional ideal.

CHAPTER 11 MODULES OVER ARTINIAN RINGS

In this chapter we consider those rings which are simplest, as measured by the complexity of the category of modules: the rings of finite representation type or, more generally (?), the right pure-semisimple rings.

A ring is right pure-semisimple iff every module over it is a direct sum of indecomposable submodules. These are precisely the rings whose every right module is totally transcendental (§1). To put this another way: a ring is right pure-semisimple iff the lattice P of pp-types has the ascending chain condition - so right pure-semisimplicity is the "with quantifiers" version of the right noetherian condition. The first section contains various equivalents to right pure-semisimplicity, as well as "local" versions of some results (i.e., applying to arbitrary theories closed under product, rather than just to T^*). I also include a proof of the fact that a right pure-semisimple ring is right artinian and that each of its indecomposable modules is of finite length.

A ring is of finite representation type if it is right pure-semisimple and if there are, up to isomorphism, only finitely many indecomposable modules. It is not known whether or not a right pure-semisimple ring must be of finite representation type. If the ring is an artin algebra, then the concepts are equivalent (though I don't prove that here): the artin algebras include the algebras which are finite-dimensional over a base field. We see (§2) that an artin algebra is of finite representation type if it has only finitely many indecomposable finitely generated modules (in fact, this is true of any right artinian ring). It is also seen that a countable ring is of finite representation type iff it has, up to isomorphism, fewer than $2^{\aleph_0}$ countable modules.

This chapter is concerned with finitely generated modules over right artinian rings. Such modules need not be pure-injective: yet we wish to use pp-types and the associated techniques, without having to step beyond the realm of finitely generated modules. In the third section, a theory of hulls is developed for finitely generated modules over right artinian rings. It is shown that if a is an element of such a module M, then there is a minimal direct summand of M containing a; this summand is unique up to a-isomorphism and depends only on the pp-type of a. We are then able to use these finitely generated analogues of hulls, more or less as we used the hulls of §4.1. Although we do not use the fact, it is nice to know that the two notions of hull fit together in the sense that the "finitely generated" hull of an element purely embeds in the hull of the element, and the one module is indecomposable iff the other is.

The main result of the fourth section is that a ring is of finite representation type iff every module over it has finite Morley rank. This second condition is equivalent to the lattice of pp-types having finite length. Thus finite representation type may be viewed as the "with quantifiers" version of the right artinian property. Our proof of the result is self-contained for artin algebras but, for general right artinian rings, we have to quote a result of Auslander which says, in effect, that if R is a ring of finite representation type then every irreducible type is neg-isolated. Indeed, this condition on irreducible types characterises the rings of finite representation type.

There is a supplementary section on the "pathologies" one encounters if no restriction is placed on the modules under consideration.

Throughout the chapter, I have made some attempt to distinguish between what is true for any totally transcendental theory closed under products and what is at least less local than this (though the correct setting for some of the results eludes me).

11.1 Pure-semisimple rings

The ring R is said to be **right pure-semisimple**, **rt. pss** for short, if every right R-module is a direct sum of indecomposable submodules. For instance, semisimple artinian rings are right pure-semisimple; for other examples, see below. As is suggested by the terminology, these generalise semisimple artinian rings, which may be seen as the quantifier-free version of right pure-semisimple rings (the modules over a semisimple artinian ring have complete elimination of quantifiers, so all embeddings are pure).

It turns out that, over a right pure-semisimple ring, every indecomposable module is finitely generated. The same class of rings is defined by the requirement that there be a cardinal κ such that every module is a direct sum of submodules each of cardinality no more than κ. It is not enough to require that there be only a set of indecomposable modules, as is shown by any regular ring which is not actually semisimple artinian.

Proposition 11.1 *Let T be a theory of modules, not necessarily complete, which is closed under products. Suppose that every model of T is a direct sum of indecomposable submodules. Then T is totally transcendental.*

Proof [Pr84; 2.1] Set $T_1 = \mathrm{Th}(\oplus\{M_{T'} : T'$ is a complete extension of T and $M_{T'}$ is any chosen model of $T'\})$. Thus T_1 is the join, in the sense of §2.6, of the various T' and is itself a complete extension of T. So, if M is a model of T then M purely embeds in a model of T_1. Thus we reduce to the case where T is complete.

So suppose that T is complete. Recall that a module is totally transcendental iff it is Σ-pure-injective (3.2). I show that every module is t.t. by establishing first that every pure-injective module is Σ-pure-injective.

So let N be pure-injective. By assumption, one has a decomposition $N = \oplus_\Lambda N_\lambda$ where each N_λ is indecomposable and, being a direct summand of N, is also pure-injective.

Then $\mathrm{pi}(N^{(\aleph_0)}) = \mathrm{pi}(\oplus_\Lambda N_\lambda{}^{(\aleph_0)}) = \mathrm{pi}(\oplus_\Lambda \oplus_{i\in\omega} N_{\lambda,i})$ say, where for each $i \in \omega$ one has $N_{\lambda,i} \simeq N_\lambda$. By assumption, $\mathrm{pi}(N^{(\aleph_0)})$ has a representation $\oplus_\Sigma M_\sigma$ (say), where again the M_σ are indecomposable pure-injectives. Now, by 4.A14 the two decompositions $\mathrm{pi}(\oplus_{\Lambda\times\omega} N_{\lambda,i})$ and $\oplus_\Sigma M_\sigma = \oplus_\Sigma \overline{M}$ are essentially the same, in the sense that there is a bijection $f: \Lambda\times\omega \longrightarrow \Sigma$ such that $N_{\lambda,i} \simeq M_{f(\lambda,i)}$ for each λ, i.

Thus:
$N^{(\aleph_0)} \simeq (\oplus_\Lambda N_\lambda)^{(\aleph_0)} \simeq \oplus_{\Lambda\times i} N_{\lambda,i} \simeq \oplus_{\Lambda\times i} M_{f(\lambda,i)} \simeq \oplus_\Sigma M_\sigma = \mathrm{pi}(N^{(\aleph_0)})$. That is, $N^{(\aleph_0)}$ is pure-injective and so N is Σ-pure-injective, as desired.

Thus every module purely embeds in a t.t. module (namely its pure-injective hull). Hence every module is t.t. (3.7); as required. □

Applying this to T^*, we obtain the following characterisation of right pure-semisimple rings (the direction "$\Leftarrow$" by 3.14).

Theorem 11.2 *The ring R is right pure-semisimple iff every right R-module is a totally transcendental.* □

So we obtain the following list of equivalents.

Corollary 11.3 *The following conditions on the ring R are equivalent:*

(*i*) *R is right pure-semisimple;*

(*ii*) *every module is pure-injective;*

(*iii*) *every direct sum of pure-injective modules is pure-injective;*

(*iv*) *every pure-injective module is Σ-pure-injective.* □

I now go on to derive further equivalent characterisations of and information about such rings. I should say at this point that I am not going to try to carefully assign credits for these results, since they evolved over some time around the early/mid 70's and in a number of papers: see [Pr84] for references. I do, however, take the opportunity to add the following to the list of references in [Pr84]: [Aus71], [Gri70a], [KS75], [She77].

The next theorem replaces indecomposables by modules of bounded size: but the result is the same. The injective case is the Faith-Walker theorem [FW67; 1.1].

Theroem 11.4 [Gar80; Lemma 4] *Suppose that* $T=T^{\aleph_0}$ *is a theory of modules. Then the following conditions are equivalent:*

(i) T *is totally transcendental;*

(ii) *there is a cardinal* κ *such that every model of* T *is a direct sum of submodules each of cardinality no more than* κ;

(iii) *there is a cardinal* κ *such that every model of* T *purely embeds in a direct sum of modules each of which may be purely embedded in* $\tilde{M}$ *and is of cardinality no more than* κ.

Proof As in the first result, we may assume that T is actually complete. Already we have (i)⇒(ii) by **3.14** - the existence of κ is obvious since each summand may be taken to be the hull of a single element. Also (ii)⇒(iii) is trivial.

(iii)⇒(i) Since T is closed under products, it will be enough to show that T is superstable (3.3). This will be done by directly verifying the defining condition (3.A and comment following that). So let A be any subset of the monster model and let $N(A)$ be a copy of the hull of A. Let us assume, for convenience, that κ is infinite.

By assumption, there is a pure embedding of $N(A)$ into $N''=\oplus_\Lambda N_\lambda$ where, for each $\lambda\in\Lambda$, one has that N_λ is a direct summand of $\tilde{M}$ and $|N_\lambda|\leqslant\kappa$. Since T is closed under products it follows that N'' also purely embeds into $\tilde{M}$. In particular, $N(A)$ has the same pp-type whether regarded (algebraically) as a pure submodule of N'' or as sitting inside $\tilde{M}$. So we may as well suppose that we are working inside the monster model.

Each element $a\in A$ is contained in a finite sub-sum of N''. Hence A is contained in a direct summand, $N'=\oplus_{\Lambda'}N_\lambda$, of N'' with $|\Lambda'|\leqslant|A|+\aleph_0$ and hence with $|N'|\leqslant\kappa(|A|+\aleph_0)$.

Since A is contained in the pure submodule N' of $\tilde{M}$, one has that its pp-type is the same, whether measured in $N(A)$, $\tilde{M}$ or N'. So the morphism $f:N(A)\longrightarrow N'$, which is the composition of the pure embedding of $N(A)$ in N with the canonical projection from N'' to N', preserves the pp-type of A. By **4.14** it follows that f preserves the pp-type of $N(A)$. In particular, f is an embedding of $N(A)$ into N'. Hence $|N(A)|\leqslant|N'|$, so $N(A)$ has cardinality bounded by $\kappa|A|$.

We show that T is superstable, by counting types over A. Set $\tilde{M}=N(A)\oplus M$ for some M. If p is a 1-type over A then take any realisation $c=(a_0,b)\in N(A)\oplus M$ of it.

Suppose also that the 1-type q, over A, is realised by $c'=(a_0,b')\in N(A)\oplus M$ where $\text{tp}(b)=\text{tp}(b')$. It is shown that q must equal p (so a bound on $|S_1(A)|$ will be obtained). Let $\varphi(v,\bar{a})$ be pp with $\bar{a}$ in A. Then the following assertions are equivalent: $\varphi(v,\bar{a})\in p(v)$; $\varphi(c,\bar{a})$ holds; $\varphi(a_0,\bar{a})\wedge\varphi(b,\bar{0})$ holds (on projecting); $\varphi(a_0,\bar{a})\wedge\varphi(b',\bar{0})$ holds (by assumption); $\varphi(c',\bar{a})$ holds (on adding); $\varphi(v,\bar{a})\in q(\bar{v})$. Thus $p^+=q^+$ and so (2.17) $p=q$, as desired.

There are at most $|N(A)|\leqslant\kappa(|A|+\aleph_0)$ choices for a_0 and at most $|S_1(0)|\leqslant 2^{|T|}$ choices for $\text{tp}(b)$. Hence $|S_1(A)|\leqslant\kappa(|A|+\aleph_0).2^{|T|}$. In particular, if $|A|\geqslant\kappa.2^{|T|}$ (a constant) then $|S_1(A)|\leqslant|A|$ - so T is superstable, as required. ▫

Corollary 11.5 *Suppose that* T *is a complete theory of modules such that there exists a cardinal* κ *with* $|N(A)|\leqslant\kappa|A|$ *for every subset* A *of the monster model. Then* T *is superstable.*

Proof This was shown in the last part of the proof of **11.4**. ▫

Corollary 11.6 *The following conditions on the ring* R *are equivalent:*

(i) R *is right pure-semisimple;*

(ii) *there exists a cardinal* κ *such that every module is (or even, purely embeds in,) a direct sum of modules, each of cardinality bounded by* κ;

(*iii*) *every module is totally transcendental;*
(*iv*) T^* *is totally transcendental;*
(*v*) *the lattice* $\mathrm{P}(R)$ *of pp-1-types has the ascending chain condition.*

Proof The equivalence of (iv) and (v) is immediate from 3.1(c); that of (iii) and (iv) is clear by 3.7. Setting $T=T^*$ in 11.4 yields (ii)$\Leftrightarrow$(iv). The equivalence of (i) and (iii) is 11.2. ▫

The equivalence of (i) and (v) in 11.6 says that right pure-semisimplicity is the pp-version of the right noetherian condition. It will turn out that finite representation type is the corresponding strengthening of the right artinian condition.

Setting $T=T^*$ in 11.5 and applying 11.6 yields the next result.

Corollary 11.7 *The ring* R *is right pure-semisimple iff there exists* κ *such that for every* A $(\subseteq \tilde{M} \models T^*)$ *one has* $|N(A)| \leqslant \kappa|A|$. ▫

The corollary above is not "purely algebraic", since it explicitly refers to the pp-type of A; but there is the following stronger result.

Corollary 11.8 *The ring* R *is right pure-semisimple iff there is a cardinal* κ *such that for every module* M *one has* $|\overline{M}| \leqslant \kappa|M|$.

Proof The direction "$\Rightarrow$" is clear by what has been shown already. So suppose that the cardinality restriction is satisfied: we verify that the corresponding condition of 11.7 also holds. Given a subset A of the monster model of T^*, there exists a pure submodule, B, of $\tilde{M}$ which contains A and has cardinality no more than $|A|.|T|$ (add in witnesses for every pp formula with parameters in $A=A_0$; this gives A_1; repeat; ... and set $B=\bigcup_\omega A_i$). Then $|N(A)| \leqslant |\overline{B}| \leqslant \kappa|T|.|A|$ - so 11.7 does apply. ▫

One knows ([Sab70a; Cor2]) that, in any case, $|\overline{M}| \leqslant |M|^{(|R|+\aleph_0)}$. If R is regular one has complete elimination of quantifiers for T^* (16.16) and so, using 16.B, a special case of 11.7 is the following.

Corollary 11.9 *Suppose that* R *is regular and non-artinian. Then for every cardinal* κ *there exists a module* M *such that* $|E(M)| > \kappa|M|$. ▫

The condition of right pure-semisimplicity is very strong. The next result details some of its consequences.

Theorem 11.10 *Suppose that* R *is right pure-semisimple. Then:*
(*a*) R *is right artinian;*
(*b*) *there are, up to isomorphism, at most* $|R|+\aleph_0$ *indecomposable modules;*
(*c*) *every indecomposable module is finitely generated.*

Proof (c) Let N be indecomposable and choose a non-zero element a of N. Let p be its pp-type in N and let φ be a pp formula which generates p modulo T^* (φ exists since T^* is t.t.). Then (cf. 8.4) there is some finitely generated submodule, M, of N in which a lies and in which the pp-type of a is p.

Since M is, by 11.6, pure-injective it has, as a direct summand, some copy N' of the hull of a. Since M is finitely generated, so is its direct summand N'. But N' is isomorphic to N: hence N is finitely generated (it follows easily that, in fact, $N=M$).

(b) This is immediate from (c) (or use that every indecomposable pure-injective is the hull of a pp-1-type and, since T^* is t.t., there are at most $|R|+\aleph_0$ of these).

(a) The original proof is [Ch60; 4.4]. I give one slightly closer to that in [Fai76].

Notice first that R is right noetherian: the lattice of right ideals embeds in the lattice of pp-1-types which, by 11.6, has acc.

Let N be the **nilradical** of R (the sum of all the nilpotent ideals). Then the module $E(R/N)$ is finitely generated. For, by assumption, $E(R/N) = \oplus_I E_i$ for suitable indecomposable injectives E_i. Since R/N is finitely generated it is contained in some finite sub-sum which, by (c), is itself finitely generated. So $E(R/N)$ is also finitely generated, being a direct summand of this last injective module.

Now let E' be the injective hull of R/N as an R/N-module. Since E', as an R-module, is an essential extension of R/N it follows (see §1.1) that $R/N \leq E' \leq E(R/N)$ (as R-modules). Therefore, since E' is a submodule of a finitely generated R-module and since R is right noetherian, E' is itself finitely generated as an R-module (hence as an R/N-module).

Now, by Goldie's Theorem (see [St75; §2.2]), E' has the structure of a semisimple artinian ring of fractions of R/N. Let $c \in R/N$ be a regular element. Then, inside E', there is an inverse c^{-1} for c and one has $R/N \leq c^{-1}(R/N) \leq c^{-2}(R/N) \leq \ldots$ (note $1 = c^{-1}.c \in c^{-1}(R/N)$). Since E' has acc on submodules, there is some integer n and some $b \in R/N$ such that $c^{-(n+1)} = c^{-n}b$. Hence $c^{-1} = b \in R/N$: that is, c already is invertible in R/N.

Thus, every regular element of R/N has an inverse in R/N. Hence R/N is its own ring of fractions. So, by Goldie's Theorem, R/N is a semisimple artinian ring.

Furthermore, since R is right noetherian there is an integer k with $N^k = 0$. Each factor N^i/N^{i+1} (an R/N-module) is finitely generated as an R-module, hence as an R/N-module. Therefore R is a finitely generated right R/N-module. Since the latter is artinian, it follows that R is right artinian. □

The property of being semisimple artinian is a two-sided one: but the corresponding question for pure-semisimplicity is open.

Open question 1 If R is right pure-semisimple is R left pure-semisimple?

A positive answer to this would have further consequences; for right and left pure-semisimplicity together imply finite representation type (see §2). A weaker (possibly - see **[Sim77a]**) question, suggested by **11.10**(a) is the following one.

Open question 2 If R is right pure-semisimple, is R necessarily left artinian?

Of course a positive answer to the first question entails an affirmative answer to the second. At least for hereditary rings (**[Sim81]**), a positive answer to the second question would imply a positive answer for the first: indeed, Simson reduces the question for hereditary rings to that for rings of the form $\begin{pmatrix} F & M \\ 0 & G \end{pmatrix}$ where F and G are division rings and M is a bimodule. Furthermore, Simson showed (**[Sim77a]**) that if every right pure-semisimple and left artinian ring is also left pure-semisimple, then *every* right pure-semisimple ring is left artinian (this follows by **8.A**).

Note the model-theoretic reformulation of the first open question.

Open question 1′ If every right R-module is totally transcendental, does it follow that every left R-module is totally transcendental?

In connection with the result of Simson mentioned above, one may show the following (perhaps it can be established more easily, but the proof does illustrate how one might use the results of §8.3). If one could show the result below with the cardinality restriction replaced by one on the dimensions, then it would, by **[Sim81**; 3.3**]**, imply that every right pure-semsimple hereditary ring is of finite representation type (the proof could hardly be described as delicate, but any extension of it would seem to need consideration of the geometry involved).

<u>Proposition 11.11</u> [Pr83; 1.18] *Let R be the matrix ring $\begin{pmatrix} F & M \\ 0 & G \end{pmatrix}$, where F and G are infinite division rings and M is an (F,G)-bimodule. Suppose that $|F|<|G|$. Then R is not right pure-semisimple.*

Proof It will be enough to produce a sequence of pp formulas, with $\ldots\varphi_n \to \varphi_{n-1} \to \ldots \to \varphi_1 \to \varphi_0$ (in every module) and none of the implications reversible. By the results of §8.3, it is equivalent to produce a sequence of matrices $A_n = \begin{pmatrix} \bar{r}_n \\ S_n \end{pmatrix}$ such

that, for no matrices E,H is $\begin{pmatrix} 1 & \\ & E \\ 0 & \end{pmatrix} . A_{n+1} = A_n H$ and where A_{n+1} is formed from A_n by adding columns.

Therefore, the kind of equation that we want to avoid is $\begin{pmatrix} 1 & \bar{c} \\ 0 & D \end{pmatrix} \cdot \begin{pmatrix} \bar{r}_{n+1} \\ S_{n+1} \end{pmatrix} = \begin{pmatrix} \bar{r}_n \\ S_n \end{pmatrix} . H$,

which re-arranges to $\begin{pmatrix} \bar{r}_{n+1} \\ 0 \end{pmatrix} = \begin{pmatrix} \bar{c} & \bar{r}_n \\ D & S_n \end{pmatrix} \cdot \begin{pmatrix} -S_{n+1} \\ H \end{pmatrix} \ldots (*)$.

Let us assume inductively that A_n has coefficients from M: so c and D may be assumed to have entries from F, and H to have entries from G. Also assume, inductively, that S_n is diagonal with non-zero entries on the diagonal: so if X is a matrix with entries in G then $S_n X = 0$ implies $X=0$. Choose any non-zero element a of $\begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ and set $S_{n+1} = \begin{pmatrix} S_n & 0 \\ 0 & a \end{pmatrix}$.

Let $\mathcal{H}$ be the set of those matrices H with entries in G such that there exists a solution D to $DS_{n+1} = S_n H$. Note that, for $H \in \mathcal{H}$, the map $H \mapsto D$ is 1-1 (by assumption on S_n and choice of a). So, since D has entries in F, it follows that $|\mathcal{H}| \leq |F|$.

Now, we wish to choose $\bar{r}_{n+1} = (\bar{r}_n, x)$ such that the equation $(*)$ has no solution. That is, given H, choose $\bar{r}_{n+1}$ not equal to $-cS_{n+1} + \bar{r}_n H$. There are only $|F|$ possibilites for H and, for each of these, no more than $|F|$ possibilities for c so, since $|G| > |F| \geq \aleph_0$, a suitable choice of r_{n+1} may be made, as required. ◻

11.2 Pure-semisimple rings and rings of finite representation type

The ring R is said to be of **finite representation** type, FRT, if it is right pure-semisimple and if there are, up to isomorphism, only finitely many indecomposable modules. Thus, every R-module has the form $N_1^{(\kappa_1)} \oplus \ldots \oplus N_t^{(\kappa_t)}$, where $N_1,\ldots,N_t$ represent the finitely many isomorphism types of indecomposables and $\kappa_1,\ldots,\kappa_t$ are cardinals.

It is known that a ring is both right and left pure-semisimple iff it is of finite representation type (see §8.4). In particular, finite representation type is a two-sided property. The following question is, however, open and is, by the result just quoted, equivalent to the Open Question 1 of §1.

<u>Open question 1</u> If R is right pure-semisimple is R necessarily of finite representation type?

The answer is known to be affirmative for certain types of ring, and this will be discussed below.

Example 1

(a) Any semisimple artinian ring is of finite representation type.

(b) One kind of semisimple artinian ring is the group ring, $K[G]$, of a finite group G over a field K, where the characteristic of the field does not divide the order of the group. Even if the characteristic, p, of the field does divide the order of the group, it may be that $K[G]$ is of finite representation type. Specifically, $K[G]$ is of finite representation type iff each Sylow p-subgroup (maximal p-subgroup) is cyclic (see [CR62; 64.1]). Thus, for example, $F_2(\mathbb{Z}_2\times\mathbb{Z}_2)$ is not of finite representation type but $F_2(S_3)$ is, where F_2 denotes the field with two elements and S_3 is the symmetric group on three symbols (exercise: exhibit infinitely many finitely generated indecomposables over the first [See [CR62; 64.3]]).

Example 2 The ring $\mathbb{Z}_4$ is of finite representation type. Indeed, every $\mathbb{Z}_4$-module has the form $\mathbb{Z}_2{}^{(\kappa)} \oplus \mathbb{Z}_4{}^{(\lambda)}$ for suitable κ, λ.

Example 3 The ring $\begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$ of upper-triangular 2×2 matrices over the field K is of finite representation type: it is just the path algebra of the Dynkin quiver A_2 (see §13.2). I justify this statement by describing all the modules.

Let e_{11}, e_{12}, e_{22} be the usual matrix units in R. Let M be any R-module. Then, as a K-vectorspace, M decomposes as $M.1 = Me_{11} \oplus Me_{22}$. Consider the annihilator, A, in Me_{11} of the element e_{12} - so $A = \{m \in M : m = me_{11} \text{ and } me_{12} = 0\}$. It is trivial to check that A is a submodule of M. It is not much more difficult to see that A is injective and is a direct sum of copies of the unique indecomposable (indeed simple) injective module $S = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix} / \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$.

Thus every R-module decomposes as a direct sum of copies of S and a module, M' say, in which the K-linear map $-\times e_{12}: M'e_{11} \longrightarrow M'e_{22}$ is monic with image $M'e_{12} = M'e_{11}e_{12}$. Such a module M' may usefully be thought of as a pair $(U = M'e_{22}, W = M'e_{11})$ of K-vectorspaces together with a specified embedding $W \longrightarrow U$ (and every such pair arises in this way from an R-module).

It is easy to show that direct-sum decompositions of R-modules without injective direct summands are equivalent to direct-sum decompositions (in the obvious sense) of the corresponding pairs of vectorspaces (cf. §3.A). And it is not difficult to see that every pair (U, W) is a direct sum of copies of just two indecomposable pairs: $(K, 0)$ and (K, K).

The indecomposable $(K, 0)$ corresponds to the simple projective module

$$P_1 = \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix} \simeq \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$$

and (K, K) corresponds to the indecomposable projective

$$P_2 = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}$$

(exercise).

Thus, every R-module has the form $S^{(\kappa)} \oplus P_1{}^{(\lambda)} \oplus P_2{}^{(\mu)}$ for suitable cardinals κ, λ, μ. In particular, R is of finite representation type.

Exercise 1 Show that the ring $\begin{pmatrix} K & K & K \\ 0 & K & 0 \\ 0 & 0 & K \end{pmatrix}$ is of finite representation type, but that $\begin{pmatrix} K & K & K & K & K \\ 0 & K & 0 & 0 & 0 \\ 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & K & 0 \\ 0 & 0 & 0 & 0 & K \end{pmatrix}$ is not. [Hint: c.f. Ex13.2/1, 17.5.]

Exercise 2 [Bau75; Thm3] (R countable) Show that every R-module is $\aleph_0$-categorical iff R is a finite ring of finite representation type (cf. §4.C).

In order for a right artinian ring to be of finite representation type, it is enough that there be, up to isomorphism, only finitely many indecomposable finitely generated modules ([Tac73; §9]). This is false for arbitrary rings (cf. Ex16.2/3)). I include a proof of this fact for the (important) special case of Artin algebras, after giving some related results (all from [Pr84]).

Proposition 11.12 [Pr84; 2.5] *Suppose that there is a totally transcendental module C such that every finitely generated module purely embeds in C. Then R is right pure-semisimple.*

Proof Let M be any non-zero module and take a non-zero element a of M. Let p be the pp-type of a in M. Set A_0 to be the submodule generated by a and let p_0 be the pp-type of a in A_0 (so $p_0 \leq p$).

Choose, if possible, a sequence $A_0 < A_1 < \ldots < M$ of finitely generated submodules of M such that, if p_i is the pp-type of a in A_i, then $p_0 \subset p_1 \subset \ldots \subset p$.

By hypothesis there is, for each i, a pure embedding $f_i : A_i \longrightarrow C$; by purity $\mathrm{pp}^C(f_i a_i) = p_i$. Since C is t.t., eventually $p_i = p_{i+1}$ - contradiction.

So there is some finitely generated submodule, A, of M with $\mathrm{pp}^A(a) = \mathrm{pp}^M(a)$ (since R is not assumed to be noetherian, 8.4 cannot be applied to conclude that p is finitely generated). Since A purely embeds in C, it is t.t. (but p being finitely generated in A need not imply that p is finitely generated in M, so we continue...). In particular A is pure-injective. So there is a copy $N(a)$ of the hull of a, which is a direct summand of A.

It is claimed that $N(a)$ is actually a direct summand of M; but this is immediate from 4.14, since $\mathrm{pp}^{N(a)}(a) = \mathrm{pp}^M(a)$. Now we use an argument already seen in 3.14. Let $\mathfrak{F}$ be a family $\{N_i\}_i$ of finitely generated direct summands of M, such that the sum of the family is direct and pure in M, and which is maximal such (clearly Zorn's Lemma applies to give existence). Since $\bigoplus_i N_i$ can be embedded in some power of C - still a t.t. module - this direct sum is itself t.t. so, in particular, is pure-injective. Therefore $M = (\bigoplus_i N_i) \oplus N'$ say. If N' were non-zero then what was shown first would give a non-zero finitely generated direct summand of N' - contradicting maximality of $\mathfrak{F}$.

Thus every module is a direct sum of finitely generated submodules. So by 11.6 (with $\kappa = |R| + \aleph_0$) one concludes that R is indeed right pure-semisimple. □

Corollary 11.13 [Pr84; 2.6] *The ring R is right pure-semisimple iff every direct sum of finitely generated modules is totally transcendental.*

Proof ⇒ This is by 11.6.

⇐ This is by 11.12. □

Corollary 11.14 [Pr84; 2.7] *Suppose that every finitely generated R-module is totally transcendental and that there are only finitely many indecomposable finitely generated modules. Then R is of finite representation type.*

Proof Let C_1 be the direct sum of one copy of each of the finitely many indecomposable finitely generated modules. Since the sum is finite, the hypothesis implies (3.5) that C_1 is itself totally transcendental. Hence $C = C_1^{(\aleph_0)}$ is t.t. (3.4).

But every finitely generated module, being t.t., is a direct sum of indecomposable submodules (3.14) and so is isomorphic to a direct summand of C. Therefore the result follows by 11.12 and the definition of finite representation type. □

A ring is said to be an **Artin algebra** if its centre is an artinian ring and if the ring is finitely generated as a module (on the right, equally on the left) over its centre. The main examples are finite-dimensional algebras over fields. Observe that an Artin algebra must be both right and left artinian.

Lemma 11.15 *Suppose that R is an Artin algebra. Then every finitely generated R-module is totally transcendental of finite Morley rank (so is in particular pure-injective) with pp-rank bounded by its length as a module over the centre $C(R)$ of R.*

Proof Let M_R be finitely generated. Since R is finitely generated over $C(R)$, $M_{C(R)}$ is finitely generated. This implies, since $C(R)$ is of finite length, that $M_{C(R)}$ is of finite length. By 2.1, every pp-definable subgroup of M_R is a $C(R)$-module. Hence the length of M as a $C(R)$-module is a (finite) bound on the maximum length of a chain of pp-definable subgroups of M_R. □

Lemma 11.16 *Suppose that R is a finite-dimensional algebra over the algebraically closed field K and let M be an indecomposable finite-dimensional R-module. Then the pp-rank (and hence the Morley rank) of M is equal to the K-vectorspace dimension of M.*

Proof By 4.53, if $\varphi(M) > \psi(M)$ is a minimal pair of pp-definable subgroups of M then the quotient $\varphi(M)/\psi(M)$ is a 1-dimensional vectorspace over $\mathrm{End}M/\mathrm{JEnd}M$. Since M is finite-dimensional over K, this division ring is a finite extension of K so, since K is algebraically closed, it equals K and so $\varphi(M)/\psi(M)$ is a 1-dimensional vectorspace over K. Thus the result follows (the pp-rank equals the Morley rank since K is infinite!). □

Example 4 If K is not algebraically closed, then the conclusion of 11.16 may fail. Take K to be the real field and consider the simple module $K[X]/\langle X^2+1\rangle$. This is two-dimensional over K but is 1-dimensional over its endomorphism ring (the field of complex numbers). Therefore it has no proper non-trivial pp-definable subgroups, so it has pp-rank 1 but K-dimension 2.

Corollary 11.17 *Let R be an Artin algebra. Suppose that there are, up to isomorphism, only finitely many indecomposable finitely generated R-modules. Then R is of finite representation type.*

Proof This is immediate from 11.14 and 11.15. □

In fact, it is enough to assume just that R is right artinian, rather than an Artin algebra, in order to obtain the conclusion of 11.17, but the proof for this general case is rather different [Tac73; §9].

Exercise 3 [War78; 1.2] Show that a commutative noetherian ring over which every countably generated module is a direct sum of indecomposables is an artinian principal ideal domain - these are just the commutative rings of finite representation type.

The countable rings of finite representation type may be characterised in terms of the number of countable modules.

Theorem 11.18 [BM82] *Suppose that* R *is a countable ring. Then the following conditions are equivalent:*
(*i*) R *is of finite representation type;*
(*ii*) *there are only countably many countable modules up to isomorphism;*
(*iii*) *there are fewer than* $2^{\aleph_0}$ *countable modules up to isomorphism;*
(*ii*)^{o}-(*iii*)^{o}: *as* (*ii*)-(*iii*), *but for left modules.*

Proof Since R is not assumed to be right artinian, an appeal to 11.17 would be unjustified: so there really is something to prove.

As stated above, (i) is a right/left-symmetric condition (see 8.24); (i)⇒(ii) is clear; trivially (ii)⇒(iii). So the implication (iii)⇒(i) remains.

First, it will be shown that every finitely generated module is totally transcendental. Assuming that this is not the case, consider some non-t.t. module A generated by $a_1,\ldots,a_n$ (say). Since A has an infinite descending chain of pp-definable subgroups (by 3.1), it must be that the number of 1-types modulo Th(A) over A is $2^{\aleph_0}$ (compare proof of 3.1(c)(i)⇒(iii)′). Since every element of A may be expressed as a term, using only $a_1,\ldots,a_n$ as parameters, one has $S_1(A)=S_1(a_1,\ldots,a_n)$. So there are $2^{\aleph_0}$ 1-types over $\{a_1,\ldots,a_n\}$. It follows (exercise) that there are $2^{\aleph_0}$ $(n+1)$-types modulo Th(A) over 0. But every $(n+1)$-type over 0 may be realised in a countable model of Th(A). So there must be $2^{\aleph_0}$ countable models to hold all these realisations - contrary to hypothesis.

Thus every finitely generated module is totally transcendental. Were there infinitely many non-isomorphic indecomposable finitely generated modules, $\{N_i : i\in\omega\}$ say, then one would have $2^{\aleph_0}$ countable modules: the $\oplus\{N_i : i\in I\}$, for $I\subseteq\omega$, provide $2^{\aleph_0}$ non-isomorphic (by 4.A14) countable modules -contradiction.

The conclusion now follows by 11.14. □

11.3 Finite hulls over artinian rings

It has been seen that if the ring R is right pure-semisimple or if it is an Artin algebra, then every finitely generated module is pure-injective. Therefore, when dealing only with finitely generated modules over such rings, we have the full machinery of hulls available. However, a finitely generated module over a right artinian ring need not be pure-injective (see Ex14.2/1).

One may ask over what rings the finitely generated modules are pure-injective (and hence, if R is countable, totally transcendental)? In the two cases above where one has this property one also has existence of almost split sequences (see §13.1) on one side: is there a connection between finitely generated modules being pure-injective and existence of almost split sequences? (and, therefore, a connection with whether right pure-semisimple implies finite representation type).

What, then, are we to use in the general artinian case, since hulls are not available if one restricts to finitely generated modules? Is there still a connection between irreducible types and indecomposable modules? The material of §8.2 relates to this but, to get the best results (and the relativisation goes through almost completely), one must restrict to modules of finite length. No doubt, at least some of this section could be carried over to finitely presented modules over left perfect rings (cf. [Sab71b]).

Therefore I show first that if R is right artinian then, within the context of finitely generated modules, there are finitely generated analogues of hulls.

Throughout this section, R is assumed to be right artinian.

I begin with two useful, well-known, results about modules of finite length.

Proposition 11.19 (Fitting's Lemma) *If* M *is a module of finite length and if* f *is an endomorphism of* M, *then* $M=\mathrm{im}f^k\oplus\ker f^k$ *for some* $k\in\omega$.

Proof (outline) Use the (obvious) fact that f is monic iff it is epi iff it is an automorphism. We have $\mathrm{im} f^k = \mathrm{im} f^{k+1} = \ldots$ and $\ker f^k = \ker f^{k+1} = \ldots$, where k is the length of M. So, if $a \in M$, then $f^k a = f^{2k} b$ for some $b \in M$: thus $a = f^k b + (a - f^k b) \in \mathrm{im} f^k + \ker f^k$. Also, f^k is epi on $\mathrm{im} f^k$ (since the latter equals $\mathrm{im} f^{2k}$); so $\mathrm{im} f^k \cap \ker f^k = 0$. ◻

Proposition 11.20 (Harada-Sai Lemma) [HaSa71] *Suppose that the modules $M_0, \ldots, M_{n-1}$ are indecomposable and all of length bounded by $b \in \omega$. For each $i = 0, \ldots, n-2$, let $f_i : M_i \longrightarrow M_{i+1}$ be a non-isomorphism. Suppose that the composition $f_{n-2} \ldots f_1 f_0$ is non-zero. Then $n < 2^b$.*

Proof (outline) The following statement is proved by induction on k: $l(\mathrm{im} f_{2^k-2} \ldots f_1 f_0) \leqslant b-k$, where "$l$" denotes the length of a module. For $k \leqslant 1$ this is clear, (since f_0 is a non-isomorphism, and $l(M_0) \leqslant b$).

Suppose that the statement is true for the particular value k. Set $f = f_{2^k-2} \ldots f_1 f_0$ and $h = f_{2^{k+1}-2} \ldots f_{2^k}$. If either of these has image of length strictly less than $b-k$ then $l(\mathrm{im} f_{2^{k+1}-2} \ldots f_0) = l(\mathrm{im}\, h f_{2^k-1} f) \leqslant b-(k+1)$, as required. So suppose otherwise and set $g = f_{2^k-1}$. Then suppose, for a contradiction, that $l(\mathrm{im}\, hgf) = b-k$.

By the induction hypothesis applied to f, we see that $\mathrm{im} f \cap \ker hg = 0$ $(*)$. By additivity of length, we have that $l(\mathrm{im} f) = b-k$, $l(\ker hg) = l(M_{2^k-1}) - l(\mathrm{im}\, hg)$ and $l(\mathrm{im}\, hg) = b-k$. Combining these equations, we see from $(*)$ that M_{2^k-1} is the direct sum of $\mathrm{im} f$ and $\ker hg$. But this module is indecomposable, so $\ker hg$ must be zero, and g is monic.

Similarly, one shows that g is epi. But that contradicts g being a non-isomorphism. ◻

Let $p \in \mathrm{P}_n{}^{\mathrm{f}}$ be a finitely generated pp-n-type. There is (**8.4**) a finitely generated module M containing a realisation $\bar{a}$ of p. Since R is right artinian, M has finite length; so there is a direct summand, $H(\bar{a})$, of M which contains $\bar{a}$ and which is minimal such. I show that $H(\bar{a})$ is unique up to $\bar{a}$-isomorphism: in fact, $H(\bar{a})$ is determined by p (as is $N(p)$).

So suppose that $\bar{b}$ in the finitely presented (so finite length) module M' realises p. Choose $H(\bar{b})$ in M' in the same way that $H(\bar{a})$ was chosen. Since $H(-)$ is a direct summand, one has $\mathrm{pp}^{H(\bar{a})}(\bar{a}) = p = \mathrm{pp}^{H(\bar{b})}(\bar{b})$. Therefore, by **8.5**, there are morphisms $f : H(\bar{a}) \longrightarrow H(\bar{b})$ and $g : H(\bar{b}) \longrightarrow H(\bar{a})$ with $f\bar{a} = \bar{b}$ and $g\bar{b} = \bar{a}$. In particular, gf fixes $\bar{a}$.

By Fitting's Lemma (**11.19**) there is $k \in \omega$ such that $H(\bar{a}) = \mathrm{im}(gf)^k \oplus \ker(gf)^k$. Since $\bar{a}$ is in $\mathrm{im}(gf)^k$ and since this module is a direct summand of $H(\bar{a})$, so of M, minimality of $H(\bar{a})$ gives $H(\bar{a}) = \mathrm{im}(gf)^k$. Therefore g must be epi: also $\ker(gf)^k = 0$ - so f must be monic. Then, by the symmetry of the situation, one concludes that f and g are mutually inverse. This establishes the following result.

Proposition 11.21 [Pr83; §1] (R *right artinian*) *Let p be a finitely generated pp-n-type, and suppose that $\bar{a}$ in M and $\bar{b}$ in M' are realisations of p in finitely generated modules. Let $H(\bar{a})$ be a minimal direct summand of M containing $\bar{a}$ and let $H(\bar{b})$ be a minimal direct summand of M' containing $\bar{b}$. Then there is an isomorphism between $H(\bar{a})$ and $H(\bar{b})$ taking $\bar{a}$ to $\bar{b}$.* ◻

The module $H(\bar{a})$ may be called the **finite hull** or, if no confusion should arise, simply the **hull** of $\bar{a}$. If $p = \mathrm{pp}^{H(\bar{a})}(\bar{a})$ then $H(p) = H(\bar{a})$ is the (**finite**) **hull** of p.

Another question which arises is that of the relationship between $H(p)$ and $N(p)$. It should be fairly clear that p is irreducible iff $H(p)$ is indecomposable (use **8.7**); so $N(p)$ is indecomposable iff $H(p)$ is indecomposable. Nevertheless, this does not immediately relate the two modules $H(p)$ and $N(p)$ (although, for our purposes, **8.7** is all that is needed). An obvious question is: "What is the pure-injective hull of $H(p)$? Is it $N(p)$?". Equivalently: "Is the pure-injective hull of $H(p)$ indecomposable?". The next result answers this question (affirmatively).

<u>Proposition 11.22</u> [Pr83; 2.12] *(R right artinian) Let p be any finitely generated n-type. Then the pure-injective hull of $H(p)$ is $N(p)$.*

Proof Let $\bar{a}$ realise p: it will be enough, by **4.6**, to show that if $\bar{b}$ is in $H(\bar{a})$ then $\mathrm{pp}(\bar{b}/\bar{a})$ is a maximal pp-type over $\bar{a}$ (where the over-theory may be taken to be that of $H(\bar{a})$).

Notice that $\mathrm{pp}(\bar{b}/\bar{a})$ is finitely generated. For, by **8.4**, $\mathrm{pp}^{H(\bar{a})}(\bar{b}\frown\bar{a})$ is finitely generated - say by the pp formula $\varphi(\bar{v},\bar{w})$. Then the formula $\varphi(\bar{v},\bar{a})$ generates $\mathrm{pp}(\bar{b}/\bar{a})$. So if $\mathrm{pp}(\bar{b}/\bar{a})$ were not maximal there would be a finitely generated pp-type over $\bar{a}$ strictly containing it. Then there would be a finitely generated module, M, containing $\bar{a}$ in such a way that $\mathrm{pp}^M(\bar{a})=p$, and containing a tuple $\bar{c}$ in M with $\mathrm{pp}(\bar{c}/\bar{a})\supset\mathrm{pp}(\bar{b}/\bar{a})$.

By **8.5** there would be morphisms $f:H(\bar{a})\longrightarrow M$ and $g:M\longrightarrow H(\bar{a})$, the first taking $\bar{a}\frown\bar{b}$ to $\bar{a}\frown\bar{c}$, the second fixing $\bar{a}$. Consider the endomorphism gf of $H(\bar{a})$. This morphism fixes $\bar{a}$ but is strictly pp-type increasing on $\bar{b}$. From the fact that $\bar{a}$ is fixed, we deduce quickly that gf is an isomorphism (an application of Fitting's Lemma, just as before **11.21**) and so cannot strictly increase the pp-type of $\bar{b}$ - contradiction.

Thus the pp-type of $H(\bar{a})$ over $\bar{a}$ is maximal. So, by **4.6** and **4.14**, $H(\bar{a})$ is purely embedded into $N(\bar{a})$, as required. □

<u>Corollary 11.23</u> *(R right artinian) Let p be any finitely generated (pp-)type. If $f:H(\bar{a})\longrightarrow M$ is such that $\mathrm{pp}(f\bar{a})=p$ then f is a pure embedding.*

Proof This follows by **11.22** and **4.14**. □

Exercise 1 One may give a more algebraic proof of the fact that if $H(p)$ is indecomposable then so is its pure-injective hull.

Suppose that R is right noetherian such that every finitely generated module is a direct sum of indecomposable submodules, each with local endomorphism ring (e.g., suppose that R is right artinian or a principal ideal domain; also see [Bra79; 9.2]). Let M_R be finitely generated and indecomposable; then $\bar{M}$ is indecomposable.
[Let M be generated by $\bar{b}$ and suppose that $\bar{M}$ has a non-trivial decomposition as $N_1\oplus N_2$. Choose non-zero elements $a_i\in N_i$ and pp formulas φ_i linking a_i to $\bar{b}$ $(i=1,2)$. Then there is $M'=M_1\oplus M_2$ say, a finitely generated submodule of $\bar{M}$ with $a_i\in M_i$ and with M' containing $\bar{b}$ together with witnesses for the quantifiers in the $\varphi_i(a_i,\bar{b})$. Clearly M is pure in M' (it is even pure in $\bar{M}$) and so, since M'/M is finitely presented, M is a direct summand of M' (Exercise 2.3/1). By locality of endomorphism rings, one has the exchange property (see, e.g., [Fai76; 18.17]): so (decompose then recompose), $M'=M\oplus M''\oplus M_2$ say. On projecting $\varphi_2(a_2,\bar{b})$ to M_2 one obtains a contradiction to $\neg\varphi_2(a_2,\bar{0})$.]

Exercise 2 Give yet another proof of **11.22** using **8.4** and **8.7**.

<u>Corollary 11.24</u> [Pr83; 2.13] *Suppose that R is right artinian and let M be finitely generated.*
(a) M is indecomposable iff $\bar{M}$ is indecomposable.
(b) M is a direct sum of n indecomposable modules iff the same is true of $\bar{M}$. □

<u>Corollary 11.25</u> *(R right artinian) Let p be a finitely generated pp-type. Suppose that $H(p)$ is a direct sum of k indecomposable submodules. Then the algebraic weight of p is k (cf. §6.4).* □

<u>Corollary 11.26</u> [Pr83; 2.14] *If R is right artinian then every finitely generated pp-type over 0 has finite weight.* □

One does need the type to be over 0 in **11.26** (see after **6.27**); and some condition on the ring is necessary - consider $R=M=\mathbb{Z}$, where the pp-type of the element $1_{\mathbb{Z}}$ has weight $\aleph_0$. The next result shows that if R is a right artinian ring then the correspondence $N \mapsto \overline{N}$ between indecomposable finitely generated modules and their pure-injective hulls, is 1-1 (modulo isomorphism).

Proposition 11.27 *Suppose that R is right artinian and that N and N' are indecomposable finitely generated R-modules. If $\overline{N} \simeq \overline{N'}$ then $N \simeq N'$.*

Proof Take non-zero elements a, a' in N, N' respectively. Suppose that their pp-types are respectively generated (modulo T^*) by the pp formulas φ, φ'. We may consider N and N' as both purely embedded in the same copy $\overline{N}=\overline{N'}$ of the pure-injective hull. By **11.24**(a), $\overline{N}$ is indecomposable so, by **4.11**, there is a pp formula $\psi(v,w)$ such that $\overline{N} \models \psi(a,a') \wedge \neg\psi(a,0)$.

Since N is pure in $\overline{N}$, it satisfies $\exists w\,(\psi(a,w) \wedge \varphi'(w))$: say $b \in N$ witnesses "w". Since $\overline{N}$, so N, satisfies $\neg\psi(a,0)$, b is non-zero. Also, since b satisifes $\varphi'(w)$ and since this formula generates the pp-type of a', there is (**8.5**) a morphism $f: N' \longrightarrow N$ taking a' to b.

If f is an isomorphism, then we finish. Otherwise, repeat the argument with a' and b in place of a and a'. Since N and N' are of finite length, eventually we find that they are isomorphic or we reach a contradiction. ◻

The next lemma, which is immediate if R is an Artin algebra, will be useful later.

Lemma 11.28 *Suppose that R is right artinian. Let M be indecomposable and finitely generated. Then the sub-poset $\{p \in \mathrm{P}_n{}^{\mathrm{f}} : H(p) \simeq M\}$ of $\mathrm{P}_n{}^{\mathrm{f}}$, consisting of those pp-n-types realised in M, has the acc - even has finite length.*

Proof Any strict chain in this subset of $\mathrm{P}_n{}^{\mathrm{f}}$ induces, by **8.5**, a sequence of non-isomorphisms from M to itself. But if the length of M is k then (**11.20**) any such chain has length bounded by 2^k, as required. ◻

One should beware that **11.28** does not say that M has the dcc on pp-definable subgroups (for then M would be t.t. - in contradiction with Ex14.2/1): not every pp-definable subgroup of M need have the form Sa where $S=\mathrm{End}M$ and $a \in M$.

Exercise 3 (cf. Exercise 2.3/3) Let M be a finitely presented module (over any ring) and let S be its endomorphism ring.

(i) If $a \in M$ and if φ is a pp formula equivalent to the pp-type of a in M, then $\varphi(M)=Sa$.

(ii) Every finitely generated S-submodule of M is a pp-definable subgroup of M_R.

(iii) If ${}_SM$ is noetherian then the S-submodules of M are precisely the pp-definable subgroups of M_R.

(iv) If M_R is weakly saturated and if every S-submodule of M is pp-definable, then ${}_SM$ is noetherian.

Exercise 4 [Pr83; 3.13] Prove the following slight strengthening of the Harada-Sai Lemma (**11.20**). Let $f_i: M_i \longrightarrow M_{i+1}$ $(i=0,\ldots,n-2)$ be a sequence of non-isomorphisms and let $\overline{a}_0$ be in M_0 such that, if we set $\overline{a}_1=f_0\overline{a}_0$, $\overline{a}_2=f_1\overline{a}_1$, ..., then M_i is the finite hull of $\overline{a}_i$. If each M_i has length no more than b then $n < 2^b$.

"What makes a module indecomposable": this is a well known question of Auslander. On the basis of the results of this section, we can give one answer to this: a module (finitely generated over an artinian ring) is indecomposable iff, for every two non-zero elements a, b of it, there is a system of linear equations and a solution vector of the form $(a\ b\ \overline{x})$ but no solution vector of the form $(a\ 0\ \overline{y})$.

11.4 Finite Morley rank and finite representation type

It is shown in this section that the ring R has finite representation type iff every R-module has finite Morley rank. Essentially this is done by showing that both conditions are equivalent to the requirement that all irreducible types be isolated. In the important case of Artin algebras the proof given here is self-contained: for the general artinian case we need to quote a result of Auslander for one direction.

Actually, that every module having finite Morley rank implies that the ring is of finite representation type (**11.29**) already follows from 7.23 (or [Zg84; 8.12]) **5.13** and **5.18**.

Theorem 11.29 *If the Morley rank of the largest theory, T^*, of R-modules is finite, equal to n say, (that is, if the length of the lattice $P_1(R)$ is n) then R has no more than n indecomposable modules up to isomorphism. In particular, R is of finite representation type.* □

Another result, from which **11.29** follows, is **9.4**.

The converse to **11.29** is easy if R is an Artin algebra.

Theorem 11.30 *Suppose that the ring R is such that every finitely generated module has its lattice of pp-definable subgroups of finite length. If R is of finite representation type then the Morley rank of T^*, and hence of every module, is finite.*

Proof Let $N_1,...,N_k$ be the finitely generated indecomposable modules. The assumption on R implies that the Morley rank of $(N_1 \oplus ... \oplus N_k)^{(\kappa)}$ is finite for every cardinal κ. But every module is (since R is right pure-semisimple) a direct summand of such a sum of copies of $N_1 \oplus ... \oplus N_k$. Hence the result follows. □

Corollary 11.31 [Pr84; 3.9] *Let R be an Artin algebra. Then the following conditions are equivalent:*

(i) R is of finite representation type;

(ii) $P_1(R)$ has finite length;

(iii) every module has finite Morley rank (and there is a uniform bound);

(iv) $MR(T^*)$ is finite.

Proof Clearly (iii) and (iv) are equivalent by **5.21**; the equivalence of (ii) and (iv) is by **5.13** and **5.18**. Then, by **11.15**, **11.30** applies to give (i)⇒(iv). Finally **11.29** gives us (iv)⇒(i). □

Example 1 Let R be the path algebra of the quiver A_2 (see Ex 11.2/3). There are three indecomposables; two with pp-rank 1 and one with pp-rank 2. It is left as an exercise to show that the Morley rank of T^* is 4 (see **11.39** below).

Exercise 1 Let R be the ring of $n \times n$ matrices over some division ring. What is $MR(T^*)$?

In fact **11.31** does not need the assumption that R is an Artin algebra: R being right artinian will do (note that all the conditions of **11.31** imply right pure-semisimplicity, so imply right artinian). But the general case involves more work and brings to light the importance of the property of irreducible (pp-)types being isolated. This property links up with elementary cogeneration (as has been seen in §9.4), with finite presentation of certain functors (§12.2) and hence ([Aus74a; 2.7]) with existence of almost split sequences.

The next result was proved first for totally transcendental theories [Pr84; 3.6]. Then the global case (i.e., R rt. pss) was generalised to the case where every finitely generated module is totally transcendental [Pr83]. Using the machinery of §3 one can now give a proof for arbitrary right artinian rings.

<u>Proposition 11.32</u> *Suppose that R is right artinian and let p be a finitely generated irreducible type. Then the following conditions are equivalent:*
(i) *p is isolated (as a pp-type, equivalently as a type);*
(ii) *if $\{M_\lambda\}_\lambda$ is a set of indecomposable finitely generated modules and if $H(p)$ purely embeds in $\prod_\lambda M_\lambda$ then $H(p)\simeq M_\lambda$ for some λ;*
(iii) *if $\{M_\lambda\}_\lambda$ is a set of indecomposable finitely generated modules and if $H(p)$ purely embeds in $\prod_\lambda M_\lambda$ then, for some λ, the morphism $H(p)\longrightarrow M_\lambda$ induced by the projection $\pi_\lambda:\prod_\mu M_\mu\longrightarrow M_\lambda$, is an isomorphism.*

Proof (i)⇒(iii) Let $\bar{a}$ in $H(\bar{a})=H(p)$ realise p. Inside the product $M=\prod M_\lambda$ one has $\bar{a}=(\bar{a}_\lambda)_\lambda$ for suitable $\bar{a}_\lambda$ in M_λ; set p_λ to be the pp-type of $\bar{a}_\lambda$ in M_λ. Since $H(\bar{a})$ purely embeds in M, one obtains $p=\bigcap_\lambda p_\lambda$. Since p is irreducible and isolated in P^{f} (8.7) it must be that $p=p_\lambda$ for some λ. Therefore, the projection π_λ preserves the pp-type of $\bar{a}$, hence (by **11.23**) is a pure embedding of $H(\bar{a})$ into M_λ. Since $M_\lambda/H(\bar{a})$ is finitely presented, it follows (Exercise 2.3/1) that $H(\bar{a})$ is a direct summand of M_λ But M_λ is indecomposable; so $\pi_\lambda:H(\bar{a})\longrightarrow M_\lambda$ is an isomorphism, as required.

(iii)⇒(ii) This is immediate.

(ii)⇒(i) Suppose, for a contradiction, that p is not isolated but that, nevertheless (ii) holds. Then there is a representation $p=\bigcap\{p_\lambda:\lambda\in\Lambda\}$, where the p_λ may be taken to be irreducible, finitely generated and with $p_\lambda>p$ for all λ (exercise: use the fact that if $q>p$ then there is a finitely generated pp-type p' with $q\geqslant p'>p$, and p' is a finite intersection of irreducible finitely generated pp-types).

Set $H_\lambda=H(\bar{a}_\lambda)$ where $\bar{a}_\lambda$ is some realisation of p_λ. Then, if $\bar{a}=(\bar{a}_\lambda)_\lambda\in\prod H_\lambda=H$ (say), one has that the pp-type of $\bar{a}$ in H is $\bigcap_\lambda p_\lambda=p$. So, by **8.5**, there is a morphism from $H(\bar{a})$ into H which, since it preserves the pp-type of $\bar{a}$, is a pure embedding (by **11.23**). Then from (ii) it follows that one has $H(\bar{a})\simeq H(\bar{a}_\lambda)$ for some λ.

If p_λ were not itself isolated then one could repeat the above argument with p_λ replacing p: and so on. Noting that $p<p_\lambda$ and that $H(p)=H(p_\lambda)$, we see that **11.28** guarantees the termination of this process after a finite number of steps. Therefore there is a finitely generated pp-type q with $H(q)=H(p)$ and with q isolated. Let $\bar{b}$ in $H(\bar{a})$ realise q.

By (i)⇒(iii) applied to q and H, some projection π_μ preserves the pp-type of $\bar{b}$. Hence (**11.23**) π_μ must preserve the pp-type of $\bar{a}$. That is, $p=p_\mu$ - contradiction (for p_μ is isolated), as required. ◻

<u>Corollary 11.33</u> *Suppose that R is right artinian. Let p and q be finitely generated irreducible pp-types with $H(p)\simeq H(q)$. Then p is isolated (with respect to a given T) iff q is isolated (with respect to T).*

Proof This follows from **11.32** since condition (ii) holds equally for p and q. Of course, if $H(p)$ is a t.t. module (e.g., if R is an Artin algebra) then this is immediate from the fact that the prime model realises exactly the isolated types.

Alternatively the result follows easily by **9.26** and **9.24** (say). ◻

Next, I give the totally transcendental version of **11.32** which overlaps with, but also diverges from that result.

<u>Proposition 11.34</u> [Pr84; 3.6] *Suppose that T is totally transcendental. Then the following conditions are equivalent for any irreducible type p over 0:*
(i) *p is isolated;*
(ii) *if $\{N_\lambda\}_\lambda$ is any set of indecomposable pure-injectives and if $N(p)$ is a direct summand of the product $\prod_\lambda N_\lambda$, then $N(p)\simeq N(p_\lambda)$ for some λ;*

(iii) if $\{N_\lambda\}_\lambda$ *is any set of indecomposable pure-injectives and if* $N(p)$ *is a direct summand of the product* $\prod_\lambda N_\lambda$ *then, for some* λ, *the canonical projection* $\pi_\lambda : \prod_\mu N_\mu \twoheadrightarrow N_\lambda$ *induces an isomorphism* $N(p) \simeq N_\lambda$.

Proof First note that none of the conditions (i), (ii), (iii) is changed by assuming that $T = T^{\aleph_0}$ (by **4.39** and (say) **9.26**).

(i)⇒(iii) and (iii)⇒(ii) are just as in the proof of **11.32**. A proof of (ii)⇒(i), very much like that in **11.32**, may be given using the t.t. condition in place of **11.23**.

Alternatively; let $M_0 = \bigoplus_\lambda N_\lambda^{(\kappa_\lambda)}$, with the N_λ being the hulls of all the isolated irreducible 1-types, be the prime model of T (**4.62**). By **9.33**, M_0 is an elementary cogenerator: so there is κ with $N(p)$ purely embedding into M_0^κ. Since $M_0^\kappa = (\bigoplus_\lambda N_\lambda^{(\kappa_\lambda)})^\kappa$ is pure in $\prod_\lambda N_\lambda^{\kappa_\lambda \times \kappa}$, one has the failure of (ii) if p is non-isolated (for then $N(p)$ cannot be isomorphic to any N_λ). ▫

The next result, needed for the converse of **11.29**, is due to Auslander.

<u>Theorem 11.35</u> [Aus76; 2.4] *Suppose that the ring* R *is of finite representation type. If* $\{N_\lambda\}_\lambda$ *is a set of indecomposable modules and if* N *is an indecomposable which purely embeds in the product* $\prod_\lambda N_\lambda$ *then, for some* λ, *one has* $N \simeq N_\lambda$. ▫

<u>Corollary 11.36</u> *If* R *is a ring of finite representation type then every irreducible type (in finitely many free variables) is isolated.*

Proof Since finite representation type implies right pure semisimplicity, hence every pp-type finitely generated, this follows by **11.35** and **11.32**. ▫

Of course, even if R has finite representation type one cannot expect all types in finitely many free variables to be isolated - for then T^* would be $\aleph_0$-categorical (assuming $|R| \leqslant \aleph_0$).

Example 2 Take R to be the ring $\mathbb{Q}$ of rationals; so $T^* = \mathrm{Th}(\mathbb{Q})$. Since T^* is not $\aleph_0$-categorical (for $\mathbb{Q} \equiv \mathbb{Q} \oplus \mathbb{Q}$) there must be non-isolated types (necessarily of weight $\geqslant 2$). Since there are only two 1-types (the type of the zero element, and that of any non-zero element), or by **11.36**, we must look to 2-types for non-isolation. In $\mathbb{Q} \oplus \mathbb{Q}$ let a be the element $(1,0)$ and let b be $(0,1)$ - so b is not a multiple of a. Then the type of the pair (a,b) is non-isolated (exercise).

The converse of **11.29** is then provided by **6.28**.

<u>Corollary 11.37</u> *If the ring* R *is of finite representation type then* T^* *has finite Morley rank.*

Proof Otherwise, **6.28** would imply that there was a non-isolated irreducible 1-type over 0, contradicting **11.36**. ▫

Summarising, we have the following.

<u>Theorem 11.38</u> [Pr84; 3.9] *The following conditions on the ring* R *are equivalent:*

(i) R *is of finite representation type;*

(ii) T^* *has finite Morley rank;*

(iii) every R*-module (equivalently every left* R*-module) has finite Morley rank;*

(iv) $\mathrm{P}_1(R)$ *has finite length;*

(v) every irreducible type (in finitely many free variables) is isolated.

If R *is right artinian, then a further equivalent is:*

(vi) $\mathcal{I}(T^*)$ *is finite.*

Proof The equivalence of (i) and (ii) is **11.29** and **11.37**. That (ii), (iii) and (iv) are equivalent follows as in **11.6**. By **11.36**, (i) implies (v).

Suppose that (v) holds. It follows by **6.28** that T^* has finite Morley rank - that is, (ii) holds. The last statement follows by [Tac73; §9] (cf. **11.17** above). □

Example 3 If R is not assumed to be right artinian, then $\mathcal{I}(T^*)$ being finite does not imply that R is of finite representation type. Consider Ex16.2/3.

Given a ring of finite representation type, the Morley rank of T^* is easily calculated if one knows the indecomposables. It is just the pp-rank of the module M which is a direct sum of one copy of each indecomposable (so, by **11.16**, if R is a K-algebra with K algebraically closed, it is just the K-dimension of M). This follows from the next result which, in turn, is an immediate consequence of **9.3**.

<u>Lemma 11.39</u> *Let M and N be modules of finite Morley rank. Then* $\mathrm{MR}(M\oplus N)=\mathrm{MR}(M)+\mathrm{MR}(N)$. □

Thus, for example, if R is the path algebra of the quiver E_6 then $\mathrm{MR}(T^*)=156$.

Exercise 3 It's not difficult to classify the rings of finite representation type such that the Morley rank of T^* is very small.

(i) Show that if $\mathrm{MR}(T^*)\leqslant 2$ then R is semisimple artinian.

(ii) Show that if $\mathrm{MR}(T^*)=3$ then either R is semisimple artinian or else $R/J\simeq J$ is a division ring (e.g. $R=\mathbb{Z}_4$).

(iii) Show that if $\mathrm{MR}(T^*)=4$ then the only essentially new possibility is that R is of the form $\begin{pmatrix} D' & D \\ 0 & D \end{pmatrix}$ where D and D' are division rings and D has a (D',D)-bimodule structure (e.g. the path algebra of A_2). [Hint: show that $J^2=0$ - this is a useful first step in all three parts (and in the first two, pretty well the last step).]

Fix an integer d and consider the class $\mathfrak{F}_d(K)$ of all d-dimensional algebras of finite representation type over the base field K. Any d-dimensional K-algebra can be described by d^3 structure constants, which tell how the elements of a chosen K-basis multiply together. In particular, any d-dimensional K-algebra with a chosen K-basis determines a point of affine K^{d^3}-space. Different bases of the same algebra are related by invertible $d\times d$ matrices over K - that is, by elements of $GL_d(K)$. The action of $GL_d(K)$ on K^d induces an action on the points of K^{d^3} which stabilises the set, X_d, of points corresponding to members of $\mathfrak{F}_d(K)$. Under this action, the orbits are precisely the isomorphism classes of d-dimensional K-algebras of finite representation type. The question of whether there are, for each d and K, only finitely many algebras of finite representation type (up to isomorphism), was open.

Gabriel [Gab75a] showed that if K is algebraically closed then the set X_d is an open subset of the set Y_d of all members of K^{d^3} which correspond to algebras: there exist polynomials $f_1,\ldots,f_n\in K[X_i:1\leqslant i\leqslant d^3]$ such that a point $\overline{k}$ of Y_d^3 corresponds to an algebra of finite representation type iff not all of $f_1(\overline{k}),\ldots,f_n(\overline{k})$ are zero. Gabriel left open the question of whether there are such polynomials over the prime subfield.

Herrmann, Jensen and Lenzing in [HJL81] and [JL82] (also [JL80]) went some way towards answering this, by showing that the class $\mathfrak{F}_d(K)$ for any field K is finitely axiomatisable so, using the elimination of quantifiers in the case that K is algebraically closed, they deduced that, for K algebraically closed, the set X_d is constructible over the prime subfield (that is, X_d is defined by a certain finite set of equations and inequations over the prime subfield). These papers contain a number of results on axiomatisability, effectivity, bounds on the number of indecomposables and the effect of extending the base field.

On the question of the finiteness of $\mathfrak{F}_d(K)$, the further development of covering theory (see [BG82]), together with the fact that there is a bound, in terms of the dimension of the algebra, on the number of indecomposables (see [JL82: 3.6] and also [HJL81: 5.1]; alternatively, see [Bon82; §5] plus [BaBr81; §4]) shows that, outside of characteristic two (where there the map from an algebra to its Auslander-Reiten quiver need not be 1-1), there are indeed only finitely many d-dimensional K-algebras of finite representation type (there are, no doubt, a number of routes to this fact).

In any case, the multiplicative basis theorem [BGRS85] supercedes all this, at least for algebraically closed fields. The theorem says that, if R is a algebra of finite representation type over an algebraically closed field K then there is a K-basis of R in which the product of any two basis elements is either zero or a basis element. In particular, over such K, there are only finitely many orbits in $\mathfrak{F}_d(K)$. For a discussion of the situation over non-algebraically closed fields, see [Gus85].

11.P "Pathologies"

The area that I touch on here is really rather large. So what I do is to direct the reader to some review papers and just mention some results which directly impinge on what is discussed elsewhere in the text. For a more balanced presentation, the reader should consult the review works that I mention.

In [Cor63], Corner showed that every countable reduced torsionfree ring is the endomorphism ring of some countable reduced torsionfree abelian group. He then gave examples of various pathologies which can arise in abelian groups (cf. Kaplansky's "Test Problems" [Kap54; p12]).

Corner's results were extended in the work of Brenner and Butler and Corner [BB65], [Bre67], [Cor69]. Brenner and Butler showed that every associative algebra over a field K can be realised as the algebra of those endomorphisms of a K-vectorspace which leave invariant a specified set of subspaces. Brenner improved this by showing that the number of subspaces may be taken to be five, provided the algebra is countably generated over K. Set theory began to make its appearance when Corner showed that it is enough to suppose that the number of generators of the algebra is less than the first strongly inaccessible cardinal. (In fact, Corner had already noted that a "proof" of Fuchs that there are arbitrarily large indecomposable torsionfree abelian groups failed at certain large cardinals.)

Since then, these results have been extended in many directions. One direction is, of course, to wild representation type (see Chpt.13). In another direction, one is concerned with realising algebras as endomorphism rings of members of various classes of modules. So this enterprise includes finding large indecomposable modules, finding large modules with no indecomposable direct summands, and so on. Set-theoretical techniques have turned out to be essential, and set-theoretical axioms beyond ZFC have to be invoked for some results.

The other source of related material is Shelah's solution to the Whitehead Problem ([She74], see [Ek76]).

The following are some survey papers: [CG85]; [Göb83]; [Göb84]; also see the introduction to [DuGö82].

The following sort of result, this one taken from [DuGö82] (also see [CG85]), is particularly striking. First note that a complete discrete rank 1 valuation domain does not have arbitrarily large indecomposable modules.

<u>Theorem</u> *Let R be a Dedekind domain, not a field. Let κ be an infinite cardinal. Then the following are equivalent:*

(i) R is not a complete discrete valuation domain;

(ii) there exists an indecomposable R-module of rank $\geqslant 2$

(iii) there exists an indecomposable R-module of rank $\geqslant \kappa$;

(*iv*) *there exists an R-module (not pure-injective!) of rank* $\geqslant \kappa$ *with no indecomposable summand*;

(*v*) *there exist R-modules of rank* $\geqslant \kappa$ *which do not satisfy Kaplansky's Test Problems*;

(*vi*) *if A is any cotorsion-free R-algebra then there exists an R-module with endomorphism algebra isomorphic to A*. □

There have been recent extensions of these results to arbitrary rings.

The above result can be used to answer in the negative a question of Kucera [Kuc87]. For it follows that there are arbitrarily large abelian groups with local endomorphism ring: since such a group M may be as large as one desires, the weight of M in $\mathrm{Th}(M^{\aleph_0})$ need not be 1.

It is shown in [DüGo85] (also see references in §15.1) that the class of all torsion theories of abelian groups (cf. §15.1) is a proper class: this contrasts with hereditary torsion theories, of which there can be at most 2^{λ} where $\lambda = 2^{|R|}$. Also see [DFS87].

[Hu83] contains related results on reflexive modules.

There are examples of the sort of pathology one obtains, even over tame rings, by working with arbitrary ("large") modules in [BrRi76].

CHAPTER 12 FUNCTOR CATEGORIES

In the early 70's, M. Auslander initiated a novel approach to the study of modules over artinian rings: rather than dealing directly with the modules, one works in the functor category (mod-R,Ab) of additive functors from the category of finitely presented modules to the category of abelian groups. In other words, one studies modules over the category of finitely presented modules. It might seem that this is piling complication upon complication, but Auslander's approach has been remarkably successful.

One main point of this chapter is to reconsider some of Auslander's results, especially the functorial characterisation of rings of finite representation type, in terms of pp formulas and pp-types. The other main purpose is to set down the material on pp-types and functors which should be useful in the classifications of infinite-dimensional indecomposable pure-injectives over particular (classes of) algebras.

Let U be the forgetful functor from mod-R to Ab - the functor which simply forgets that an R-module is anything more than an abelian group. If φ is a pp formula in one free variable, then the assignment $M \mapsto \varphi(M)$, with the induced action on morphisms, is a functor, F_φ, from mod-R to Ab: indeed, it is a subfunctor of U. We see in §1 that every subfunctor of U is a (possibly infinite) sum of such functors induced from pp formulas. It is also shown that the F_φ are finitely presented functors and that, if one allows pp formulas in more than one free variable, the F_φ are generating. Therefore, one may use these "pp-functors" in place of the more usual representable functors $(M,-)$.

The main reason for looking at (mod-R,Ab) is that the simple functors are in bijective correspondence with the indecomposable finitely generated modules. Of course, we are also interested in certain infinitely generated indecomposable modules (*viz.* the pure-injective ones), and this functor category is not large enough to let us see these. But we can replace mod-R by a variety of other subcategories of $\mathcal{M}_R$. In particular, we can replace it by the full subcategory with objects all the pure-injective modules (or "small" parts of this). If we do this, then there are more subfunctors of the forgetful functor but, still, all are sums of functors of the form F_p, where now p may be any pp-type.

In the second section, we consider the connection between the simple objects of (mod-R,Ab) and the indecomposable finitely generated modules (R right artinian): in particular, we describe the natural bijection between them. If one replaces mod-R by a category of pure-injective modules, then there is no particular reason to suppose that every simple functor arises from an indecomposable pure-injective. What we do see is that every simple subquotient of U does correspond to an indecomposable pure-injective (and conversely). That seems to be enough for the applications that I have in mind.

Also in that section, there is a proof of the fact that a ring is of finite representation type iff every simple functor in (mod-R,Ab) is finitely presented and every non-zero functor has a simple subfunctor. We see first that a simple functor is finitely presented iff the corresponding indecomposable is the ("finitely generated") hull of a neg-isolated type. So the condition that every simple functor be finitely presented is equivalent to the condition that every *finitely generated* irreducible type be neg-isolated. The other condition - that every non-zero functor have a simple subfunctor - is seen to be equivalent to the ring being right pure-semisimple. Putting these together, we obtain the functorial characterisation of finite representation type (our proof is self-contained only in the artin algebra case).

In the third section, we turn to somewhat different matters. It has already been mentioned in Chapter 4 that there are ways of turning pure-injectives into the injectives of suitable categories, thus allowing one to apply the well-developed theory of injective objects. There is more that one way of doing this: I describe a couple. I also show how to embed the model-theoretic context $\mathcal{C}_{T^*}$ into an abelian category. I finish the section by indicating how to develop a theory of torsion-theoretic localisation at pure-injectives.

There is a supplementary section which directs the reader to related work on pure global dimension and Krull dimension of functor categories.

12.1 Functors defined from pp formulas

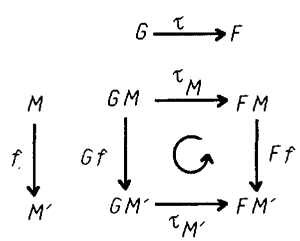
Readers who take fright at the mention of categories and functors may be relieved to learn that they have been using them all along in these notes. For, given any pp formula φ, the assignment which takes each module M_R to the pp-definable subgroup $\varphi(M)$ actually yields a functor from $\mathcal{M}_R$ to **Ab**. Let me be more precise.

Given a pp formula φ in n free variables, the functor F_φ from the category $\mathcal{M}_R$ of R-modules to the category **Ab** of abelian groups takes the module M to the abelian group $F_\varphi M = \varphi(M)$, and takes a morphism $M \xrightarrow{f} M'$ between R-modules to the induced morphism of abelian groups $F_\varphi f : \varphi(M) \longrightarrow \varphi(M')$. This morphism is well-defined, since if $\varphi(\bar{a})$ is true in M then $\varphi(f\bar{a})$ is true in M' (2.7). In practice, I will tend to blur the distinction between f and $F_\varphi f$. It is easy to check that the conditions for F_φ to be a functor are satisfied. In all this, we rely on the fact that $\varphi(M)$ is an abelian group: it may be that it is more - for example, if R is a K-algebra then it will be a K-vectorspace. In any case, it will be assumed for the moment that the image category is **Ab**, but what I say applies just as well if the image category is $\mathcal{M}_K$, for example. It should be observed for future use that if, in the above, the pp formula φ is replaced by a pp-type p, then one may define the corresponding functor F_p which takes M to $p(M)$.

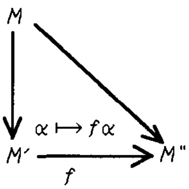
It will turn out that $\mathcal{M}_R$ is not a very good category on which to define our functors. It is too large, and infinitely generated modules display too many pathologies. Especially when R is an artinian ring, and even in general, it is more appropriate to look at functors from the category, mod-R, of finitely presented modules. These functors will be the the main topic of this section. When, however, we are lead to deal with infinitely generated modules - for example, when we look at infinitely generated pp-types - then we have to expand mod-R somewhat.

Let us note some basic points about these functors which are defined from pp formulas. The functor corresponding to the formula "$v=v$" is, of course, the **underlying, or forgetful, functor** which simply forgets the R-multiplications: we denote it by U. Observe that its powers, U^n, are the functors corresponding to formulas of the form "$\bar{v}=\bar{v}$". If φ is a pp formula in n free variables then, clearly, F_φ is a subfunctor of U^n. Moreover if $\varphi \to \psi$ then $F_\varphi \leqslant F_\psi$ (whether or not the converse is true depends on whether the domain category is large enough to distinguish between inequivalent pp formulas).

For the sake of readers who are not very happy with functorial language, let me make a few observations. If F and G are functors, then $F \oplus G$ denotes the functor which is given on objects by taking M to $FM \oplus GM$.



The relationship $G \leqslant F$ means, roughly, that for all objects M one has $GM \leqslant FM$; more precisely, there is a natural transformation $\tau : G \longrightarrow F$ such that, for every module M, the M-component of τ, τ_M, is an embedding from FM to GM. If $G \leqslant F$ then one can define the quotient or factor F/G, which is in the same functor category and is defined on objects by $M \mapsto FM/GM$ and on morphisms in the obvious way. Recall that a **natural transformation** from G to F is given by, for each object M, a morphism $\tau_M : GM \longrightarrow FM$ such that all diagrams of the sort shown commute.



There is another sort of functor which is very commonly considered. Let M be any object of the domain category $\mathcal{C}$ (which is always supposed to be additive): define the corresponding **representable** functor $(M,-)$ by: $(M,-)(M')=(M,M')$ on objects (note that our morphism sets are abelian groups or K-vectorspaces, as appropriate); and by the "obvious" action, given by composition, on morphisms (the "obvious" action should become so, on considering the diagram opposite).

One of the key facts about the representable functors is that they are projective and, taken together, form a generating set of finitely generated projectives for the functor category $(\mathcal{C},\mathrm{Ab})$. Moreover, this functor category contains a copy of $\mathcal{C}^{\mathrm{op}}$: the embedding is defined by taking each object M of $\mathcal{C}$ to the functor $(M,-)$, and taking each morphism $M\xrightarrow{f}M'$ of $\mathcal{C}$ to the "obvious" induced morphism $(f,-):(M',-)\longrightarrow(M,-)$. This actually embeds $\mathcal{C}^{\mathrm{op}}$ as a full subcategory of $(\mathcal{C},\mathrm{Ab})$. The last point means that every morphism (natural transformation) between the functors $(M',-)$ and $(M,-)$ arises in this way from some morphism $M\xrightarrow{f}M'$ in $\mathcal{C}$ (the proof is immediate from the Yoneda Lemma). For more background, see [Aus66], [Mit72] for example.

In the usual functorial approach to the representation theory of algebras, the representable functors play a prominent role: here this role will be taken over by the pp-defined functors above. Such functors appear in [Z-HZ78] under the name "p-functors" and in [Zim77] under the name "(finitely) matrizable functors" (they are also implicit in [GJ73]).

Lest all this seem overly abstract, I present the proofs for the following special case: R is any ring (eventually it will be right artinian) and the domain category, $\mathcal{C}$, is the category mod-R of finitely presented (equivalently, if R is right noetherian, finitely generated) modules. I will indicate how the results may be generalised, especially since there is another possibility for $\mathcal{C}$ which is of significance. We will see that every subfunctor of the forgetful functor is a sum, possibly infinite, of the "pp-functors" defined above. Denote the functor category (mod-R,Ab) by $\mathrm{F}=\mathrm{F}(R)$.

It is the following property (8.5) of finitely presented modules which is crucial:

(EH) if $\bar{a},\bar{b}$ are, respectively, in the finitely presented modules M, M' and are such that $\mathrm{pp}^{M}(\bar{a})\leqslant \mathrm{pp}^{M'}(\bar{b})$, then there is a morphism $M\xrightarrow{f}M'$ such that $f\bar{a}=\bar{b}$.

<u>Lemma 12.1</u> [Pr83; 3.1] *Suppose that R is arbitrary and let φ,ψ be pp formulas (both in n free variables for some n). Then $\psi\to\varphi$ iff $F_\psi\leqslant F_\varphi$ as functors in* (mod-R,Ab).

Proof If $\psi\to\varphi$ then, for every module M, one has $\psi(M)\leqslant\varphi(M)$ and so, simply by definition, $F_\psi\leqslant F_\varphi$. If, conversely, $F_\psi\leqslant F_\varphi$, then for every finitely presented module M one has $\psi(M)\leqslant\varphi(M)$. So we finish by recalling that, if there is some tuple in some module satisfying ψ but not φ, then there is such an element in a finitely presented module (8.14). □

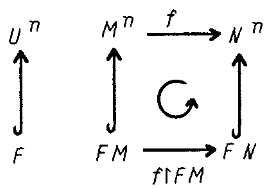
It has been observed that every functor of the form F_φ is a subfunctor of the forgetful functor U. The structure of the lattice of subfunctors of U will be a major consideration in this and the next section. I begin by showing that every subfunctor of U (and of its powers) is a sum of functors of the form F_φ: in consequence, these functors are precisely the finitely generated subfunctors of U.

Exercise 1 We are concerned here with functors *as subfunctors of U*. Show that it is possible to have functors $F,G\leqslant U$ which are distinct, but which are isomorphic as functors. [Hint: take $R=K[x_1,x_2:x_ix_j=0\ (i,j\in\{1,2\})]$ where K is a field and consider the category of functors from the amenable (see below) category of projective = free modules to Ab]

<u>Proposition 12.2</u> [Pr83; 3.2] *Let* $n \in \omega$ *and let* F *be a subfunctor of* U^n. *Then* $F = \Sigma\{F_\varphi : F_\varphi \leqslant F\}$.

Proof Set G to be the sum, $\Sigma\{F_\varphi : F_\varphi \leqslant F\}$, of those pp-functors less than or equal to F. Were G strictly below F then (by definition of these functors) there would be some finitely presented module M, and some $\bar{a}$ in M with $\bar{a} \in FM \setminus GM$.

Let $p = \mathrm{pp}^M(\bar{a})$: since M is finitely presented, there is a single formula φ equivalent to p in every (finitely presented) module (**8.4**). I claim that $F_\varphi \leqslant F$ - contradicting that F_φ is not below G.



So let N be finitely presented and let $\bar{b} \in \varphi(N)$: it must be shown that $\bar{b} \in FM$. Since $\bar{b}$ satisfies φ, and hence satisfies $\mathrm{pp}^M(\bar{a})$, one has $\mathrm{pp}^N(\bar{b}) \geqslant \mathrm{pp}^M(\bar{a})$. So, by the property (EH), there is a morphism $M \xrightarrow{f} N$ taking $\bar{a}$ to $\bar{b}$. Since $\bar{a} \in FM$ the image, $\bar{b}$, of $\bar{a}$ under $f^{(n)}$ is in FN (consider the diagram shown).

Thus $\varphi(N) = F_\varphi(N) \leqslant F(N)$, as required. □

We will now see that, like the representable functors, the pp-functors form a generating set of finitely presented objects of **F**. A functor F is **finitely generated** if, whenever it is expressed as a sum of subfunctors, it is equal to the sum of finitely many of them: in other words, it cannot be expressed as a directed union (or sum) of proper subfunctors. The functors F_φ are finitely generated: for if we have $F_\varphi = \Sigma F_i$ for some $F_i \leqslant F_\varphi$ then, taking $\bar{a}$ in M be a free realisation of φ (in a finitely presented module M (§8.3)), we deduce that there are finitely many elements $\bar{a}_j \in M$ with $\bar{a}_j \in F_j(M)$ and with $\bar{a} = \Sigma \bar{a}_j$. Then, since each element in $\varphi(N)$ for any (finitely presented) N is, by (EH), the image of $\bar{a}$ under some morphism, it follows that F is the sum of the corresponding finitely many functors F_j, as required.

The functor F is said to be **finitely presented** if (it is finitely generated and) there is an exact sequence $H \hookrightarrow G \twoheadrightarrow F$ with G and H finitely generated. Just as with modules, it follows that if F is finitely presented then, in any exact sequence of the above form, if G is finitely generated, so is H.

<u>Proposition 12.3</u> [Pr83; 3.4] *If* φ *is a pp formula then the functor* F_φ *is finitely presented.*

The set of all such functors, F_φ, *is generating in* (mod-R,Ab): *that is, for every functor* F:mod-$R \longrightarrow$ Ab *there is an exact sequence* $G \hookrightarrow H \twoheadrightarrow F$ *with each of* G, H *a coproduct (direct sum) of functors of the form* F_φ.

Proof Since the representable functors are known to be finitely generated, it will be enough, for the first part, to show that for every φ there is an exact sequence of the form $(M',-) \hookrightarrow (M,-) \twoheadrightarrow F_\varphi$ with $M, M' \in$ mod-R.

We take M to be any finitely presented module which contains a free realisation, $\bar{a}$ (say), of φ. An epimorphism from $(M,-)$ to F_φ is defined as follows: it is the natural transformation $\tau:(M,-) \longrightarrow F_\varphi$ which is given component-wise by $\tau_N:(M,N) \longrightarrow \varphi(N)$, where τ_N is defined to take $f \in (M,N)$ to the value, $f\bar{a}$, of f at $\bar{a}$. By (EH), these components τ_N are epi (cf. proof of **12.2**): this implies that τ is epi, as required. (Details of checking that τ is a natural transformation are left as an exercise in the definitions.)

So the kernel of τ has to be identified. Suppose that $\bar{a}$ is $(a_1, \ldots, a_n)$: set $L = \Sigma_1^n a_i R \leqslant M$. The natural projection $\pi: M \twoheadrightarrow M/L$ defines, by composition, a monomorphism $(M/L,-) \hookrightarrow (M,-)$. The claim is that this is the kernel of τ. Let $N \in$ mod-R and let $f \in (M,N)$. Observe that $\tau_N f = 0$ iff $f\bar{a} = 0$; that is, iff $fa_1 = \ldots = fa_n = 0$ - which is equivalent to $fL=0$. The last occurs precisely if f factors through M/L - that is iff f lies in the image of $(M/L,N) \hookrightarrow (M,N)$, as required.

Next, it is shown that the functors F_φ are generating. Since the representable functors are known to be generating, it will be enough to show that every representable $(M,-)$ is isomorphic to a quotient of a pp-functor.

Take any finite generating set $a_1,\dots,a_n$ for M, and let p be $\mathrm{pp}^M(\bar{a})$: as noted already, F_p has the form F_φ for some pp formula φ. As in the first part, there is an exact sequence $(M/L,-) \hookrightarrow (M,-) \longrightarrow F_\varphi$ where $L = \Sigma a_i R = M$: thus $(M,-)$ is even isomorphic to F_φ, as required. □

At the end of the above proof we saw that every representable functor is isomorphic to a functor of the form F_φ, where φ is a pp formula in a finite number of variables. The result would be neater if every representable were an image of (so, by projectivity, a direct summand of) a functor arising from a pp formula in one free variable. If the representables corresponding to cyclic modules were generating, then the above proof would show this: however, that is the case iff every finitely presented module embeds into a direct sum of cyclic modules. This property is satisfied by Prüfer domains - indeed, these are exactly the commutative domains over which every finitely presented module is a direct summand of a direct sum of cyclic modules [War69; Prop 5]. On the other hand it will be seen that every simple functor is an image of a functor F_φ where φ has just one free variable, and that will be enough for us here.

It follows from what has been said that if a functor F has a presentation $F_\psi \hookrightarrow F_\varphi \twoheadrightarrow F$ by pp-functors then it is finitely presented (this is one justification for our use of pp-functors in place of representable functors).

Corollary 12.4 [Pr83; 3.3] *Let $F \leqslant U^n$ for some $n \in \omega$. Then F is finitely generated iff it is of the form F_φ for some pp formula φ in n free variables.*

Proof One direction has been established already, so suppose that $F \leqslant U^n$ is finitely generated. By **12.2**, F is a sum of functors of the form F_φ and, being finitely generated, it is a sum of finitely many of them. So the only point to be checked is that if $\varphi_1,\dots,\varphi_k$ are pp formulas (in n free variables) then the sum $G = \Sigma F(\varphi_i)$ is of the same form; but indeed (**8.1**), it is just F_φ where φ is $\bigwedge_1^n \varphi_i$. □

The point used at the end of the last proof is contained in the next result, the proof of which is left as an easy exercise.

Lemma 12.5 *The map $\langle\varphi\rangle \mapsto F_\varphi$ defines a lattice anti-isomorphism from the lattice, P^{f}, of finitely generated pp-types to the lattice, $\mathrm{Latt}^{\mathrm{f}}(U)$, of finitely generated subfunctors of $U^{(n)}$. In particular, the intersection of two finitely generated subfunctors of $U^{(n)}$ is again finitely generated.* □

The last point is in contrast with the situation for right ideals.

Also, the situation for infinitely generated pp-types is rather more complicated. Clearly, there is a well-defined morphism $\mathrm{P} \longrightarrow \mathrm{Latt}(U)$, given by taking p to F_p; but, in general, this need be neither mono nor epi. An example which shows that it is not mono is obtained by taking the ring to be $\mathbb{Z}$, and considering the pp-type $p(v)$ which says that v is annihilated by the prime q and is divisible by every positive power of q: F_p is the zero functor on mod-$\mathbb{Z}$. For some consideration of the extent to which $p \mapsto F_p$ is epi, see after **12.10**.

Corollary 12.6 *If the elementary Krull dimension (see* §10.5*) of the largest theory of R-modules is defined, then so is the Krull dimension of the lattice of finitely* *generated subfunctors of* U, *and the values are equal. In particular,* U *has the dcc on subfunctors iff* R *is right pure-semisimple.* □

Before going on to look in more detail at the subfunctors of U, I indicate how the results of this section may be generalised. Say that a full subcategory $\mathcal{C}$ of $\mathcal{M}_R$ is **amenable** ([Pr83; §3]) if it is closed under finite direct sums and direct summands and if it satisfies the weak injectivity/extension-of-morphism property (EH), which was defined above. Of course mod-R is amenable, but there are other important examples - the full subcategory whose objects are: the pure-injectives; or the pure-injective summands of models of T $(T=T^{\aleph_0})$; or the pure-injectives of finite weight in $\mathcal{P}(T)$, where $(T=T^{\aleph_0})$; or the finite abelian groups; or the finitely generated projective modules. Some examples which I will refer to later are the full subcategories of $\mathcal{M}_R$ whose objects are: all the pure-injective modules - I use the notation PI(R); or the hulls of finite sets of elements - I use the notation PIF(R). I drop the "R" when convenient. Also, given a complete theory $T=T^{\aleph_0}$, there are the corresponding categories for direct summands of models of T, and I denote these by PI(T) and PIF(T) respectively. Of course, PI(R) and PI(T) are not sets, so one might prefer to restrict the size of their objects by some large enough cardinal (and one would lose little by doing so).

Clearly, if $\mathcal{C}$ is amenable then so is $\oplus\mathcal{C}$ - the full subcategory whose objects are the arbitrary coproducts of members of $\mathcal{C}$.

Given an amenable category $\mathcal{C}$, denote by $\mathrm{P}_n\mathcal{C}$ the sub-poset of P_n consisting of all pp-n-types of elements of members of $\mathcal{C}$. The special notation $\mathrm{P}_n{}^{\mathrm{f}}$ has already been introduced for P_n(mod-R). It is easy to see that $\mathrm{P}\mathcal{C}$ is $\wedge$-closed in P and that, if $\mathcal{C}$ is closed under arbitrary products, then $\mathrm{P}\mathcal{C}$ is $\bigcap$-closed in P. On the other hand, $\mathrm{P}\mathcal{C}$ need not be $\vee$-closed (Exercise: take $\mathcal{C}$ to be $\mathcal{P}(T)$ where T is the theory of $M=(\mathbb{Z}_2\oplus\mathbb{Z}_3)^{\aleph_0}$, take $p=\mathrm{pp}^M(\bar{1},\bar{0})$, $q=\mathrm{pp}^M(\bar{0},\bar{1})$, and compare the join of p and q in P and in $\mathrm{P}\mathcal{C}$).

Let me now indicate how much of the above goes in the general case and, also, in the cases of particular interest to us.

Replacing elements of P^{f} by elements of $\mathrm{P}(\mathcal{C})$, **12.1** holds for PI and PIF and, indeed, for any amenable category, provided one replaces the pp formulas by members of $\mathrm{P}\mathcal{C}$.

Similarly, **12.2** goes through if one replaces the functors of the form F_φ by those of the form F_p where $p\in\mathrm{P}\mathcal{C}$; in particular, if $\mathcal{C}$ is PI or PIF, then one has that every subfunctor of the forgetful functor is a sum of functors of the form F_p for $p\in\mathrm{P}$ (observe, by the way, that since R need not be pure-injective, the forgetful functor need not be representable, although if R is an artin algebra, then it will be).

The statement of **12.3** breaks into three parts. The first is that the functors F_p should be finitely generated. This is true for PI and PIF and, indeed, in general, because we have defined the members of $\mathrm{P}\mathcal{C}$ to be just those pp-types which have a "free realisation" in some member of $\mathcal{C}$.. The next point is whether the F_p are finitely presented. The argument of **12.3** shows that this is still true for PI, however it may fail for PIF - consider PIF($\mathbb{Z}$) and take p to be the pp-type of $1\in\mathbb{Z}$ - since $\bar{\mathbb{Z}}/\mathbb{Z}$ is not the hull of a finite number of elements, one has that F_p is not finitely presented. A necessary and sufficient condition for all the functors F_p to be finitely presented is

(*) for every C in $\mathcal{C}$ and every finitely generated submodule L of C, the functor $(C/L,-)$, when restricted to $\mathcal{C}$, is finitely generated.

The remainder of **12.3**, that the functors F_p $(p\in\mathrm{P}\mathcal{C})$ are generating, is a rather strong condition. If satisfied, then each $(C,-)$ for $C\in\mathcal{C}$ is an image of, hence, by projectivity, is a direct summand of, a finite direct sum of F_p's. In particular, each $(C,-)$ embeds in some power U^n of the forgetful functor. This is a strong condition: for example, if $R\in\mathcal{C}$ so that

$U \simeq (R,-)$, this implies, by the Yoneda Lemma, that there is an epi from R^n to C, i.e., that C is finitely generated.

Turning to 12.4: the categories $\mathcal{C}$ for which a subfunctor of U is finitely generated iff it is of the form F_p for some $p \in \mathrm{P}\mathcal{C}$ are just those for which the F_p are finitely generated. Finally, 12.5 also goes through for such categories, provided one replaces P^f by $\mathrm{P}\mathcal{C}$.

I finish this section by considering the relationship between $\mathrm{P}\mathcal{C}$ and the lattice $\mathrm{Latt}U$ of all subfunctors of the forgetful functor (we know already that every such functor is a sum of functors of the form F_p). I will do this only in the two special cases $\mathcal{C} = \text{mod-}R$ and $\mathcal{C} = \mathrm{PIF}$: let me denote the respective functor categories by F and PIF So let us consider the map $F : p \mapsto F_p$.

The first point to note is that "more" of the elements of $\mathrm{Latt}U$ are finitely generated in the case of PIF than in the case of F: indeed every F_p is finitely generated for PIF, whereas only the F_p, with p equivalent in every finitely presented module to some F_φ, are finitely generated for F.

In the case of PIF, it is immediate that the function F is 1-1; this is not true for F. For consider the case $R = \mathbb{Z}$ and let p be the pp-type of an element of order 2 in $\mathbb{Z}_{2^\infty}$. It is easy to see that if $a \in p(M)$, where M is a finitely presented abelian group, then a must be zero. Thus $F_p = 0$ and so the function F is not 1-1. In contrast, we will see that if R is an artin algebra then F is 1-1. But, first I show that F commutes with infinite joins in P.

Consider F first. By "φ" I will denote the pp-type generated by φ. Let $p \in \mathrm{P}$. Then $p = \bigvee\{\varphi : \varphi \in p\}$. I show that $F_p = \bigcap\{F_\varphi : \varphi \in p\}$; let G be this intersection of subfunctors of U. If $\varphi \in p$ then one has that any element satisfying p (in any module) also satisfies φ. Hence $F_p \leqslant F_\varphi$ and so $F_p \leqslant G$. If one had $F_p < G$ then, by 12.2, there would be some pp formula ψ with $G \geqslant F_\psi$ but F_p not containing F_ψ. Then, for any $\varphi \in p$, one would have from $F_\psi \leqslant G \leqslant F_\varphi$ that $\varphi \leqslant \psi$ in P (this follows by 12.1): that is, ψ implies every formula in p. Hence, one would have $\psi \geqslant p$ and so $F_\psi \leqslant F_p$ - contradiction. This has shown that F commutes with arbitrary joins of finitely generated pp-types. Since every pp-type may be expressed as a join of finitely generated pp-types, it follows that F does commute with arbitrary joins in P.

Now consider PIF: suppose that $p = \bigvee\{q_\lambda : \lambda \in \Lambda\}$. Since F commutes with finite joins (by 2.3(iii)), it may be assumed that the join is directed. Let $G = \bigcap\{F(q_\lambda) : \lambda \in \Lambda\}$. Since $F_p \leqslant F(q_\lambda)$ for each λ (by definition of the functors) one has $F_p \leqslant G$. But also, G is a sum of functors of the form F_φ, so the argument in the previous paragraph applies to show that F commutes with joins.

Now let R be an artin algebra. Let $p > q$. By the above, one has $F_p = \bigcap\{F_\varphi : \varphi \in p\} \leqslant \bigcap\{F_\psi : \psi \in q\} = F_q$. To show that the inclusion is strict, take any $\varphi \in p \backslash q$; if F_p and F_q were equal, one would have $F_q \leqslant F_\varphi$. By 13.2 (the proof may be read now), there is an element a, with pp-type q, lying in a direct product of finitely presented modules: so a has the form $(a_\lambda)_\lambda$ where, for each λ, a_λ lies in a finitely presented module. Since a satisfies q, so does each a_λ; since a_λ lies in a finitely presented module it therefore satisfies φ; since each component of a satisfies φ, so does a. That is, $\varphi \in q$ - contradiction, as required. Thus F is 1-1 on F if R is an artin algebra.

Next, I consider irreducible pp-types: an obvious question is whether, if p is an irreducible pp-type, the functor F_p is $+$-irreducible.

For PIF this is obvious (by 4.30).

For F; again let R be an artin algebra. Let $p \in \mathrm{P}$. Then, in P, $p = \bigcap\{\psi : \psi \to p\}$. This is immediate from 13.2 (cf. the above paragraph). Suppose that p is irreducible in P and that one has $F_p = G + H$ in $\mathrm{Latt}U$. By 12.2, $G = \sum_\varphi F_\varphi$ and $H = \sum_\psi F_\psi$ for suitable pp-formulas. Put $q = \bigcap_\varphi \langle\varphi\rangle$ and $q' = \bigcap_\psi \langle\psi\rangle$ in P. Then, since F is 1-1, $F_q = G$ and $F_{q'} = H$. So one has $F_p = F_q + F_{q'}$ and hence, since $p \mapsto F_p$ is 1-1, it follows that $p = q \cap q'$. So $p = q$ (say) and therefore $F_p = H$, as required.

Finally, let us consider the relationship between F_p and the functors which lie below it.

Consider F first. Let $p \in$ P; then one has $F_p = \Sigma\{F_\psi : \psi \to p\}$. For certainly if ψ implies p then $F_\psi \leqslant F_p$. Conversely, if $F_\psi \leqslant F_p$ then, for every $\varphi \in p$, one has (as before) that, in every finitely presented module φ is a consequence of ψ; therefore this is true in every module and so does ψ imply p, as required. One may ask whether it is, further, the case that F commutes with arbitrary meets in P. So suppose that $p = \bigcap_\lambda q_\lambda$ in P. Then certainly $\Sigma_\lambda F(q_\lambda) \leqslant F_p$. Let φ be such that $F_\varphi \leqslant F_p$: in order to show that F does commute with arbitrary meets, it will suffice (by 12.2) to show that $F_\varphi \leqslant \Sigma_\lambda F(q_\lambda)$. Again, it seems that we have to assume that R is an artin algebra: then each q_λ may be represented as a meet of the form $\bigcap_\mu \psi_{\lambda\mu}$ for suitable pp formulas $\psi_{\lambda\mu}$; hence p is the meet of all these finitely generated pp-types as λ and μ vary. So, by what has already been shown, F_p is the sum of all the $F(\psi_{\lambda\mu})$. By the Grothendieck property and since F_φ is finitely generated (cf. proof of 12.9), it follows that F_φ is contained in a finite sum of such $F(\psi_{\lambda\mu})$'s and hence is contained in a finite sum of $F(q_\lambda)$'s, as required. So, if R is an artin algebra then F commutes with arbitrary meets in P. We will see that this is not so for PIF.

So consider PIF and let p be an irreducible pp-type. Then (by 4.30) the inclusion $F_p > \Sigma\{F_q : q > p\}$ is strict. But if p is not neg-isolated then $p = \bigcap\{q : q > p\}$. Now, unless R is of finite representation type, there are irreducible pp-types which are not neg-isolated (11.38) and so it follows that F does not commute with arbitrary meets in P. It is clear, in any case, that $\Sigma\{F_q : q > p\}$ is actually the radical JF_p (see §2) and, unless p is neg-isolated, it is not of the form $F_{p'}$ for any pp-type p'.

So we see that, for an artin algebra, the "gaps" between functors of the form F_p (p irreducible) and their radicals are "closed" when we work in the functor category (mod-R,Ab) but they open up when we expand to the category PI(F) reflecting the presence of new indecomposables and their corresponding simple functors (see the next section for these).

12.2 Simple functors

Throughout this section the ring R is assumed to be right artinian.

Also, I will work mainly in the functor category F = (mod-R, Ab), although I will sometimes refer to the category (PI(F), Ab).

A major insight of Auslander [Aus74a; §2] is that over a right artinian ring there is a bijection between the simple functors in F and the indecomposable finitely presented modules. We begin the section by examining this connection from the point of view of pp-types.

For any functor F the (Jacobson) **radical** of F, denoted JF, is defined to be the intersection of all maximal proper subfunctors of F (if F is finitely generated then this intersection will not be empty). The functor F is said to be **local** if JF is the unique maximal subfunctor of F, in which case JF also is the sum of all proper subfunctors of F.

Theorem 12.7 [Aus74a; 2.3] (*R right artinian*) *Let $p \leftrightarrow \varphi$ be a finitely generated pp-type. Then F_p is local iff p is irreducible: in this case, denote by S_p (or $S(p)$) the simple quotient functor F_p/JF_p.*

Every simple functor in F *has the form S_p for some such pp-type.*

Proof [Pr83; 3.6] If p is reducible (in P^f, equally in P, by 8.8) then there are $q, q' \in P^f$ strictly containing p and such that $q \cap q' = p$. Put in terms of the corresponding functors, this says that F_p is the sum of the two proper subfunctors F_q and $F_{q'}$, and so F_p is not local.

If, on the other hand, p is irreducible and if F_p were not local, then p would be a sum of proper subfunctors and, since by **12.3** it is finitely generated, it would be a sum of finitely many (so two) of them. By **4.30**, one obtains a contradiction.

Suppose next that S is a simple functor in (mod-R,Ab). Choose $N \in$ mod-R of minimal length such that $SN \neq 0$. Since S preserves finite direct sums, N certainly is indecomposable. Let a be a non-zero element of N and set $p = \mathrm{pp}(a)$. I show that $S \simeq S_p$.

By the proof of **12.3**, there is an exact sequence $(N/aR,-) \hookrightarrow (N,-) \twoheadrightarrow F_p$ in F. Mapping this to S yields the left exact (exercise) sequence in Ab $(F_p, S) \hookrightarrow ((N,-), S) \longrightarrow ((N/aR,-), S)$. By the Yoneda Lemma, this is equivalent to the following left exact sequence: $(F_p,S) \hookrightarrow SN \longrightarrow S(N/aR)$. By choice of N, the last term in the sequence is zero: so we conclude $(F_p,S) \simeq SN$. Since the latter is non-zero, it follows that there is a non-zero morphism from the local functor F_p to S. Hence $S \simeq F_p/JF_p \simeq S_p$, as required. □

Exercise **1** Prove the second part of the above by using the fact that the functors of the form F_φ are generating (**12.3**).

From the above proof, one quickly deduces the explicit action of any simple functor $S \simeq S_p$. For, if $N \simeq N_p$ is the indecomposable on which S is non-zero then, by that proof, we have $SN \simeq (F_p, S_p) = (F_p, F_p/JF_p) \simeq D_p$, where D_p is the division ring $\mathrm{End}N_p/J\mathrm{End}N_p$.

<u>Corollary 12.8</u> [Aus74a; §2] (*R right artinian*) *Suppose that p and q are irreducible finitely generated pp-types (not necessarily in the same number of free variables). Then $S_p \simeq S_q$ iff the hulls of p and q are isomorphic.*

The action of the simple functor S_p is given on objects by $N \mapsto D_p^{(\kappa)}$ where κ is the multiplicity of N_p in N.

Proof If $S_p \simeq S_q$ then $S_qN_p \neq 0$ since $S_pN_p \neq 0$: say $a \in F_qN_p \setminus JF_qN_p$. Then one has $q^+(a)$ and, for every $q' > q$, not $q'^+(a)$: thus $\mathrm{tp}(a) = q$.

The other points are obvious from what has gone before. □

If R is an artin algebra then **12.7** provides us with a bijection between isomorphism types of indecomposable (pure-injective) finitely generated modules and isomorphism types of simple functors in (mod-R, Ab). Moreover, every simple functor is a sub-quotient of the forgetful functor U. This means that one may, in principle, classify the indecomposable finitely generated pure-injectives by looking at "filtrations" of U by pp formulas (*viz.* maximal chains of subfunctors of U). This is precisely the method used (though expressed in slightly different ways) by Gelfand and Ponomarev, Ringel, Gabriel and others ([Gab75], [GP68], [GP72], [Ri75], [Ri75b], [BuRi87]) to classify the indecomposable finite-dimensional representations of certain finite-type and tame algebras. The success of this method suggests that one should try to use the same kind of techniques to classify all the indecomposable pure-injectives over tame finite-dimensional algebras. In §13.3 I will briefly report on some limited success in this direction; here, I will at least establish the basic connection between indecomposable pure-injectives and simple functors in an appropriate functor category.

The obvious functor category to try is (PI, Ab) or, rather, one where PI is replaced by a small part of it, such as PIF. The specific choice will not matter for what I say here, so let F′ be a functor category of this sort. Since, as we have seen in §12.1, the functors F_p need not be generating in F′, there seems to be no particular reason to suppose that every simple functor in F′ is of the form F_p/JF_p for some irreducible pp-type p. Probably this does not matter: what we do seem to need, in order to apply the method, is that every simple subquotient of the forgetful functor U should be of this form. I proceed to show this.

Theorem 12.9 *Let R be any ring, let F' be a functor category of the sort above, and let S be a simple subquotient of the forgetful functor. Then there is an irreducible pp-type p such that $S \simeq F_p/JF_p$.*

Proof By 4.38, every (continuous) pure-injective is a summand of a product of indecomposable pure-injectives. So there is an indecomposable pure-injective N with $SN \neq 0$. Suppose that S has the form F/G where $U \geqslant F > G$. Let $a \in FN \setminus GN$ have type p: then one has (by (EH)) $F = F_p + G$ and so $S \simeq F_p/G \cap F_p$ necessarily is isomorphic to S_p. □

Although we began this section by considering finitely generated functors, we have actually been lead in 12.7 to consider infinitely generated ones - for the radical of a finitely generated functor need not be finitely generated. In fact, whether JF_φ is or is not finitely generated, is of considerable significance.

First we see what it means, in terms of the corresponding pp-types, for JF_φ to be finitely generated. Then we will go on to derive a proof of the result of Auslander that a ring is of finite representation type iff every simple functor is finitely presented and every non-zero functor has a simple subfunctor.

Theorem 12.10 [Pr83; 3.8] *Let R be any ring. Suppose that p is an irreducible finitely generated pp-type. Then the following conditions are equivalent:*
(i) the simple functor S_p is finitely presented;
(ii) p is isolated in $\mathsf{P}^{(f)}$;
(iii) the type $\breve{p}$ is isolated (in the space of all 1-types for T^*).

Proof The equivalence of (ii) and (iii) for finitely generated irreducible types has been seen before in 8.8.

Consider the presentation $J_p \hookrightarrow F_p \twoheadrightarrow S_p$ of S_p. It has been remarked already that S_p is finitely presented iff J_p is finitely generated; that is (by 12.4) iff J_p has the form F_q for some finitely generated pp-type q. Such a type q (if it exists) is the minimal pp-type above p in P^{f}, and hence in P - so the equivalence of (i) and (ii) follows. □

Note that if p is irreducible (f.g. or not) and not isolated in P, then there is no pp-type going to JF_p under the mapping described in §12.1. For, if $p = \bigcap_\lambda q_\lambda$ is a representation of p as an infinite intersection of strictly greater pp-types then, certainly one has, for each λ, that $F(q_\lambda) \leqslant JF_p$; then it follows by 12.2 that, in fact, $JF_p = \Sigma F(q_\lambda)$ but there is no single q_λ with $JF_p = F(q_\lambda)$.

The next result follows by 11.32 (compare with 11.38).

Corollary 12.12 [Pr83; 3.9, 3.11] *Let R be right artinian. Set $\mathsf{F} = (\text{mod-}R, \mathbf{Ab})$. Then the following conditions are equivalent:*
(i) every simple functor in F is finitely presented;
(ii) every finitely generated irreducible (pp-)type is isolated;
(iii) whenever N, N_λ $(\lambda \in \Lambda)$ are indecomposable finitely generated modules such that N purely embeds in the product of the N_λ then, for some λ, one has $N \simeq N_\lambda$. □

Corollary 12.13 [Aus74a; 1.11] *If the ring R is of finite representation type, then every simple functor in $(\text{mod-}R, \mathbf{Ab})$ is finitely presented.* □

This third corollary is immediate from 12.10 and 11.38.

An important class of rings satisfying the equivalent conditions of 12.12 are the algebras finite-dimensional over a field (or more generally the Artin algebras) - see Chapter 13 for more on these.

Corollary 12.14 *Suppose that* $T = T^{\aleph_0}$ *is totally transcendental. Then the following are equivalent for an irreducible type* p:
(i) *the simple functor* S_p *in* $(\mathcal{P}(T), \mathrm{Ab})$ *is finitely presented*;
(ii) *the pp-type* $p \in \mathrm{P}^T$ *is isolated in* P^T;
(iii) *the type* $\tilde{p}$ *is isolated in* $S^T(0)$. □

Corollary 12.15 *Suppose that* $T = T^{\aleph_0}$ *is totally transcendental. Then the following are equivalent*:
(i) *every simple functor in* $(\mathcal{P}(T), \mathrm{Ab})$ *is finitely presented*;
(ii) T *has finite Morley rank*. □

The two "local" results above are obtained by working with the amenable category $\mathcal{P}(T)$ in place of mod-R (and applying 6.28 for the second result).

If the ring R is of finite representation type then one can say more than 12.13: indeed, one has the following result of Auslander.

Proposition 12.16 *Suppose that* R *is of finite representation type. Then every finitely generated functor in* (mod-R, Ab) *is of finite length.*

Proof By 11.36 and 6.28, P_n has finite length for each n. Hence each underlying functor U^n has finite length, and so the same is true for all the F_φ. Since every finitely generated functor is an image of one of these (see 12.3), the result follows. □

In fact, Auslander showed that R is of finite representation type if, and only if, every finitely generated functor in (mod-R, Ab) is of finite length [Aus74a; 1.11]. This latter condition breaks down into two components, as follows.

Lemma 12.17 [Aus74a; 1.11] *The following conditions on* F = (mod-R, Ab) *are equivalent*:
(i) *every finitely generated object of* F *is of finite length*;
(ii) (a) *every simple functor in* F *is finitely presented, and*
 (b) *every non-zero functor in* F *has a simple subfunctor.* □

Clearly (b) follows from (i). Also, if S is simple, with presentation $K \hookrightarrow F \twoheadrightarrow S$ where F is finitely generated then, since F has finite length, it follows that K is finitely generated. For the converse (that (ii) implies (i)), see [Aus74a; §1] (the proof uses the functor F defined in the proof of 12.18(i)⇒(ii) below). The converse also follows from 12.19 below (for an artin algebra our proof of this is self-contained - see 11.31).

An obvious next step is, therefore, to consider the condition that every non-zero (finitely generated) functor has a non-zero socle: in fact this condition is really dual to that which requires every simple functor to be finitely presented. It was proved in [Aus76] (also see [Aus74a; 3.4] - note that Auslander is "working on the other side" there) that this property characterises the right pure-semisimple rings among the right artinian ones: I give a proof of this here, using the techniques of this chapter.

Theorem 12.18 *Let* R *be arbitrary and set* F = (mod-R, Ab). *Then the following conditions are equivalent*:
(i) *every non-zero functor in* F *has a simple subfunctor*;
(ii) R *is right pure-semisimple.*

Proof The proof comes from [Pr83; 3.12], but it is actually quite close to [Aus74a; §1].

(i)⇒(ii) Suppose that R is not right pure-semisimple; i.e. (11.2) suppose that the largest theory T^* of modules is not totally transcendental. A functor in F is produced which has no simple subfunctor.

Let $\Psi = \{\psi : \psi$ is pp and the interval $[\psi(v), v=0]$ has the dcc$\}$: in other words Ψ is the set of all finitely generated subfunctors of U which are artinian. Let F be the sum of these functors - it is, of course, also their directed union. The assumption that T^* is not t.t. is just that F is a proper subfunctor of U. Consider, therefore, the non-zero functor $\overline{F} = U/F$: we will see that the assumption that $\overline{F}$ has a simple subfunctor leads to a contradiction.

So suppose that $G > F$ is a subfunctor of U such that $\overline{G} = G/F$ is a simple subfunctor of $\overline{F}$. There is a finitely presented module M and an element $a \in M$ such that $a \in GM \setminus FM$. Let φ be a pp formula equivalent to the pp-type of a in M: thus $F_\varphi \leqslant G$ but $\varphi \notin \Psi$. We may in fact suppose that the pp-type of a is irreducible: write M as a direct sum of indecomposables and decompose φ accordingly; say as $\varphi = \varphi_1 + \ldots + \varphi_n$. Since $[\varphi : v=0]$ does not have the dcc this is true for at least one of the φ_i - with which we replace φ.

Since $\varphi \notin \Psi$ there is $\theta < \varphi$ with $\theta \notin \Psi$: in terms of functors this gives $F < F + F_\theta \leqslant F + F_\varphi \leqslant G$ (the first inclusion is strict by the Grothendieck property and since F_θ is finitely generated). By simplicity of G/F one concludes $F + F_\theta = F + F_\varphi = G$. Therefore $F_\varphi = F_\varphi \cap (F + F_\theta)$: writing F as a directed union $\vec{\bigcup} F_\psi$ one has, by the Grothendieck property, $F_\varphi = F_\varphi \cap (\vec{\bigcup} F_\psi + F_\theta) = \vec{\bigcup}(F_\varphi \cap F_\psi + F_\theta)$. Since F_φ is finitely generated it follows that $F_\varphi = F_\varphi \cap F_\psi + F_\theta$ for some $\psi \in \Psi$. But then irreducibility of (the type generated by) φ implies that either $F_\varphi = F_\varphi \cap F_\psi$ or $F_\varphi = F_\theta$ - each of which is impossible. This is the required contradiction.

(ii)⇒(i) By 11.6, the forgetful functor and its powers U^n have the descending chain condition on subfunctors. Therefore, if $U^n \geqslant F' > F$ then there is a functor minimal with respect to being contained in F' but not equal to F: hence F'/F has a simple subfunctor. Since (12.3) the functors $F_\varphi \leqslant U$ generate F, it follows that every non-zero functor does indeed have non-zero socle, as required. □

Exercise 2 Find conditions on the amenable category $\mathcal{C}$ under which, if every non-zero functor in $(\mathcal{C}, \mathrm{Ab})$ has a simple subfunctor, then $\mathrm{P}^{\mathcal{C}}$ has the acc.

<u>Theorem 12.19</u> [Aus74a; 1.11] *Suppose that the ring R is right artinian, and let* F *be the category* (mod-R, Ab). *Then the following conditions are equivalent.*

(*i*) R *is of finite representation type.*

(*ii*) (*a*) *every simple functor in* F *is finitely presented, and*

(*b*) *every non-zero functor in* F *has a simple subfunctor.*

(*iii*) (*a*) *every* (*finitely generated*) *irreducible type is isolated, and*

(*b*) R *is right pure-semisimple.*

(*iv*) T^* *is totally transcendental of finite Morley rank.*

Proof This follows from 12.10, 12.12, 12.13 and 11.38. □

Note that the only part which we have not yet proved here (outside of the context of artin algebras) is the fact that if R is of finite representation type, then every irreducible type is isolated. For more on the topics discussed in these first two sections, see [Aus82].

12.3 Embedding into functor categories or, how to turn pure-injectives into injectives

My original sub-title for this section was "or, 101 ways of turning pure-injectives into injectives". Perhaps that does exaggerate the situation somewhat, but it is true that many authors have found contexts related to the category of R-modules, together with methods of embedding $\mathcal{M}_R$ into these contexts, such that the pure-injective R-modules become precisely the injectives of the new category. These ways do turn out to be more or less equivalent - in some cases they literally are equivalent as functors. One could give an alternative presentation of the model-theory of modules by making use of these functors and what is known of injective objects.

In this section I will describe two methods of "turning pure-injectives into injectives". The first embeds $\mathcal{C}_{T^*}$ into $(R\text{-mod},\text{Ab})$ (R-mod is the category of finitely presented left R-modules); the second embeds $\mathcal{C}_{T^*}$ into $(\text{mod-}R,\text{Ab})^{\text{op}}$.

The reader may observe, in what has just been said, that each embedding introduces an "op" (recall that R-mod is mod-R^{op}). Therein lies a source of annoyance: are we to work with covariant or with contravariant functors? with a category or with its opposite? The choice is largely a matter of taste, but one's preference may depend on one's purposes. For example, in the first two sections of this chapter I have preferred to work with covariant functors, but then I have to put up with the fact that the natural embedding which takes an object M to the corresponding representable functor $(M,-)$ is an embedding of the opposite category into the functor category. On the other hand, in this section I am concerned with embedding a category into a larger context, and so it is more natural to use a covariant embedding. Therefore, if one wants to relate some results of this section to those of the previous two, it is necessary to spin all arrows through 180^0. This is almost an argument for using contravariant functors in §1,2 but surely the functors $M \longmapsto \varphi(M)$ are too natural to distort.

Subsection 1

I follow Gruson and Jensen [GJ73] (also see [GJ81] - I will use the latter for reference) in defining the category $\text{D}(R)$ to be the functor category $(R\text{-mod},\text{Ab})$. This is of course a Grothendieck abelian category, with a generating set of projectives being given by the representable functors $(L,-)$ for ${}_RL \in R\text{-mod}$. In the special case that there exists a duality $D: R\text{-mod} \simeq (\text{mod-}R)^{\text{op}}$, $\text{D}(R)$ may be regarded as $((\text{mod-}R)^{\text{op}},\text{Ab})$ - the category of contravariant functors from mod-R to Ab (that is, the category of right "Modules" over mod-R, as opposed to the category $(R\text{-mod},\text{Ab})$ of left Modules over R-mod). In particular this comment applies if R is a finite-dimensional algebra or, more generally, if R is an artin algebra.

Now, there is a naturally defined functor [GJ73] from the category $\mathcal{M}_R$ of all right R-modules to $\text{D}(R)$; namely that induced by $-\otimes-: \mathcal{M}_R \times R\text{-mod} \longrightarrow \text{Ab}$. That is, we have the functor $X: \mathcal{M}_R \longrightarrow \text{D}(R)$ given on objects by $X(M) = M \otimes_R -$. Thus, the right module M is taken to the functor $M\otimes_R-$ which takes a (finitely presented) left module L to the abelian group $M\otimes L$, and which has the natural effect on morphisms between left modules. Then there is an obvious way of defining X on morphisms between right modules, so that X gives a full and faithful embedding of $\mathcal{M}_R$ into $\text{D}(R)$ ([GJ81; §1]). Recall that tensor product does not in general preserve embeddings: in fact, one has that Xf is monic in $D(R)$ iff f is a pure embedding in $\mathcal{M}_R$. Using this (all of which is given in more detail in [GJ81]) one sees the following.

<u>Theorem 12.20</u> [GJ73] *Let M be a right R-module. Then the functor $M\otimes-$ is an injective object of* $\text{D}(R)$ *iff M is a pure-injective R-module.* □

This, combined with the fact that X is full and faithful, is extremely useful, since there is a well-develped theory of injective objects in Grothendieck abelian categories which therefore becomes immediately applicable to pure-injective modules.

For instance, the endomorphism ring of an indecomposable injective is local (4.A13). Hence the endomorphism ring of an indecomposable pure-injective is local - a fact also proved independently, and by a different method, in [Z-HZ78; Thm 9] (for a model-theoretic proof see [Zg84; 4.3] and also 4.27 in these notes). Furthermore, every injective splits into a discrete and a continuous part as described in **4.A5**: hence the same is true of pure-injectives (shown in [Fis75; 7.21] and [Zg84; §6]). See also Facchini [Fac85], who points out that this allows one to carry over the von Neumann decomposition into "Types" and the related dimension theories (see §16.C).

A second method for turning pure-injectives into injectives will be considered later. Here, I show how to use the above functor to embed our model-theoretic context into a more algebraic one; in particular one sees how the theory of hulls of §4.1 may be developed as a theory of injective hulls.

Recall (§5.4) that a suitable context in which to consider sets with their pp-types is that of $\mathcal{C}_{T^*}$ - the category whose objects are subsets of the monster model of T^* (so pp-types are specified along with subsets), or, if one prefers to restrict to submodules, one works with $\mathcal{C}^*_{T^*}$. It may be remarked that, in the latter category, the injective objects are just the sufficiently saturated pure-injective direct summands of the monster model (cf. proof of **15.19**). The category $\mathcal{C}^*_{T^*}$ is generally not abelian (exercise: take $R=\mathbb{Z}_4$), and the fact that one does not have kernels and cokernels makes it difficult to do algebra within that category.

The "dependence on context" seen in $\mathcal{C}^*=\mathcal{C}^*_{T^*}$ can be reflected in $\mathrm{D}(R)$ by embedding the former in the latter; but to be satisfactory there should be a description of the image. In fact there is such a description (**12.23**). We begin with the following result, which was briefly described in [Pr81] and is also in [Fac?; §§2,3,5] (Facchini's category of "filtered" modules is essentially $\mathcal{C}^*_{T^*}$).

<u>Theorem 12.21</u> *There is a full and faithful embedding of* $\mathcal{C}^*_{T^*}$ *into* $\mathrm{D}(R)$.

The proof of this theorem follows. First the embedding is described: it is a pp-version of tensor product.

Let (A,P_A) be an object of $\mathcal{C}^*=\mathcal{C}^*_{T^*}$: such an object will be more simply denoted by A. Choose a $\mathcal{C}^*$-embedding $j:A\longrightarrow M$, where M is pure in the monster model (so $P_A=P_M\restriction jA$). Although any choice for M will do, it will make life more simple if we always take M to be a copy of the hull of A.

For $L\in R$-mod set $A\otimes_{\mathrm{pp}}L$ to be the subgroup of $M\otimes L$ generated by all elements of the form $ja\otimes l$ with $a\in A$ and $l\in L$: $A\otimes_{\mathrm{pp}}L=\langle ja\otimes l : a\in A,\ l\in L\rangle$ (for the sake of readability, I often omit the "j").

Given a morphism $f:L\longrightarrow L'$ in R-mod, define $A\otimes_{\mathrm{pp}}f$ to be the restriction of $M\otimes f$ to $A\otimes_{\mathrm{pp}}L$. By definition of $M\otimes f$, one has $M\otimes f.a\otimes l = a\otimes fl$: so $A\otimes_{\mathrm{pp}}f$ is well-defined.

Thus, from A, we define an object $A\otimes_{\mathrm{pp}}-$ of $\mathrm{D}(R)$. It will follow from **12.22** that this is independent of the particular choice of embedding j; but it is perhaps of interest to see this directly, by showing that $\otimes_{\mathrm{pp}}$ is indeed a pp-variant of the usual tensor product.

Recall how the tensor product $M\otimes_R N$ of two modules is formed: one factors the cartesian product $M\times N$ by the consequences of bilinearity - that is, by the subgroup generated by the relators:

$(m,n+n')-(m,n)-(m,n')$;
$(m+m',n)-(m,n)-(m',n)$;
$(mr,n)-(m,rn)$.

To form the tensor product $A\otimes_{\mathrm{pp}}N$ of a (right) pp-type (A,P_A) and a left module N, one factors the product $A\times N$ by the relations above, together with all consequences of the pp-type of A. To be somewhat more precise: if the pp-type of A, together with the atomic diagram of N and the bilinearity relations, prove $(a,n)=0$, then (a,n) is added to the set of relators by which one is to factor. On morphisms, the action is the obvious one (well-defined, since pp formulas are preserved by morphisms).

Example 1 The standard example showing that tensor product does not preserve embeddings is obtained by taking $R=\mathbb{Z}$, $M=\mathbb{Z}$, $M'=\mathbb{Q}$, $N=\mathbb{Z}_2$. One has the embedding $\mathbb{Z}\hookrightarrow\mathbb{Q}$, but $\mathbb{Z}\otimes\mathbb{Z}_2\simeq\mathbb{Z}_2$ whereas $\mathbb{Q}\otimes\mathbb{Z}_2\simeq 0$. The reason for the last equation is that, in $\mathbb{Q}\otimes\mathbb{Z}_2$, one has $a\otimes 1=(1/_2)a.2\otimes 1=(1/_2)a\otimes 2=(1/_2)a\otimes 0=0$ for all $a\in\mathbb{Q}$. Now, although $\mathbb{Z}\otimes\mathbb{Z}_2\simeq\mathbb{Z}_2$, if one considers $\mathbb{Z}$ equipped with the pp-type induced by its embedding into $\mathbb{Q}$, then one has that $\exists w\,(v=w2)$ is in the pp-type of every element $n\in\mathbb{Z}$. So $\Delta=\mathrm{pp}^{\mathbb{Q}}\mathbb{Z}\cup\mathrm{pp}_0\mathbb{Z}_2\cup$ bilinearity proves the formula $\exists w\,(v\otimes 1=w2\otimes 1)$ and hence Δ proves $v\otimes 1=0$. Thus $(\mathrm{pp}^{\mathbb{Q}}\mathbb{Z})\otimes_{\mathrm{pp}}\mathbb{Z}_2=0$.

We will see that the functors $-\otimes_{\mathrm{pp}}L$ do preserve embeddings of $\mathcal{C}^*$.

Exercise 1 Define the obvious functor $-\otimes_{\mathrm{pp}}-:\mathcal{C}^*(T^*_R)\times\mathcal{C}^*(T^*_{R^{\mathrm{op}}})\longrightarrow\mathrm{Ab}$ and investigate its properties.

It is a straightforward exercise to check that these two definitions of $A\otimes_{\mathrm{pp}}-$ agree, and hence to see that $A\otimes_{\mathrm{pp}}-$ may be defined without reference to an embedding such as j above.

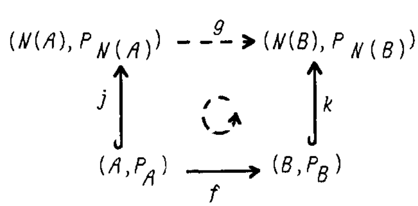
It will be seen that $A\mapsto A\otimes_{\mathrm{pp}}-$ induces a full and faithful embedding of $\mathcal{C}^*$ into $\mathrm{D}(R)$: moreover, the image of the embedding may be identified.

Let us note first that embeddings of $\mathcal{C}$ are preserved.

Lemma 12.22 *If $f:(A,P_A)\hookrightarrow(B,P_B)$ is an embedding in $\mathcal{C}^*_{T^*}$ (that is, if P_B restricted to fA equals P_A) then $f\otimes-:A\otimes_{\mathrm{pp}}\longrightarrow B\otimes_{\mathrm{pp}}-$ is monic in* $\mathrm{D}(R)$.

Proof Suppose that the embeddings $j:A\longrightarrow N(A)$ and $k:B\longrightarrow N(B)$ are used to define $A\otimes_{\mathrm{pp}}-$ and $B\otimes_{\mathrm{pp}}-$ respectively. Let L be in R-mod. The component of the natural transformation $f\otimes_{\mathrm{pp}}-$ at L is $f\otimes_{\mathrm{pp}}-:A\otimes_{\mathrm{pp}}L\longrightarrow B\otimes_{\mathrm{pp}}L$, given by $f\otimes_{\mathrm{pp}}L.ja\otimes l=kfa\otimes l$. There is a pure embedding $g:N(A)\longrightarrow N(B)$ with $gj=kf$. Since g is pure, we have $ja\otimes l=0$ iff $kfa\otimes l=0$, as required. □

It follows that the definition of $A\otimes_{\mathrm{pp}}-$ is independent of choice of embedding j.

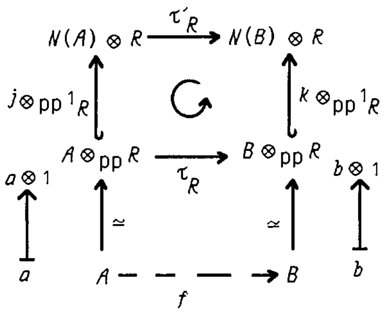


So now let us consider $-\otimes_{\mathrm{pp}}R\text{-mod}:\mathcal{C}^*_{T^*}\longrightarrow\mathrm{D}(R)$: the action on morphisms must be specified. So suppose that $f:(A,P_A)\longrightarrow(B,P_B)$ is any morphism in $\mathcal{C}^*$. Compose this with an embedding $k:(B,P_B)\longrightarrow(N(B),P_{N(B)})$ of B into a copy of its hull to get $kf:(A,P_A)\longrightarrow(N(B),P_{N(B)})$. By **4.12** this lifts to a morphism $g:(N(A),P_{N(A)})\longrightarrow(N(B),P_{N(B)})$ with $gj=kf$ as shown. Given L in R-mod, define $f\otimes_{\mathrm{pp}}L.ja\otimes l$ to be $gja\otimes l\;(=kfa\otimes l)$.

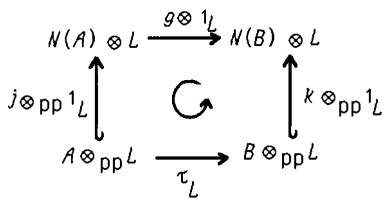
It is easy to check that this is a natural transformation from $A\otimes_{\mathrm{pp}}-$ to $B\otimes_{\mathrm{pp}}-$ and that, in this way, a functor $-\otimes_{\mathrm{pp}}R\text{-mod}:\mathcal{C}^*_{T^*}\longrightarrow\mathrm{D}(R)$ is defined.

To see that this functor is faithful, let $f:(A,P_A)\longrightarrow(B,P_B)$ be a non-zero morphism, and fix $j:(A,P_A)\longrightarrow(N(A),P_{N(A)})$ and $k:(B,P_B)\longrightarrow(N(B),P_{N(B)})$. Consider the component of $f\otimes_{\mathrm{pp}}-$ at $L={}_RR$: $f\otimes_{\mathrm{pp}}R:A\otimes_{\mathrm{pp}}R\longrightarrow B\otimes_{\mathrm{pp}}R$. This takes $ja\otimes 1$ to $kfa\otimes 1$. There is some $a\in A$ with $fa\neq 0$. So $f\otimes_{\mathrm{pp}}-\neq 0$, as required.

In order to complete the proof of **12.21** it must be shown that this functor is full. So we take any $\tau\in\mathrm{D}(A\otimes_{\mathrm{pp}}-,B\otimes_{\mathrm{pp}}-)$. It must be shown that there is some $f:(A,P_A)\longrightarrow(B,P_B)$ such that $\tau=f\otimes_{\mathrm{pp}}-$. With notation as before, injectivity of $N\otimes_{\mathrm{pp}}-\,(\simeq N\otimes-)$, plus the fact that $A\otimes_{\mathrm{pp}}-$ embeds in $N(A)\otimes-$ (and similarly for B), implies that there is a lifting of τ to some $\tau'\in\mathrm{D}(N(A)\otimes-,N(B)\otimes-)$. By considering the component of τ' at ${}_RR$, one sees that τ' is of the form $g\otimes -$ for some morphism $g: N(A) \longrightarrow N(B)$. Let f be the restriction of g to A. I claim that $\tau = f\otimes_{pp}-$. One has the diagram shown below.



From the commutativity of the upper square, one sees that $fA \leq B$ and that the lower square is commutative. Note also that, since f is the restriction to $A \longrightarrow B$ of a morphism $g: N(A) \longrightarrow N(B)$, f is in fact a morphism of $\mathcal{C}$: $f:(A, P_A) \longrightarrow (B, P_B)$.

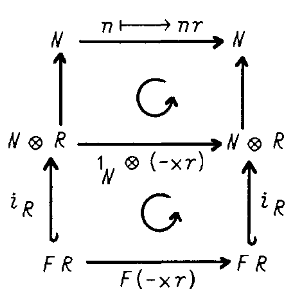


So it remains to be shown that, for any $L \in R$-mod, one has $\tau_L = f\otimes_{pp}L : A\otimes_{pp}L \longrightarrow B\otimes_{pp}L$. Since $\tau' = g\otimes -$ extends τ, we have, for each L, the diagram shown.
So $\tau_L = g\otimes 1_L \upharpoonright A\otimes_{pp}L = f\otimes_{pp}1_L$, as required. Thus the proof of 12.21 is complete. □

Those functors in $D(R)$ which arise, in the above way, from objects in $\mathcal{C}^*$ turn out to be exactly the epi-preserving functors (also see **[Fac8?**; §5**]**).

Theorem 12.23 *The functor defined above, from $\mathcal{C}^*{}_{T^*}$ to $D(R)$ (which is given on objects by taking (A, P_A) to $A\otimes_{pp}-$), is a full and faithful embedding which has, as image, the full subcategory of epi-preserving functors in $D(R)$.*

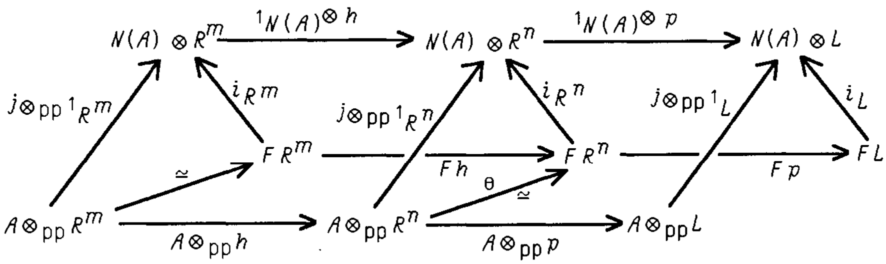
Proof First we see that every functor $A\otimes_{pp}-$ preserves epis. Fix a hull $j: A \hookrightarrow N(A)$. Let $L \xrightarrow{g} L'' \longrightarrow 0$ be exact in R-mod. Since the ordinary tensor functor is right exact, one has that the induced sequence $N(A)\otimes L \longrightarrow N(A)\otimes L'' \longrightarrow 0$ is exact. Consider the morphism $A\otimes_{pp}L \longrightarrow A\otimes_{pp}L''$. To see that this morphism is epi, note that a typical generator of $A\otimes_{pp}L''$ has the form $ja\otimes gl$ for some $a \in A$ and $l \in L$. Now $ja\otimes gl = 1_{N(A)}\otimes g . ja\otimes l$ so, since $ja\otimes l$ is an element of $A\otimes_{pp}L$, the point is established.



To show the converse, we take any epi-preserving functor F in $D(R)$. There is an embedding of F into an injective of $D(R)$, say $i: F \longrightarrow N\otimes -$, where N is a summand of the monster model. Consider the component of i at ${}_RR$: it yields a subgroup of N - $FR \hookrightarrow N\otimes R \simeq N$, where the first morphism is i_R and the second takes $n\otimes r$ to nr.
In fact, $FR = A$ is a submodule of N, for consider the diagram opposite, which compares the action of F and $N\otimes -$ on the morphism (of left R-modules) $-\times r: {}_RR \longrightarrow {}_RR$ for $r \in R$.

By commutativity of the lower square, one has that the action of $F(-\times r)$ is just right multiplication by r, and so $A = FR$ is an R-submodule of N (since F is a subfunctor of $N\otimes -$, so $F(-\times r): A \longrightarrow A$).

Next, it is claimed that $F = (A, \mathrm{pp}^{N(A)}A) \otimes_{\mathrm{pp}} -$. Essentially, we have just seen that F and $A \otimes_{\mathrm{pp}} -$ agree on R. It follows easily that they also agree on all R^n $(n \in \omega)$. So let ${}_RL$ be any finitely presented left R-module: say $R^m \xrightarrow{\cdot h} R^n \xrightarrow{p} L$ is a presentation of L. One obtains the diagram shown.



Here the top row is exact, the flanks are commutative, and the left-hand square on the bottom is commutative. It will be shown that there is an isomorphism $\tau_L : A \otimes_{\mathrm{pp}} L \longrightarrow FL$ which makes everything commutative.

First, to define τ_L, we take a typical element $a \otimes_{\mathrm{pp}} l$ of $A \otimes_{\mathrm{pp}} L$. There exists $r \in R^n$ with $l = pr$. Define $\tau_L(a \otimes_{\mathrm{pp}} l)$ to be $Fp.\theta.(a \otimes_{\mathrm{pp}} r)$. To see that τ_L is well-defined, suppose that $a \otimes r \in A \otimes_{\mathrm{pp}} R^n$ is such that $a \otimes pr = 0$. Consider $i_L.Fp.\theta(a \otimes r) = 1 \otimes p.i_{R^n}.\theta(a \otimes r) = 1 \otimes p.j \otimes 1(a \otimes r) = ja \otimes pr = 0$. Since i_L is monic, $\tau_L(a \otimes pr)$, which equals $Fp.\theta(a \otimes r)$, is zero.

To see that τ_L is monic, suppose that $\tau_L(a \otimes l) = 0$: so $Fp\theta(a \otimes r) = 0$, where $r \in R^n$ is such that $pr = l$. Then, as before, $0 = i_L.Fp.\theta(a \otimes r) = 1 \otimes p.j \otimes 1(a \otimes r) = ja \otimes pr = ja \otimes l = j \otimes 1_L(a \otimes l)$. So, since $j \otimes 1_L$ is monic, one concludes $a \otimes l = 0$, as required. Finally, to check that τ_L is onto, consider an element x of FL. Since Fp is epi, there is some y in FR^n such that $x = Fp.y = Fp.\theta(\theta^{-1}y)$. So $\tau_L(1_A \otimes_{\mathrm{pp}} p).\theta^{-1}y = Fp.y = x$, as required. □

It should be observed that the functors $(A, P_A) \otimes_{\mathrm{pp}} -$ are, in general, not right exact. For instance, take the ring to be $\mathbb{Z}_4$, take $A \hookrightarrow N(A)$ to be $\mathbb{Z}_2 \hookrightarrow \mathbb{Z}_4$, and consider the action on the exact sequence $0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$. The pp-tensor of (A, P_A) with this sequence is the non-right-exact sequence $0 \longrightarrow 0 \longrightarrow \mathbb{Z}_2 \longrightarrow 0 \longrightarrow 0$ (the point to note is that the left-hand term is indeed zero, because of the consequences of the pp-type of $\mathbb{Z}_2$ in $\mathbb{Z}_4$). The right exact functors are characterised in [GJ81; 5.3].

We now see that the theory of hulls can be developed as a theory of injective hulls. It is sufficient to show that a morphism $(A, P_A) \longrightarrow (B, P_B)$ is monic in $\mathcal{C}^*$ iff the corresponding morphism $(A, P_A) \otimes_{\mathrm{pp}} - \longrightarrow (B, P_B) \otimes_{\mathrm{pp}} -$ is monic. One direction is **12.22**. For the other, suppose that $(A, P_A) \longrightarrow (B, P_B)$ is not monic; there is no harm in supposing that these objects of $\mathcal{C}^*$ are summands of the monster model, so the $\otimes_{\mathrm{pp}} -$ functor is given simply by tensor product. So the result follows from the fact that if $N \longrightarrow M$ is a morphism then it is a pure embedding iff for every left module L, the morphism $N \otimes L \longrightarrow M \otimes L$ is monic [St75; §I.11].

In **[Fac85]** Facchini takes the following route. Let N be a pure-injective module and set S to be its endomorphism ring. Since N is an (S,R)-bimodule, the representable functor $(N_R, -)$ may be regarded as going from $\mathcal{M}_R$ to $\mathcal{M}_S$ (cf. before **17.1**). Set S' to be the regular, self-injective ring $S/J(S)$.

Given any S-module M, define the submodule $\tau_{S'}M$ to be $\bigcap\{\ker f : f \in (M_S, S'_S)\}$ (cf. §15.1). Set $RM = M/\tau_{S'}M$ - the largest quotient of M which embeds in the S-module S': RM

carries a natural S'-module structure and, with the obvious action on morphisms, this defines a functor from $\mathcal{M}_S$ to $\mathcal{M}_{S'}$. Then Facchini defines the functor $F_N : \mathcal{M}_R \longrightarrow \mathcal{M}_{S'}$ to be the composition of R with $(N_R,-)$. He shows [Fac85; Lemma1] that if N' is pure-injective then $F_N(N')$ is a non-singular S'-module (so the structure theory of Goodearl and Boyle [GB76] applies).

The functor F_N depends on N but, by choosing N large enough (specifically, by taking it to be an elementary cogenerator for the largest theory of R-modules), one sees all "small" pure-injective R-modules converted into non-singular injective modules over the regular ring S'.

Subsection 2

Now we look at another way, "dual" to that in subsection 1, in which $\mathcal{C}^* = \mathcal{C}^*{}_{T^*}$ may be embedded in an abelian category. I will also briefly indicate how localisation theory may be developed in this context. The embedding will be described by specifying it on the summands of the monster model and then extending, *via* hulls, to arbitrary members of $\mathcal{C}^*$. The sort of embedding that I have in mind here is an extension of that induced by representable functors.

So let $\mathcal{C}_0$ be the full subcategory of $\mathcal{M}_R$ whose objects are the pure-injective objects and their arbitrary direct sums. We first consider the morphism $\mathcal{C}_0 \longrightarrow (\text{mod-}R, \text{Ab})^{\text{op}}$ given on objects by $N \longmapsto (N,-)^{\text{op}}$ (the "$^{\text{op}}$" is there because we want an embedding of $\mathcal{C}_0$, rather than of $\mathcal{C}_0{}^{\text{op}}$). The problem with this is that it need not be an embedding. For example, let $R = \mathbb{Z}$; the objects of mod-$\mathbb{Z}$ are the finite-rank abelian groups: let N be any prüfer module. Then any quotient module of N, being injective, can be of finite rank only if it is zero. Thus the morphism $N \longmapsto (N,-)^{\text{op}}$ does not distinguish between the prüfer modules, hence does not induce an embedding of $\mathcal{C}_0$, let alone of $\mathcal{C}^*$. If R is an artin algebra then, by **13.2**, and one may develop the theory without replacing mod-R.

But, here, I wish to work over an arbitrary ring, so I replace (mod-R,Ab) by $(\mathcal{C}_0, \text{Ab})$. Indeed, it is just as easy to replace $\mathcal{C}_0$ by any amenable subcategory, $\mathcal{C}$, of $\mathcal{M}_R$, so the results of this subsection are initially developed in this generality.

Define F to be such a functor category $(\mathcal{C}, \textbf{Ab})$. Recall that F^{op} is a Grothendieck abelian category with, as a generating set of projectives, the representable functors $(C,-)$ for $C \in \mathcal{C}$. It follows that F^{op} is an abelian category (but, if non-trivial, cannot be Grothendieck – see [Mit65; III.1.10]) with, as a cogenerating set of injectives, the representable functors $(C,-)^{\text{o}}$, where I am using "$^{\text{o}}$" as a superscript to denote objects and morphisms of a category "when regarded as objects of the opposite category". In particular, F^{op} has enough injectives (and, if $\mathcal{C} = \mathcal{C}_0$, then F^{op} has injective envelopes). Since F has injective hulls, F^{op} has projective covers.

Denote by $S\mathcal{C}$ the category whose objects are submodules of objects of $\mathcal{C}$, and whose morphisms are the pp-type-preserving morphisms. It will be shown (**12.24**) that $S\mathcal{C}$ embeds fully and faithfully in F^{op}. Note that this provides another full and faithful embedding of $\mathcal{C}^*$ into a Grothendieck abelian category, since one may compose with the full and faithful embedding (*via* representable functors) of F^{op} into the category $(\text{F}^{\text{op}}, \text{Ab})$.

Now let $A \in S\mathcal{C}$. The "representable" functor corresponding to A, or rather, the restriction of $S\mathcal{C}(A,-)$ to $\mathcal{C}$, is defined as follows. If C is an object of $\mathcal{C}$ then $(A,C) = S\mathcal{C}(A,C)$; on morphisms, $(A,-)$ is given by composition (note that this is well-defined). Clearly this does define a functor, which I denote by $(A,-)$, in F. It is easy to verify that this process defines, in the usual way, a functor, θ, from $(S\mathcal{C})^{\text{op}}$ to F. Therefore we obtain an embedding, θ^{o}, of $S\mathcal{C}$ into F^{op}. It will be seen that the effect of all this is to embed $S\mathcal{C}$ into an abelian category which closely reflects the properties of $\mathcal{C}$ and $S\mathcal{C}$.

Proposition 12.24

(a) *Let* $f:A\longrightarrow B$ *be a morphism of* SC. *Then* $\theta f:(B,-)\longrightarrow(A,-)$ *is epi iff* $\theta^0 f:(A,-)^0\longrightarrow(B,-)^0$ *is monic iff* f *is strictly pp-type-preserving.*

(b) θ^0 *is a full and faithful embedding from* SC *to* F^{op}.

Proof Take C and D in $\mathcal{C}$ which respectively embed A and B with the correct pp-types. Part (a) is by a straightforward argument.

For part (b) we suppose that a morphism $t:(A,-)^0\longrightarrow(B,-)^0$ is given: it must be shown that some SC-morphism from A to B induces t. By injectivity of "representables", there is a morphism $t':(C,-)^0\longrightarrow(D,-)^0$ extending t. Since C and D are in $\mathcal{C}$, t' is induced by some morphism from C to D: it may be checked that the embedding of A into C, composed with this morphism, yields an SC-morphism from A to B which induces t. That θ^0 is faithful is obvious (every object of SC has a non-zero morphism into a member of $\mathcal{C}$). □

One of the objects of this sub-section is to indicate how to generalise the notion of localisation at an injective (as seen in hereditary torsion theories) to that of localisation at a pure-injective. Of course, since we have just turned pure-injectives into injectives, one might expect that there is not much more to say. But "localisation at pure-injectives" is at least a little richer than that, since one has the pp-types which play the part of right ideals: so one may work at a somewhat more concrete level. It might also be that this notion of localisation is closer to some other types of "localisation" which have been introduced, see: [Co79], [FZ82], [Ma182], [O'C84], [Ri79a], [Sc85], [SZ82].

Rather than give a detailed development, I just make a few remarks on definitions and results - filling in the details should be easy to anyone acquainted with torsion theories (see [St75], for example). For the sake of simplicity, let us suppose that $\mathcal{C}$ is the full subcategory of $\mathcal{M}_R$ whose objects are the pure-injective modules and their direct sums (so we have "hulls" and closure under products).

To specify a torsionfree class in F^{op}, it is equivalent to specify a subclass of $\mathcal{C}$ which is closed under products and direct summands. If N is an object of $\mathcal{C}$ then denote by $\mathfrak{F}_N$ the torsionfree subclass of F^{op} cogenerated by $\theta^0 N$ and let $\mathfrak{T}_N$ be the corresponding torsion class in F^{op}. Then, for instance, $\mathfrak{F}_{N'}\le\mathfrak{F}_N$ iff N' purely embeds in some power of N. One defines $\mathfrak{F}_N$-closed, $\mathfrak{F}_N$-dense and $\mathfrak{F}_N$-injective as would be expected.

Recall that SC embeds naturally in F^{op}. Then $A\in SC$ is in $\mathfrak{F}_N$ iff, for every $\bar{a}$ in A, there is a power N^K and a morphism $f\in SC(A,N^K)$ such that $\mathrm{pp}(f\bar{a})=\mathrm{pp}(a)$, that is, iff for every $\bar{a}$ in A, the pp-type of $\bar{a}$ is N-closed. The torsionfree class $\mathfrak{F}_N$ may be recovered from the lattice of closed types.

Generally, one uses (N-closed, N-dense) pp-types where one would, in the injective case, use right ideals.

12.P Pure global dimension and dimensions of functor categories

I will do no more in this section than point the reader towards certain ideas and results which have undoubted relevance to the concerns of this book, although the exact connections, beyond the obvious ones, have not been worked out. For example, the results in Chapter 13 on dimensions and representation type and the work on dimensions of functor categories described below are surely related: presumably they are connected through the lattice of pp formulas/finitely generated subfunctors of the forgetful functor (see 12.2, 12.4).

Let M be a module. Consider the exact sequence (a pure-injective resolution of M) $0\longrightarrow M=M_0\longrightarrow\overline{M}=M_1\longrightarrow\mathrm{pi}(\overline{M}/M)=M_2\longrightarrow\ldots\longrightarrow\mathrm{pi}(M_{n-1}/M_{n-2})=M_n\longrightarrow\ldots$: the morphisms are the canonical ones (projection followed by pure embedding into a pure-injective hull). If this sequence terminates, then the largest n for which $M_n \neq 0$ is said to be the **pure-injective dimension** of M, p.inj.dim$M$. The **pure global dimension** of R, p.gl.dim$R$, is sup{p.gl.dim$M$: M is an R-module}. So p.gl.dim$R = 0$ iff R is right pure-semisimple. Also, by a result of Kulikov (see **[Kap69]**), $\mathbb{Z}$ and $K[X]$ have pure global dimension 1 (i.e, if M is any module, then the quotient $\overline{M}/M$ is pure-injective - cf. Exercise 2.5/5).

There are other ways of obtaining the same dimension. Gruson and Jensen **[GJ81]**, define the **L-dimension** of M, L-dim$M$, to be the global dimension of the functor $M_R\otimes$- in (R-mod,Ab) (cf. §12.3). They show **[GJ81**; 1.1**]** that the L-dimension of a module coincides with its pure-injective dimension (recall that the functor which takes M to $M\otimes$- converts the pure-injective modules into the injectives of the functor category). This L-dimension also may be obtained from the derived functors of $\varprojlim$ (see **[GJ81**; 3.1): these derived functors were studied in **[Je72]**. In particular, L-dim$M \leqslant n$ iff, for every downwards-directed sequence $\{G_\alpha\}_\alpha$ of pp-definable subgroups of M, one has $\varprojlim^{(i)} G_\alpha = 0$, where $\varprojlim^{(i)}$ is the i-th derived functor of $\varprojlim$.

It turns out that pure global dimension is curiously dependent on the cardinality of the ring: Gruson and Jensen **[GJ73**; 1.3**]** show that if the cardinality of the ring R is $\leqslant \aleph_t$ then p.gl.dim$R \leqslant t+1$. In particular, if the ring is countable and if M is any module, then $\overline{M}/M$ is already pure-injective (so, by **2.23**, a direct summand of the monster model of the theory of M).

Baer, Brune and Lenzing relate the pure global dimension of hereditary (and related) algebras to their representation type. Okoh **[Ok80]** had already shown that, over an uncountable field, the pure global dimension of the path algebra of the extended Dynkin diagram $\tilde{A}_1$ is 2. In **[BL82]** Baer and Lenzing show that the pure global dimension of a polynomial ring in countably many, but at least two, variables (commuting or not) is $t+1$ (i.e., the maximum possible), where $\aleph_t$ is the cardinality of the base field (and t is finite). They deduce that the same is true of the path algebra of a wild quiver without relations. In **[BBL82]** the tame case is dealt with and the conclusion is that, if Γ is a connected quiver without oriented cycles and if K is a field of cardinality $\aleph_t$ (t finite), then the pure global dimension of the path algebra $K[\Gamma]$ is 0, 1 or 2 according as Γ is Dynkin, extended Dynkin or neither of these (i.e., according as $K[\Gamma]$ is of finite, tame or wild representation type - cf. §13.2).

The fact that one has to tensor up with a large field to see the pure global dimension contrasts with the fact that the link between our model-theoretic dimensions and structural properties is often best established by restricting to a countable subfield and then tensoring up (cf. e.g., after **13.7**).

A couple more related references: **[JS79]**, **[Sim77]**.

There are analogous results concerning Krull dimension of functor categories (presumably they are also just different expressions of certain finiteness conditions on the lattice of finitely generated subfunctors of the forgetful functor). For example, if R is an artin algebra, then the Krull dimension of the functor category (mod-R,Ab) is 0, 1 or 2, according as R is of finite, tame or wild representation type **[Gei85]**, **[GL8?]**. This is extended in **[Gei86]** to tilted algebras. That paper also contains the result that if R is one of Ringel's "canonical algebras" then the Krull dimension of the functor category (mod-R,Ab) is "∞": a result not unrelated (see **12.6**) to our observation in §13.3 that the m-dimension of the theory of modules over such a ring is "∞". Also see **[Bae85]**, **[Bae86]** where representation type is related to Krull and Gabriel dimension, to the noetherian condition and to the Ore condition on functor categories from various parts of mod-R to Ab for R a hereditary algebra.

CHAPTER 13 MODULES OVER ARTIN ALGEBRAS

In Chapter 11 we considered modules over arbitrary right artinian rings, and found that there are a number of simplifying features in that case. Modules over artin algebras are even better behaved, and their theory has been much developed. The main examples of artin algebras are algebras finite-dimensional over a field, and their theory has been developed farthest: this is reflected in this chapter in that, although we begin by considering general artin algebras, we soon focus on the case of algebras over a field.

Recall that a ring R is an artin algebra if it is finitely generated as a module over its centre and its centre is artinian. Such a ring is, in particular, right and left artinian. Examples are algebras finite-dimensional over a field and finite rings. Perhaps the most significant property of these rings is that there is a good duality between mod-R and R-mod.

In section 1, from the existence of almost split sequences, we deduce that the isolated points of the space of indecomposables are precisely the indecomposable finitely generated modules.

The second section contains background material on representations of quivers and representation type, as well as descriptions of the finitely generated indecomposable modules over certain algebras.

In the third section, I outline the classification of the infinitely generated indecomposable pure-injectives over the path algebras of the extended Dynkin quivers, and indicate what is known about $\mathcal{I}_R$ for certain tame non-domestic algebras.

13.1 The space of indecomposables

Let R be an artin algebra. Since every finitely generated module is totally transcendental (11.15), every finitely generated indecomposable is a point of the space, $\mathcal{I}(R)$, of indecomposable pure-injectives. We will see that every such point is in fact isolated. First we need to know something of the representation theory of artin algebras. That is a vast subject and so, throughout this chapter, I present just the material which I need. To obtain a more balanced and detailed picture of this area, the reader is advised to begin with, say, the survey article by Reiten [Rei85], and more detailed surveys by Gabriel [Gab80] and Ringel [Ri80], [Ri80a], [Ri86].

An exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is said to be **almost split** (or to be an **Auslander-Reiten** sequence) if it is not split and, for every module X and every morphism $h: X \longrightarrow C$ which is not a split epi, there is a factorisation $k: X \longrightarrow B$ with $gk = h$. It is equivalent to require of the sequence the dual property that, whenever $h: A \longrightarrow Y$ is a morphism which is not a split embedding, there is a factorisation $k: B \longrightarrow Y$ with $kf = h$. One sees from the definition that the term "almost split" is a singularly appropriate one. It is a theorem [AR75] that (over an artin algebra) any almost split sequence of finitely generated modules is determined up to isomorphism by either of its ends. This allows one to use, without ambiguity, a notation $C = \tau^{-1}A$ and $A = \tau C$ in the above situation: τ is called the **Auslander-Reiten translation**.

An extremely useful theorem of Auslander and Reiten [AR75; 4.3] is that: if M is an indecomposable non-projective module over an artin algebra then there is an almost split sequence ending with M: in particular, τM is defined. Dually, if M is an indecomposable non-injective module over an artin algebra, then there is an almost split sequence beginning with M (so $\tau^{-1}M$ is defined). The proof uses the existence of a duality between mod-R and R-mod for artin algebras (there are examples - see [Ri80; p126]) which show that over an arbitrary right artinian ring one need not have global existence of almost split sequences in the above sense. Although it is not immediately obvious how one makes use of this, it is a key point about artin algebras - see the references above, for example.

An associated idea is that of an irreducible morphism. A morphism $f:M \longrightarrow N$ between indecomposable finitely generated modules is **irreducible** if it is not an isomorphism and has no non-trivial factorisation, in the sense that, given any factorisation $f=hg$ through a finitely generated module, either g is a split mono or h is a split epi. It is easy to see that an irreducible morphism must be either mono or epi. Given an almost split sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$, with $B = B_1 \oplus \ldots \oplus B_n$, the components (with respect to this decomposition) of f and g are irreducible monos and epis. For more on all this see, for example, [Gab80].

There seems to be at least some superficial connection between irreducible morphisms and almost split sequences on the one hand and, on the other, the structure of the lattice of finitely generated pp-types. Also, Auslander's notion of (universally) minimal element [Aus74a] (also see [Aus82]) is reminiscent of that of critical type. I have not been able to clarify the connections (if, indeed, they exist), so I do not elaborate upon these points here, though the results of §12.2 have some bearing on this.

We will make use in this section, of the functorial version of the existence theorem for almost split sequences (actually the existence proof is carried out in the setting of functor categories). For **13.1**, we need only the fact that every non-injective indecomposable module begins an almost split sequence: consider what this means in terms of pp-types.

Let p be a finitely generated irreducible pp-type: say p is equivalent to the pp formula φ. Let a in the finitely generated indecomposable A realise p (so A is the hull of a). Consider the almost split sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ which begins with A, and let q be the pp-type of fa in B (if A is injective, then one proceeds more directly). Since B is finitely presented, q is finitely generated (**8.4**); is equivalent to ψ (say). I claim that F_ψ is the radical, JF_φ, of the functor F_φ (see §§12.1, 12.2 for these functors) and so, in particular, JF_φ is finitely generated. For, let q' be any finitely generated pp-type strictly above p: realise q' by c in the finitely generated module X. By **8.5** there is a morphism from A to X taking a to c. By the defining property of almost split sequences, there is a factorisation from B to X which, in particular, takes fa to c. Therefore $q = \mathrm{pp}(fa) \leqslant q'$. It follows (§12.2) that F_ψ is indeed the radical of F_φ and, furthermore, $\varphi \wedge \neg\psi$ isolates p. Therefore the following result is an immediate consequence of the existence of almost split sequences (on one side).

<u>Proposition 13.1</u> [Pr85a; §2] *Let R be an artin algebra and let N be a finitely generated indecomposable module over R. Then N is an isolated point of the space $\mathcal{I}(R)$ of indecomposable pure-injectives.* □

The converse will now be established by showing that the finitely generated points are dense in $\mathcal{I}(R)$. It is the existence of a good duality which is the key to the proof.

Let R be a finite-dimensional K-algebra, where K is a field. Then the functor which is given on right R-modules by $DM = \mathrm{Hom}_K(M,K)$ and has the obvious action on morphisms, is a duality between left and right modules and restricts to a natural equivalence $D: \text{mod-}R \longrightarrow (R\text{-mod})^{\mathrm{op}}$. Over a general ring, one may use the "duality" $DM = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. In either case, the duality has the following properties. A short exact sequence is pure iff its dual is split exact. For any right module M, the dual DM is a pure-injective left R-module. Since, for any module M, the canonical morphism $M \longrightarrow D^2M$ ("D^2" means the right duality followed by the left duality, the latter is also denoted by "D") is a pure embedding, this provides a method of obtaining pure-injective hulls (the hull of M will be a direct summand of D^2M).

Now, let $\mathcal{F}$ contain one copy of each finitely presented module. Fix a module N. For each $F \in \mathcal{F}$ let (F,N) denote the set of all R-morphisms from F to N. Set $G = \oplus_{F\in\mathcal{F}} \oplus_{f\in(F,N)} F$ and let $g:G \longrightarrow N$ be the morphism whose component at (F,f) is f. Certainly g is epi, since there is even an epi from an free module onto N. But also, the sequence $\ker g \longrightarrow G \longrightarrow N$ is pure-exact. For, given any morphism from a finitely presented module to N there is a lifting to G (by construction of G). Now, apply this to the left R-module DM, where M is any right R-module, and then apply the duality D. We end up with a pure embedding of the pure-injective module D^2M into DG, which is a product of duals of finitely presented left modules. Applying this in the case that M is pure-injective, we reach the conclusion that, over any ring:

(*) every pure-injective module is a direct summand of a direct product of duals of finitely presented left modules.

Over an artin algebra, the dual of a finitely presented module is finitely presented (this is obvious for an algebra finite-dimensional over a field). Therefore the next result is obtained. It may be noted that there is no duality over $\mathbb{Z}$ which takes finitely presented modules to finitely presented modules, since $\mathbb{Z}_{p^\infty}$ is not a direct summand of a direct product of finitely presented modules (exercise: cf. 9.29): that is, the dual of $\mathbb{Z}$ is not finitely presented.

Proposition 13.2 [Ok80a; Thm 1], [Len83; p.735], [Fac85; proof of Thm 1] also cf. [Cou78; §2] *Suppose that R is an artin algebra and let N be any pure-injective module over R. Then N is a direct summand of a direct product of finitely generated modules.* □

Corollary 13.3 [Pr85a; §2] *Suppose that R is an artin algebra. Then the finitely generated points of $\mathcal{I}(R)$ are dense in $\mathcal{I}(R)$.* □

Corollary 13.4 [Pr85a; §2] *Let R be an artin algebra. Then the isolated points of $\mathcal{I}(R)$ are precisely the finitely generated points.* □

Corollary 13.5 *Suppose that R is an artin algebra and let N be an infinitely generated indecomposable pure-injective R-module. Then N does not realise a neg-isolated in (T^*) type.* □

Thus, there is no sequence beginning with N, with the minimality property of an almost split sequence and with the middle term a direct sum of only finitely many indecomposable pure-injectives (where one is working in (say) the category of pure-injective modules). That is to say, the simple functor corresponding to N (§12.2) is not finitely presented, even if we allow infinitely generated pp-types. This contrasts to some extent with examples such as $\mathbb{Z}$ and $K[X]$, where there are infinitely generated indecomposable pure-injectives (namely the torsion injectives) which are hulls of neg-isolated types.

Let N be a finitely generated indecomposable over a finite-dimensional K-algebra. Suppose that $\varphi(N)/\psi(N)$ is an N-minimal pair. By 9.6 one has that the quotient $\varphi(N)/\psi(N)$ is 1-dimensional over the division ring $D_N = \mathrm{End}N/\mathrm{JEnd}N$. This division ring is a finite extension of K. If K is algebraically closed, then $D_N = K$ and so all "minimal gaps" are 1-dimensional over K.

These comments do not apply to all the infinite-dimensional indecomposables. Take, as illustration, the infinite-dimensional pure-injectives over any of the extended Dynkin diagrams (these are described in §3 below). One sees that if N is the unique point of $\mathcal{I}(T)$ of CB-rank 2, then N has no proper non-trivial pp-definable subgroup: indeed D_N is the K-extension of transcendence degree 1, $K(X)$. On the other hand, if K is algebraically closed then for every other indecomposable pure-injective N one does have $D_N = K$.

If K is not algebraically closed then a finite-dimensional indecomposable may well have its "minimal gaps" of K-dimension greater than 1. For instance, take $K=\mathbb{R}$ to be the real field and let R be the path algebra of an extended Dynkin diagram over $\mathbb{R}$. Let R' be $R\otimes\mathbb{C}$. Then the functor $-\otimes\mathbb{C}:\mathcal{M}_R\longrightarrow\mathcal{M}_{R'}$ takes almost every indecomposable pure-injective to the indecomposable pure-injective which is defined "in the same way" over R'. The exceptions are the regular modules (cf. §2 below) corresponding to the irreducible non-linear polynomial X^2+1: when tensored with $\mathbb{C}$, each of these splits into two indecomposable factors (the regular modules corresponding to the linear factors of X^2+1).

As another example, one may take R to be the (hereditary) algebra corresponding to a species (one with multiple edges) in the sense of [DR76]. Then one may have even (finitely generated) projective modules with endomorphism ring other than the base field.

13.2 Representation type of quivers

Path algebras of quivers form a large and significant class of algebras: indeed, over an algebraically closed field every finite-dimensional algebra is Morita equivalent to the path algebra of a quiver with relations (see [Gab80]). I will begin by describing what a quiver (without relations) is. It is perhaps appropriate at this point to say that some of the background in this section is treated more fully in [Pr8?] (conversely, some points are treated more fully here!).

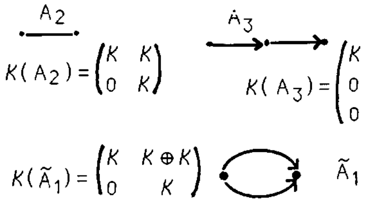
By a **quiver** I will normally mean a finite directed graph which has no oriented cycles (on occasion these requirements may be relaxed).

A **representation** of the quiver Γ over the field K (a K-**representation** of Γ) is an assignment of: to each vertex x of Γ, a K-vectorspace $V(x)$; to each edge $e:x\longrightarrow y$ of Γ, a K-linear transformation $V(e):V(x)\longrightarrow V(y)$ between the corresponding vectorspaces. Examples of quivers and their representations may be seen below.

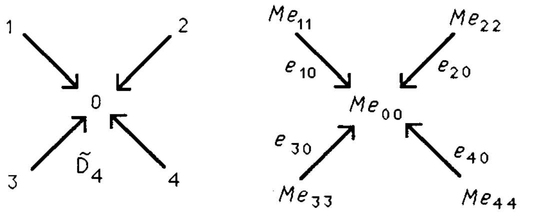
The **path algebra** over K of the quiver Γ is the K-algebra $K(\Gamma)$ described as follows (note that a K-algebra is just a K-vectorspace with a K-linear multiplication defined on it, for which there is a "1"). As a K-vectorspace, $K(\Gamma)$ has basis the set of all oriented paths in the quiver Γ, where one includes a path of length zero ("starting and ending") at each vertex of Γ. To describe the multiplication, it is enough to describe it on basis elements (and then extend to the whole algebra by K-linearity): it is simply composition of paths (where defined, 0 otherwise) - the paths of length zero compose as "local identities".

For instance, the path algebra of the quiver which has just one vertex and one loop is the algebra $K[X]$. Of course, this is not a finite-dimensional algebra (which is why one usually excludes oriented cycles from a quiver) but one obtains a finite-dimensional algebra by imposing a relation such as "$X^n=0$" where "X" denotes the (basis element corresponding to the) arrow of the quiver. This latter is an example of a quiver with relations: a quiver together with certain sums of paths between points declared to be zero - the representations of such a quiver then have to satisfy the corresponding zero-relations and the corresponding path algebra $K(\Gamma)$ is obtained by constructing the path algebra as above and then factoring by the ideal generated by the relations.

It may be seen that a K-representation of a quiver (with relations) Γ is really just a module over the corresponding path algebra $K(\Gamma)$, and *vice versa*.



Example 1 The diagram opposite shows the quivers A_2, A_3 and $\tilde{A}_1$ with the corresponding path algebras. The problem of classifying the K-representations of $\tilde{A}_1$, equivalently modules over its path algebra over K, is essentially the problem of classifying pairs of matrices over K up to simultaneous conjugacy.



For another example, consider any $K(\tilde{D}_4)$-module M ($\tilde{D}_4$ as shown). As a vectorspace, it decomposes as the direct sum $M = \oplus_0^4 Me_{ii}$. The element e_{10} then acts as the zero morphism on each component except Me_{11}, where it acts as a linear transformation from Me_{11} to Me_{00}: similarly with the other e_{ij} (e_{ii} acts as the identity on Me_{ii} and as zero elsewhere). Thus we obtain a $\tilde{D}_4$-representation from M. It is clear that the process may be reversed, so that from a $\tilde{D}_4$-representation we may piece together a $K(\tilde{D}_4)$-module.

There is a geometrical way in which one may think of representations of this quiver.

Fix a non-negative integer n. One may ask how n subspaces may be placed within a K-vectorspace. Of course, the dimensions of neither the space nor its n subspaces have been specified, so there are infinitely many possibilities; but still, it may be asked whether there is a classification which gives us some understanding of these structures (which are naturally described in the language of K-vectorspaces together with n predicates for the subspaces; they are examples of abelian structures in the sense of Fisher - see §3.A).

Certainly there is a classification for $n=0$, since then our structures are simply K-vectorspaces and every K-vectorspace is isomorphic to a direct sum of copies of K, regarded as a vectorspace over itself.

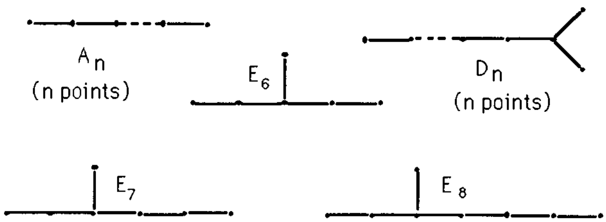
For $n=1$ the situation is not much different. One may see that every (space; subspace) pair may be represented as a direct sum of copies of the two indecomposable pairs $(K; K)$ and $(K; 0)$. The situation for $n=2$ and for $n=3$ is similar. Thus one sees that vectorspaces with $n(\leqslant 3)$ specified subspaces are examples of abelian structures (in the sense of §3.A) of finite representation type.

Actually these abelian structures (for any n) are "really just" modules: let us consider the case $n=4$ - we may then call our structures **quadruples**. Any quadruple $(U; U_1, U_2, U_3, U_4)$ gives a $\tilde{D}_4$-representation: place U at the sink vertex (labelled "0" above) and take the four morphisms to be the canonical inclusions of the U_i in U. Conversely, any $\tilde{D}_4$-representation with all four morphisms monic "is" a quadruple. In fact, every $\tilde{D}_4$-representation decomposes as the direct sum of such a quadruple and copies of the four simple injective $\tilde{D}_4$-representations - these last being the representations with a copy of K at one of the source vertices and zeroes elsewhere.

In contrast to the cases with $n \leqslant 3$, $\tilde{D}_4$ is of infinite representation type.

The following theorem, characterising the quivers of finite representation type, is due to Gabriel **[Gab72]**.

Theorem 13.A *Let Γ be a connected quiver and let K be an algebraically closed field. Then $K(\Gamma)$ is of finite representation type iff Γ, when the orientation of its arrows is ignored, is a Dynkin diagram.*
The Dynkin diagrams (that is, those without multiple edges) are shown below.



It is sufficient to consider connected quivers, since if Γ is the disjoint union of quivers Γ' and Γ'' then $K(\Gamma) \simeq K(\Gamma') \times K(\Gamma'')$ and so every $K(\Gamma)$-module is the direct sum of a uniquely defined $K(\Gamma')$-module and a uniquely defined $K(\Gamma'')$-module.

It should be noted, and it is quite typical in this area, that the representation type is independent of the underlying field, and is also independent of the particular orientation of the arrows (with the, actually minor, proviso that there be no oriented cycles).

Observe that the 3-subspace problem (and the simpler ones) are covered by Gabriel's result (explicitly, by A_1, A_2, A_3 and D_4, with appropriate orientations).

Beyond finite representation type there is a dichotomy. There are quivers (algebras) whose finite-dimensional representations (modules) may be, in some sense, parametrised by those of $K[X]$ - these are of "tame" representation type. And there are those whose finite-dimensional representations "include" those of $K\langle X,Y\rangle$ - these are of "wild" representation type, and it is generally considered that their finite-dimensional representations are unclassifiable. I will now give precise definitions.

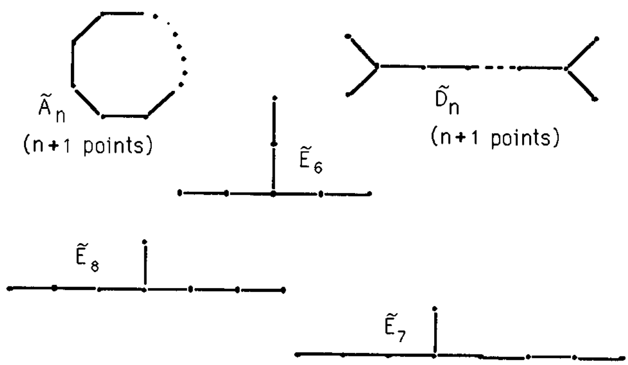
An algebra R (or corresponding quiver) is said to be of **wild** representation type if its category of finite-dimensional representations admits a functor from mod-$K\langle X,Y\rangle$ which preserves and reflects indecomposability and isomorphism, and which is of the form $- \otimes_{K\langle X,Y\rangle} M$, where M is a $(K\langle X,Y\rangle, R)$-bimodule which is finitely generated and free as a left $K\langle X,Y\rangle$-module. Thus, if R is wild one may consider that the category of finite-dimensional R-modules is at least as complex as that of finite-dimensional $K\langle X,Y\rangle$-modules. There are good reasons for believing that the latter category is "unclassifiable" (but also see **[C-B87]**): in particular, a classification of it would probably yield a classification of mod-R for *every* finite-dimensional algebra R (compare with the situation in **[Mek81]**).

The algebra R is said to be **tame** if R is not of finite representation type and if, for each integer $d \geqslant 1$, there exist $f_1(X), \dots, f_{n(d)}(X) \in K[X]$ and there exist $(K[X], R)$-bimodules Q_i for $i = 1, \dots, n(d)$, which are free as $K[X]$-modules, such that the union of the images of the tensor functors $F_i = - \otimes_{K[X]} Q_i$, from mod-$K[X, f_i(X)^{-1}]$ to mod-R includes all but finitely many d-dimensional indecomposable R-modules. See **[DS86]** for variants on the definition. (If K is a finite field then the definition is vacuous, so tensor up with the algebraic closure of K in order to see the representation type of R.)

It is a theorem that every algebra which is finite-dimensional over an algebraically closed field and is of infinite representation type is either tame or wild but not both **[Dro79]**. Thus one has, in increasing order of complexity, a trichotomy for finite-dimensional algebras: finite type; tame; wild. It appears that this correlates with an increase in model-theoretic complexity.

There is an analogue to 13.A - see [DF73], [DR76], also [Naz73].

<u>Theorem 13.B</u> *Let* Γ *be a connected quiver and let* K *be a algebraically closed field. Then* $K(\Gamma)$ *is of tame representation type iff* Γ, *when the orientations of its arrows are forgotten, is one of the extended Dynkin diagrams (also called Euclidean diagrams). These are:*



It follows that the four-subspace problem is a tame one but that the five-subspace problem is wild - thus one cannot expect to classify even the *finite-dimensional* vectorspaces with five specified subspaces.

Over an algebraically closed field, any hereditary finite-dimensional algebra is Morita equivalent to the path algebra of a quiver without relations. But if the field is not algebraically closed then there are other possibilities for hereditary algebras (see [DR76]).

It should be emphasised that, whatever the field K, if Γ is a connected quiver which is not Dynkin or extended Dynkin then $K(\Gamma)$ is of wild representation type (and the Dynkin (extended Dynkin) diagrams are still of finite (tame) type).

Let me now say something about the **Auslander-Reiten quiver**, $\Gamma(R)$, of a K-algebra R. This quiver has, for its vertices, the isomorphism types of finite-dimensional indecomposables. There is an arrow from (the vertex corresponding to) M to (that corresponding to) N iff there is an irreducible morphism from M to N. In fact, one puts in as many arrows as the dimension over K of the space of irreducible morphisms (which is naturally identified with the radical, modulo the radical squared, of the functor $(M,-)$ applied to N) but, in the examples I discuss, the multiplicity of each directed edge is always 1.

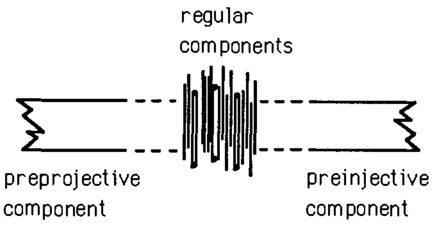
Unless the algebra is of finite representation type, this will be an infinite quiver; but it is locally finite, in the sense that every vertex is incident with only finitely many edges. This graph contains a good deal of information: for example the modules occurring in the almost split sequences can be read off from it. I refer the reader to the articles mentioned at the beginning of §1 for more detail.

I now say a little about the shape of the Auslander-Reiten quivers of the (tame) path algebras of the extended Dynkin diagrams. Let Γ be one of the extended Dynkin diagrams of 13.B. Then the Auslander-Reiten quiver of $K(\Gamma)$ falls into the following infinite components:
a component which contains all the projective indecomposables - the **preprojective** component;
a component which contains all the injective indecomposables - the **preinjective** component;
infinitely many **regular** components. The set of regular components is parametrised by the projective line over K, with finitely many points having finite multiplicity (if K is not

algebraically closed, interpret the projective line to mean the projectivisation of the space of maximal ideals of $K[X]$).

Indecomposable modules are named according to the type of component to which they belong.

All components, except those corresponding to the singularities on the parametrising curve, are planar (the singular ones may be drawn on cylinders), with the non-singular regular components being essentially semi-infinite lines (the regular components are termed "tubes").

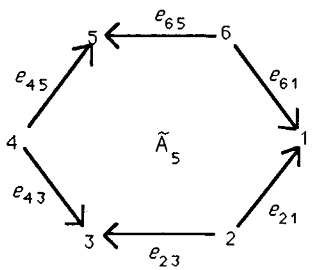


The preprojective and preinjective components are one-way infinite bands of bounded "width". They are arranged as shown, where all non-zero morphisms go from left to right and where there is the strong factorisation property that any morphism from a preprojective to a pre-injective may be factored through any chosen regular component - see [Ri84; §3.2].

(I have had to leave a great deal unsaid: see [Ri79], [Ri84]).

With regard to the definition of tame: a single functor from mod-$K[X,X^{-1},(1-X)^{-1}]$ to mod-R is enough to cover almost all indecomposables in each dimension (in fact, the definition below of band modules gives the functor on objects). The exceptions, in each dimension, are some regular modules on the singular tube plus all the preprojectives and preinjectives of that dimension (there is nothing over $K[X]$ corresponding to the latter modules).

I will next describe the finite-dimensional indecomposables over the path algebra of $\tilde{A}_5$. The finite-dimensional indecomposables over the path algebras of the other extended Dynkin quivers are, in some sense, essentially the same as over $\tilde{A}_5$: for their descriptions, see [GP72], [DF73], [DR76] (one may also consult [Bau80] and [Pr85a] for $\tilde{D}_4$).



We take $\tilde{A}_5$ to have the orientation shown: so a $K(\tilde{A}_5)$-module is given by specifying a vectorspace at each vertex and a morphism for each arrow.

The finite-dimensional indecomposables over $R=K(\tilde{A}_5)$ come in two forms: the so-called strings and bands ("type I and type II" in the terminology of [GP68]).

The easiest way to describe a string module is not to give the six vectorspaces and morphisms all at once, but to build them up gradually: in fact we specify a basis. Choose any "starting point" $i\in\{1,...,6\}$. Put a basis vector v_0 into V_i: this will be one end of the string we construct. There are two possibilities. If i is a sink (1, 3 or 5) then we choose one of the two arrows coming into i and give v_0 a pre-image under the corresponding morphism by putting a basis vector v_1 into $V_{i\pm 1}$ (as appropriate): at no point do we give v_0 a pre-image under the other morphism (since it is to end the string). Now v_1 is at a source vertex: we have made v_0 its image under one arrow; let v_2 be its image under the other. Repeat the process with v_2: it is at a sink, and we have a pre-image (v_1) under one arrow, so we add a pre-image, v_3, for it under the other, etc.. We continue in this way: one may think of the basis vectors and the morphisms connecting them as lying on a string which is wound round $\tilde{A}_5$. We specified a starting point and a direction: since the result is to be finite-dimensional we must specify another end for the string. The string may be terminated at any point, either by sending the last basis vector to zero (e.g. replacing ν_2 by 0) or by omitting to add a pre-image (e.g. stopping the construction at ν_0), as appropriate. In this way we have specified a representation of $\tilde{A}_5$ since, by the end, we have given a basis for each U_i and have described the actions of the morphisms on these basis vectors. The other possibility was that i was a source, in which case we choose one of the morphisms at i to be the zero morphism, and then proceed essentially as before.

It may be shown that these **string** modules are all indecomposable. As to their kinds: a string with two non-epi (resp. non-mono) ends is a preprojective (resp. preinjective), and a string with one end of each type is a regular module corresponding to one of the singular points on the parametrising curve.

The band modules are described as follows: choose a finite-dimensional indecomposable $K[X, X^{-1}, (1-X)^{-1}]$-module - that is, a finite-dimensional vectorspace U on which "X" acts as an indecomposable linear transformation, f, which has neither 0 nor 1 as an eigenvalue. Place a copy of U at each vertex of $\tilde{A}_5$: for the morphisms, take all but e_{61} to be the "identity" and take e_{61} to "be" f.

It may be shown that the result is indecomposable - a typical **band** module. No new module is got by replacing e_{61} by some other e_{ij} in the above construction since the "twist" may be considered to be at any of the six morphisms - by re-naming! These are the remaining regular modules.

It was shown by Donovan and Freislich [DF73] (also see [DR76]) that these are all the indecomposable finite-dimensional $K(\tilde{A}_5)$-representations. They also gave complete lists over the other extended Dynkin diagrams (the first case to be treated was that of $\tilde{D}_4$ in [GP70]). They considered each with a preferred orientation - that is no restriction since, by an operation on the category of modules (called tilting - see [BGP73], [BB80] or [Ri84]), one converts "almost all" of it to "almost all" of the category of modules over the same quiver, endowed with any other specific orientation.

The diagram $\tilde{A}_1$ had previously been treated by Weierstrass and especially Kronecker (see [Di46]). For the classification of representations of $\tilde{A}_1$ is just the problem of finding a canonical form for pairs of matrices, where the pair (A, B) is regarded as isomorphic to the pair (A', B') iff there are invertible matrices P, Q with $PAQ = A'$ and $PBQ = B'$. (The analogous problem, where for "isomorphism" we insist that $Q = P^{-1}$, is none other than the wild problem of classifying $K\langle X, Y\rangle$-modules!) Also see Dieudonne [Di46], especially for the basic method of analysing the modules so as to obtain a classification.

The quiver $\tilde{D}_4$ was dealt with by Gelfand and Ponomarev [GP70] (also see [Bre74], [Naz67]) - their paper contains a number of ideas and techniques which were to be developed into central tools in the representation theory of specific finite-dimensional algebras.

Within the tame case there is a further distinction in complexity which depends on how the modules are parametrised by $K[X]$-modules. Let us consider the parametrisation which is implicit in the above description of the indecomposable $\tilde{A}_5$-representations.

Given $i \in \{1, \ldots, 6\}$, define the functor $F: \text{mod-}K[X] \longrightarrow \text{mod-}K[\tilde{A}_5]$ as follows (cf. the description of the band modules). A $K[X]$-module M is simply a K-vectorspace together with a specified linear transformation "X". Given such a module, define FM to be the $\tilde{A}_5$-representation which has the vectorspace M_K at each vertex and where each morphism is the identity on M, except that between vertex i and vertex $i+1$, which is to be the linear transformation "multiplication by X". Given a morphism $f: M \longrightarrow N$ between $K[X]$-modules, the morphism $Ff: FM \longrightarrow FN$ is that whose component, $(FM)e_{ii} \longrightarrow (FN)e_{ii}$, at vertex i is given by f. It follows from the description of the indecomposable modules that this functor shows that $\tilde{A}_5$ is tame since, in every dimension, all but finitely many finite-dimensional indecomposables lie in the image of F (the functor is given by tensoring with the $(K[X], K(\tilde{A}_5))$-bimodule which is the $\tilde{A}_5$-representation which has a copy of $K[X]$ at each vertex and has every connecting morphism an isomorphism). There are some points to be made.

First: one functor suffices in all dimensions (the definition of tame allows for different functors to be used in different dimensions). Thus $\tilde{A}_5$ is actually of "domestic" representation type (see below).

Secondly; the definition of F makes perfect sense whether the $K[X]$-modules are finite-dimensional or not. So actually, we have a functor from the category of all modules over $K[X]$ to that of all modules over $K(\tilde{A}_5)$. This will be useful to us (in §13.3) because the closure under direct summands of the image of this functor is finitely axiomatisable (by the requirement that all morphisms but that between i and $i+1$ be isomorphisms).

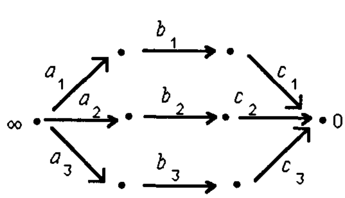
Finally, one may show that if we let i vary, then the resulting six functors suffice to cover all the regular indecomposable modules (the regular modules which are strings are images of those indecomposables, $K[X]/\langle X^n\rangle$ and $K[X]/\langle(1-X)^n\rangle$, which are lost when $X(1-X)$ is inverted). Again, this will be useful in §13.3 because we will see there (**13.6**) that every infinite-dimensional indecomposable pure-injective is in the closure (in $\mathcal{I}(T)$) of the regular indecomposables.

Refer back to the definition of tame above: as given, the parametrising functors may be different for different values of d. If it happens (as with $\tilde{A}_5$) that functors may be chosen independently of d, so that in every dimension, d, all but finitely many d-dimensional indecomposables are in the union of the images of these functors then, following Ringel, one says that R is of **domestic** representation type. If R is tame but not domestic then still it may happen that, as d increases, there is a finite bound on the number of functors which are needed - if so, then R is said to be of **finite growth**; otherwise R is of **infinite growth** ("(un)bounded growth" would be more accurate): this distinction is due to the Kiev school.

Thus domestic/finite growth/infinite growth represents an increase in the algebraic complexity of the module theory: some evidence will be presented in §13.3 to support the idea that this is linked to an increase in model-theoretic complexity.

As with $\tilde{A}_5$, the path algebras of the extended Dynkin diagrams are all domestic. Next, I give an example of an algebra of finite growth and an algebra of infinite growth and briefly describe their Auslander-Reiten quivers.

Let R be the path algebra (over some algebraically closed field K) of the quiver, with relations, below

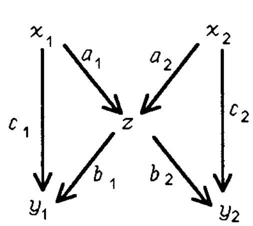


$\sum_{i=1}^{3} c_i b_i a_i = 0$

It is shown in [Ri84] that this **canonical algebra** is a tame algebra of finite growth and, there, Ringel gives there a complete description of the finite-dimensional indecomposables. I will say just a little about the shape of the Auslander-Reiten quiver of this algebra. Notice that if one omits from the quiver either the vertex labelled "0" or the vertex labelled "∞" then the remaining quiver is extended Dynkin ($\tilde{E}_6$). The module theory over R reflects this.

In particular, R has a preprojective component (corresponding to the preprojective component of the quiver obtained when "∞" is removed) and, dually, a preinjective component. Then it has a collection of sets, X_r, of tubes of "regular" modules, one for each $r \in \mathbb{Q}^+$. Each X_r is a set of tubes indexed by a projective line, with singularities, over K. (The parameter "r" indicates the relative contributions of the subquiver with "∞" removed and the subquiver with "0" removed.) The quiver has factorisation properties: if $f: M \longrightarrow N$ is a morphism between regular indecomposables with M lying in a tube in X_r and with N lying in a tube in X_t with $r<t$ then, for any $s \in \mathbb{Q}$ with $r<s<t$, there is a factorisation of f through a finite direct sum of regular modules lying in members of X_s ([Ri84; p276]). It follows immediately (from 10.6) that if T^* is the largest theory of modules over R then m-dim $T^* = \infty$.

Among algebras of infinite growth, most of those for which the classification of the finite-dimensional modules is known have indecomposables all of "string" or "band" type ([BD77] for an exception). The possibilities for the underlying strings and bands are however more complex than in the (domestic) extended Dynkin case. I illustrate with the path algebra of the butterfly quiver.



$b_2a_1 = 0 = b_1a_2$

We may form strings and bands as in the case of $\tilde{A}_5$, but now there is a binary tree of choices. For example, suppose that we are defining a string module and that we have just put in a vector v at vertex z where v is (say) the pre-image of a basis vector at y_1. We may decide that vb_2 should be non-zero and the next basis vector: or we may decide to give v a pre-image under a_1 (we cannot do both since $b_2a_1=0$). Thus, every time we come to the vertex z there is a binary choice as to which way to extend the string (the same point applies in the construction of the underlying "pattern" for a band - so one sees an exponential growth in the number of functors required to parametrise almost all indecomposables in a given dimension).

It is shown in [BuRi87] that the modules so constructed are indeed indecomposable and comprise a complete list of finite-dimensional indecomposables.

Analyses for other algebras of the same type were given in [GP68] for the Gelfand-Ponomarev algebra - $K[x,y : xy=0]$ - and in [Ri75b] for the dihedral algebra - $K\langle x,y : x^2=0=y^2\rangle$.

The Auslander-Reiten quiver for the butterfly algebra is described in [BuRi87]. The "band" modules all lie in tubes. Within each tube, the walk around the quiver, and the irreducible polynomial (cf. description of the band modules over $\tilde{A}_5$) are constant, and it is the power of the irreducible which parametrises the modules within the tube. All the other ("string") modules lie in planar sheets which extend infinitely in all directions. (One sheet is exceptional in that it has a "hole" in the middle.)

13.3 Describing the space of pure-injectives

In [Bau80] Baur proved decidability of the theory of modules over $K(\tilde{D}_4)$ (for K "sufficiently decidable"). The result, and especially his proof, suggested that it should be possible to describe the space $\mathcal{I}_{\tilde{D}_4}$. Extending the ideas of his paper, I described the infinite-dimensional indecomposable pure-injectives over any path algebra R of an extended Dynkin quiver, as well as the topology of the space $\mathcal{I}_R$. In this section I will outline the proof (for more detail, see [Pr85a]) and I will also say something concerning what is known about $\mathcal{I}_R$ for algebras of non-domestic representation types.

Let us consider the path algebra of $\tilde{A}_5$. It was seen in §2 that a single functor from mod-$K[X,X^{-1},(1-X)^{-1}]$ suffices to cover almost all modules in each dimension, but I indicated that it would be better, from the present point of view, to replace this with a finite number of functors from mod-$K[X]$, so that every regular indecomposable is covered.

Theorem 13.6 [Pr85a; §2] *If R is the path algebra of an extended Dynkin quiver, then the closure in $\mathcal{I}_R$ of the set of regular finite-dimensional indecomposables contains every infinite-dimensional point.*

Proof Although the proof requires some information about modules over the extended Dynkin quivers which I have not included in the notes, I think there is, nevertheless, some justification for putting this here. I refer to [Pr85a] or [Ri79] for more background.

Let N be any infinitely generated indecomposable pure-injective. By 13.2, N is a factor of a product of finite-dimensional indecomposables: say N is a direct summand of $P \oplus A \oplus E$ where P (resp. A, resp. E) is a direct product of preprojectives (resp. regular modules, resp. preinjectives).

It may be supposed that $P=0$. Otherwise, there is a non-zero morphism from N to a preprojective module. Therefore, since a submodule of a preprojective is preprojective (see [Ri79]), there would be an epi from N to a preprojective module. But now, by a characteristic property of preprojective modules (a kind of relative projectivity), this preprojective module would have to be a direct summand of N - contradicting indecomposability of N.

Thus N lies in the closure of a subset of $\mathcal{I}_R$ consisting only of regular and preinjective points.

So now let (φ/ψ) be any basic open neighbourhood of N. Suppose that (φ/ψ) contains infinitely many preinjectives. I show that this neighbourhood also contains a regular point. That will be enough to establish the required result (since, by 13.4, N is non-isolated and since the closure of any finite set of finitely generated points is just itself).

Consider the pp formula $\varphi(v)$: say it is $\exists \overline{w}\, \theta(v,\overline{w})$, where θ is a conjunction of linear equations. Suppose that M is any module and let $a \in M$ satisfy $\varphi(a)$; so there is $\overline{b}$ in M with $M \models \theta(a,\overline{b})$. Let M_0 be the submodule of M generated by the entries of $\overline{b}$ and a. Then $\dim_K M_0 \leqslant (1+l(\overline{w})).\dim_K R$ and, of course, $M_0 \models \varphi(a)$.

I emphasise this point (which is valid over any finite-dimensional algebra): given a pp formula φ, there is $n \in \omega$ such that, whenever M is a module and $a \in M$ satisfies $\varphi(a)$, there is a submodule of M, of K-dimension no more than n, containing a and in which $\varphi(a)$ holds.

So let M be an indecomposable preinjective in (φ/ψ). Let $a \in \varphi(M)\setminus\psi(M)$ and let M_0 be as above. Then $M_0 \models \varphi(a) \wedge \neg\psi(a)$. Since there were supposed to be infinitely many points in (φ/ψ), we may suppose that M is of large enough K-dimension that M_0 is a proper submodule of it.

Now, since M is an indecomposable preinjective, every indecomposable proper submodule of it is either preprojective or regular (this follows by the relative injectivity of preinjectives). If each indecomposable direct summand of M_0 is regular, then there in nothing more to do, since some indecomposable summand of M_0 lies in the neighbourhood (φ/ψ). If there is a preprojective summand, then we use the fact that any embedding from a preprojective to M factors through a sum of regular modules. Let M'' be M_0 with each preprojective summand replaced thus by a regular module. So we have $M_0 \hookrightarrow M'' \longrightarrow M$ with the composition being the inclusion. Since $M_0 \models \varphi(a)$, certainly $M'' \models \varphi(a)$. Since $M \models \neg\psi(a)$, certainly $M'' \models \neg\psi(a)$. Therefore some indecomposable, necessarily regular, summand of M'' lies in the neighbourhood (φ/ψ), as required. □

Now, it is easy to show that, although the images of these functors are not elementary subclasses of $\mathcal{M}_R$, their closures under direct summands are. So, if $\mathcal{C}_i$ is the closure under direct summands of the image of the functor F_i, then $\mathcal{I}(\mathcal{C}_i)$ is a closed subset of $\mathcal{I}_R$. It is an easy matter to see that F_iN is an indecomposable pure-injective $K(\tilde{A}_5)$-module iff N is an indecomposable pure-injective $K[X]$-module. So we obtain certain infinite-dimensional indecomposables in $\mathcal{I}_{\tilde{A}_5}$. The union of the $\mathcal{C}_i$ contains all the regular indecomposables so, by **13.6**, we draw the conclusion that we have found all the infinite-dimensional points of $\mathcal{I}_{\tilde{A}_5}$ - they are just the images of the infinite-dimensional points of $\mathcal{I}_{K[X]}$ (the overlaps of the images of the functors are straightforward to compute). Observe that, since we know the latter explicitly and since the actions of the functors F_i are quite clear, we obtain a completely explicit description of the infinite-dimensional indecomposable pure-injectives over $K(\tilde{A}_5)$.

Thus we have a list of the points of $\mathcal{I}_{\tilde{A}_5}$. But also, the functors behave nicely with respect to pp formulas, and it is not difficult to infer that they induce homeomorphisms of $\mathcal{I}_{K[X]}$ with its images. From this we deduce the topology of the union of the $\mathcal{C}_i$'s: that is not the whole space of indecomposables over $K(\tilde{A}_5)$, but it is easy enough to describe, from this, the topology of the whole space (the preprojective and preinjective points have to be included). Indeed, if the topology of $\mathcal{I}_{K[X]}$ is given "explicitly" then so will be that of $\mathcal{I}_{K(\tilde{A}_5)}$ (this has implications for decidability - see §17.3).

What we find, then, is that the picture of $\mathcal{I}_{K(\tilde{A}_5)}$ is very similar to that of $\mathcal{I}_{K[X]}$, but with two distinctions. The parametrising set for each of the "Prüfers" and "p-adics" is not the space of maximal ideals of $K[X]$ but rather the projective curve with singularities which comes up in the parametrisation of the regular modules. Also there are the sets of preprojectives and preinjectives - modules which have no analogues over $K[X]$.

We find in particular that the CB-rank of $\mathcal{I}_{K(\tilde{A}_5)}$ is 2 (cf. **[Gei85]**) (this also equals the elementary Krull dimension of T^* for $K(\tilde{A}_5)$), so there are no continuous pure-injectives (see §10.3, §10.4). Thus we have a classification of the pure-injective $K(\tilde{A}_5)$-modules.

The other extended Dynkin diagrams are treated in an exactly analogous way. The pictures obtained of the spaces of indecomposables differ only in the number and multiplicities of the singular points on the parametrising curve.

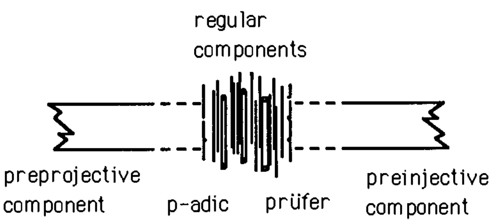
(I mention here that the original proofs were considerably more complicated and were more in the line of **[Di46]**: the key to the simpler proof outlined here is the "density" of the regular points (**13.6**) - this vastly eases the task of showing that every infinite-dimensional point is in the image of one of the functors.)

Lenzing has independently obtained related results on this domestic classification problem. Also, using the results of **[Ri79]**, Okoh showed in **[Ok80]** that there are no continuous pure-injectives, and in **[Ok80a]** obtains the broad classification of the indecomposable pure-injectives (in particular, that the infinite-dimensional ones fall into three classes, called "prüfer", "p-adic" and "rank 1 torsionfree divisible").

The infinite-dimensional points that we found are actually limits and colimits of certain natural families of finite-dimensional indecomposables. This is not too surprising since the Prüfer and p-adic abelian groups are respectively limits of monos and colimits of epis of the sort $\mathbb{Z}_p \hookrightarrow \mathbb{Z}_{p^2} \hookrightarrow \dots \hookrightarrow \mathbb{Z}_{p^n} \hookrightarrow \dots$ and $\dots \twoheadrightarrow \mathbb{Z}_{p^n} \twoheadrightarrow \dots \twoheadrightarrow \mathbb{Z}_{p^2} \twoheadrightarrow \mathbb{Z}_p$ respectively. This translates to the "Prüfer" and "p-adic" R-modules (R one of the tame path algebras) being respectively limits of monos and colimits of epis taken along the tubes which comprise the regular components of the AR-quiver.

But we get more than this. It turns out that the p-adics are pure-injective hulls of limits of natural series of monos within the preprojective component; and the Prüfers are colimits of natural series of epis within the preinjective component.

In fact we get the following picture, with factorisation properties, extending that described in §2. The factorisation properties are: that every morphism from an indecomposable preprojective to an indecomposable regular may be factored through any of the "p-adics" and that every morphism from an indecomposable regular to an indecomposable preinjective may be factored through any of the "Prüfers". Thus, one may say that the infinite-dimensional pure-injectives "glue together" the components of the AR-quiver (also, in some sense, the point of CB-rank 2 glues together the regular components, cf. [Ri79] and [Ri79a]).



The factorisation of any morphism $f:P \longrightarrow M$ from a preprojective to a regular module depends on the fact that if N is any one of the indecomposable "p-adic" modules, then N may be represented as the pure-injective hull of the union of a countable increasing chain of indecomposable preprojectives.

Then one uses the Auslander-translate τ which, over these algebras, is a functor, plus the fact that the regular modules are τ-periodic. Similarly for the dual "prüfer" modules.

I turn now to the non-domestic case, where considerably less is known. Consider first the canonical algebra defined in §2. The fact that there is a set of morphisms which is "densely ordered" with respect to factorisation means that the lattice $P(R)$ does not have m-dimension, and so, by **10.22** (for K countable; tensor up for the general case), the space $\mathcal{I}_R$ is uncountable and does not have Cantor-Bendixson rank (cf. [Gei86]). Nevertheless, I would be surprised if there were any continuous pure-injectives: that is, I conjecture that the theory of R-modules has width. One may also make some guesses as to where to find the infinite-dimensional indecomposable pure-injectives: almost certainly there is one at the "top" and one at the "bottom" of each tube, as for the extended Dynkin quivers. It also seems reasonable to expect at least one (perhaps more?) family, parametrised by the projective line over K at each "irrational cut" in the ordered set, $\{X_r : r \in \mathbb{Q}\}$, of families of tubes.

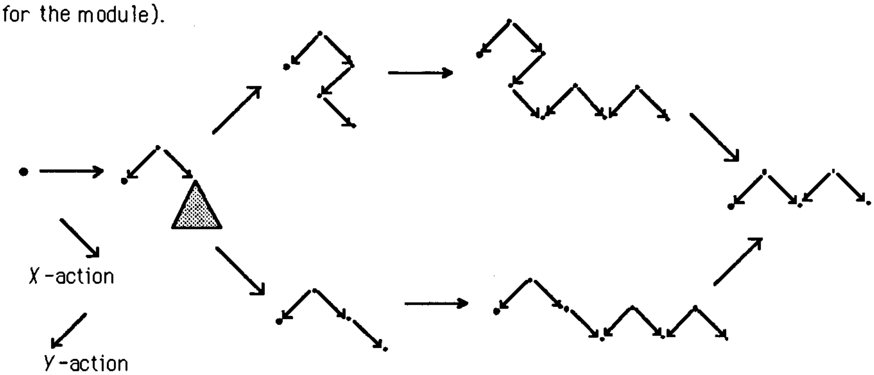
Turning to the butterfly quiver, the situation seems to be at least as complicated. There appears to be no way to use functors from mod-$K[X]$ as in the domestic case: for infinitely many would have to be used and, perhaps more to the point, their images are "interlaced" in an essential fashion - one which gives rise to infinite points which do not arise from any finite number of images. Both Point and I have worked on the problem of classifying the pure-injectives over such algebras.

It seems that the functorial methods, pioneered by Auslander, and used for tame algebras by Ringel in [Ri75b] (see also [Ri75] and appendix to [Gab75]) should lead to a solution. That technique (see §12.2) is to look at the category of functors from mod-R to the category of abelian groups, since the simple functors are in natural bijective correspondence with the indecomposable finite-dimensional modules. Every simple functor is a quotient of subfunctors of the forgetful functor: so the problem comes down to understanding the subfunctors of the forgetful functor (i.e., pp formulas - §12.1). Since we are interested in the infinitely generated indecomposables, we have to move to a functor category in which indecomposable pure-injectives correspond naturally to the simple subquotients of the forgetful functor (see **12.9**).

It seems more than likely than the approach of [Ri75b] can be made to yield a description of $\mathcal{I}_R$ for these algebras (a method successful for one will surely apply to them all). Point and I have taken this line: we believe that we know the list of indecomposables but we have not yet been able to establish that our list is correct. We can however say the following about the complexity of $\mathcal{I}_R$ for these algebras.

The space of indecomposables does not have CB-rank (even if K is countable there are $2^{\aleph_0}$ indecomposables), and the elementary Krull dimension is undefined. Nevertheless, the theory of R-modules does have width: so (**10.9**) there are no continuous pure-injectives (and we believe that the indecomposables are classifiable). The space is not even "T_0": indeed there are ($2^{\aleph_0}$ sets of) $2^{\aleph_0}$ non-isomorphic indecomposables which are elementarily equivalent (hence topologically indistinguishable). There are analogues of the Prüfers and p-adics, and they do sit nicely in relation to the components of the AR-quiver (they are however far from typical - indeed there are only $\aleph_0$ of them!).

Finally, I turn to the wild case. The theory of $K\langle X, Y\rangle$-modules does not even have width (and so, **10.14**, there are non-zero continuous pure-injective $K\langle X, Y\rangle$-modules). To establish this, we consider "string" modules. We will actually show that the theory of modules over the finite-dimensional algebra $A = K\langle X, Y\rangle/(Y^2, X^3, YXY, YX^2, X^2YX)$ does not have width and then, since these modules form a (finitely) axiomatisable subclass of $\mathcal{M}_R$, it follows that the theory of $K\langle X, Y\rangle$-modules does not have width. (One may note that A has, as a factor ring, the algebra B^{op} on Ringel's list as given in §17.2(3). Hence A is wild.) The proof that A-modules do not have finite width is contained in the diagram which follows. Rather than deal directly with pp formulas, I deal with their realisations. Each diagram below defines a string module: the vertices are basis elements and the arrows show the actions of X and Y (zero where no arrow is shown). The triangle depending from one dot indicates that the submodule generated by the basis element at that vertex is as free as possible (no product is zero unless it is forced to be so by the relations which hold between X and Y in the ring). Corresponding to the circled element in a given module, there is the pp-formula which describes that element (the module has a finite presentation and the circled element is a certain linear combination of generators for the module).



(To prove that the pp-types of the circled elements are strictly increasing, equivalently that the morphisms are not split, requires some work.)

The essential feature of the diagram is that it shows a situation - a diamond of morphisms (corresponding to a diamond of pp-definable subgroups) which is "self-reproducing" (cf. Ex10.2/1): each side of the diamond contains a diamond of exactly the same sort as the original one, which in turn .... Thus (see 10.8) the theory of A-modules does not have width.

The idea is as follows: a certain element a (Y times the pre-image under X of the circled element) is to be annihilated by both X and Y and made divisible by Y. Before making $aX = 0$ we may just make $aY = 0$, make aX divisible by Y and thereafter do quite complicated things, before finally declaring aX to be zero. We may also do something similar with X and Y interchanged. These paths meet only when both aX and aY become zero. The "complicated things" which may be done include repetitions of this whole process, and so a continually refining diamond is obtained.

Another reflection of the complexity in this case is the fact that even the classification of the "finite" points of the space of indecomposables is considered to be impossible (although see [C-B87]).

As for transferring this complexity: let R be any wild algebra. Then, by definition, there is a $(K\langle X,Y\rangle, R)$-bimodule P which is finitely generated and free on the left, such that the tensor functor $-\otimes_{K\langle X,Y\rangle} P : \text{mod-}K\langle X,Y\rangle \longrightarrow \text{mod-}R$ preserves and reflects indecomposability and isomorphism.

Consider the pattern of indecomposables and morphisms between them which was used to show $\mathrm{w}(T^*_{K\langle X,Y\rangle}) = \infty$. We may apply the tensor functor to this, and we obtain a pattern of R-modules and morphisms between them. If R is countable then the only possible obstacle to applying Ziegler's result **10.14**, now over R, is that some of the morphisms may be isomorphisms or zero. The first possibility is excluded by the definition of wild: the second is also easily dispensed with. For let $f: N \longrightarrow M$ be any non-zero morphism between $K\langle X,Y\rangle$-modules. Then, since P is free on the left, it must be that $f\otimes 1_P : N\otimes P \longrightarrow M\otimes P$ also is non-zero (exercise or see [St75; p.27]) - as required. Thus have the conclusion:

<u>Theorem 13.7</u> *Let R be any wild (finite-dimensional) algebra. Then* $\mathrm{w}(T^*_R) = \infty$. *Hence there exists a continuous pure-injective R-module.* □

Strictly speaking, the second conclusion is immediate (from **10.14**) only if R is countable. To deal with the general case, one may work over the restriction, R', of R to a countable subfield, K', of K which is relatively algebraically closed in K, such that every ring element appearing in one of the pp formulas in the "width ∞ pattern" lies in this subring. Then there is an R'-module, N, the lattice of pp-definable subgroups of which has width ∞. Consider the R-module $N\otimes_{K'}K$: this has isomorphic lattice of pp-definable subgroups so, in particular, this lattice is countable. Hence **10.13** applies, and we conclude that **13.7** does hold without any cardinality restriction on R.

CHAPTER 14 PROJECTIVE AND FLAT MODULES

This chapter is devoted to the model theory of projective and, more generally, flat modules. As with the dual case of injective and absolutely pure modules (Chapter 15), one obtains relatively "complete" results. In both cases, the key step is the description of the particular form taken by the pp-definable subgroups.

If a module M is flat, then every pp-definable subgroup of it has the form $\varphi(M)=M.\varphi(R_R)$: indeed, this property characterises the flat modules. It follows that the model-theoretic complexity of a flat module can be no greater than that of the ring. We see (§1) that, if the ring is left coherent, then its pp-definable subgroups are precisely the finitely generated left ideals. We deduce that the class of flat modules is axiomatisable iff the ring is left coherent.

It follows from the results of §1 that a ring which is left coherent is totally transcendental as a module over itself iff it is right perfect: but we do not have a general algebraic characterisation of the totally transcendental rings. We then note that the left coherent, right perfect rings are precisely those over which the class of projective modules is elementary (over such a ring, every flat module is projective). The section finishes with a characterisation of those rings over which the free modules form an elementary class.

14.1 Definable subgroups of flat and projective modules

A major step in understanding the model theory of any particular class of structures is the characterisation of the definable sets. It is shown below that if M is a flat module and if φ is a pp formula, then $\varphi(M)=M.\varphi(R)$. Therefore finiteness conditions on the lattice of pp-definable subgroups of the ring immediately carry over to all flat modules. More generally, the model theory of flat modules is thus somehow reduced to that of the ring. So what are the pp-definable subgroups of R? For left coherent rings the answer is: precisely the finitely generated left ideals; and in fact this answer characterises such rings, as does the axiomatisability of the notion of flatness. Particularly in the first part of this section, I follow **[Rot83a]**.

Some notations will facilitate our progress. Suppose that L is a submodule of the *left* module R^n $(n\in\omega)$. For any (*right*) module M, denote by ML the subgroup of M^n generated by the set $\{ar=(ar_1,\dots,ar_n) : a\in M,\ r=(r_1,\dots,r_n)\in L\}$.

<u>Lemma 14.1</u> *Suppose that L is a subgroup of the left module R^n and let M be any module.*
(*a*) *ML is an* End(M)*-submodule of M^n (via the diagonal action).*
(*b*) *If L is left finitely generated then ML is a subgroup of M^n, pp-definable in M.*

Proof (a) Take $a\in M$, $r\in L$, $f\in\text{End}M$. Then: $f.ar=f(ar_1,\dots,ar_n)=(f.ar_1,\dots,f.ar_n)=(fa.r_1,\dots,fa.r_n)=fa.r$. Hence ML is closed under the digonal action of End$M$ and so (a) is clear.

(b) Supppose that $L=\sum_{j=1}^m Rr^j$ where $r^j=(r_1^j,\dots,r_n^j)$: so $ML=\sum_{j=1}^m Mr^j$. Let $\varphi(v_1,\dots,v_n)$ be the formula $\exists w_1,\dots,w_m \bigwedge_{i=1}^n v_i=\sum_{j=1}^m w_jr_i^j$. Thus, if $(a_1,\dots,a_n)\in M^n$, then $\varphi(a_1,\dots,a_n)$ holds iff there are $b_1,\dots,b_m$ in M with $a_i=\sum_{j=1}^m b_jr_i^j$ for each i. Any such tuple, $(a_1,\dots,a_n)$, is therefore in the subgroup, ML, generated by the tuples $(b_jr_1^j,\dots,b_jr_n^j)=b_jr^j$. Conversely, if $r\in L$, say $r=\sum_{j=1}^m s_jr^j$, and if $a\in M$, then $ar=\sum_{j=1}^m as_j.r^j$ $(\in\sum_{j=1}^m Mr^j=ML)$ is a typical generator of ML and it clearly satisfies φ, with the as_j witnessing w_j in φ. Thus $ML=\varphi(M)$, as required. □

Note that the formula $\varphi(\bar v)$, appearing in the proof above, is conveniently expressed in matrix notation as $\exists\bar w\,(\bar wH=\bar v)$ where $l(\bar v)=n$, $l(\bar w)=m$, $H={}_m(h_{ji})_n$ with $h_{ji}=r_i^j$.

Observe that, although any pp-definable subgroup of M^n - say $\varphi(M^n)$ - is pp-definable in M (being $\varphi(M)^n$), the converse is false: take a field K for R; then the diagonal submodule of

K^n is pp-definable in K but not in K^n). In particular, ML as above need not be definable in M^n.

A particular situation of the sort described in 14.1 occurs when L is itself a subgroup of R^n pp-definable in R - say $L=\varphi(R)$. Then the question arises of the relation between $\varphi(M)$ and $M.\varphi(R)$. We always have one inclusion.

Lemma 14.2 [Rot83a] *Let* $\varphi(\bar{v})$ *be a pp formula and let* M *be any module. Then* $M.\varphi(R)\leq\varphi(M)$.

Proof Since $\varphi(M)$ is a group, it is enough to check that the additive generators of $M.\varphi(R)$ lie in $\varphi(M)$. So suppose that m is an element of M and $R\models\varphi(r_1,\dots,r_n)$. Also let us suppose that $\varphi(v_1,\dots,v_n)$ is $\exists w_1,\dots,w_l \bigwedge_{j=1}^{m}\sum_{i=1}^{n} v_i t_{ij}+\sum_{k=1}^{l} w_k s_{kj}=0$ for suitable t_{ij},s_{kj} in R.

Then, from $\varphi(r_1,\dots,r_n)$ one deduces the existence of elements, $s_1',\dots,s_l'$ of R, with $\bigwedge_j \sum_i r_i t_{ij}+\sum_k s_k' s_{kj}=0$. Therefore one has $\bigwedge_j m(\sum_i r_i t_{ij}+\sum_k s_k' s_{kj})=0$: that is, $\bigwedge_j \sum_i mr_i.t_{ij}+\sum_k ms_k'.s_{kj}=0$. Hence $M\models\varphi(mr_1,\dots,mr_n)$ and so $m(r_1,\dots,r_n)$ lies in $\varphi(M)$, as required. ▫

A point which may be seen in the proof of the result above, but which is somewhat obscured by its statement, may be seen more readily by using matrix notation. So let $\varphi(\bar{v})$ be of the form $\exists\bar{w}\,(\bar{v}\,\bar{w})H=0$. Now, "$(\bar{v}\,\bar{w})H=0$" defines a subgroup of M^{n+l} where $n=l(\bar{v})$ and $l=l(\bar{w})$. For any module M denote by $\pi^M(n+l,n)$, or even by π, the canonical projection from M^{n+l} to M^n. Then one has the following.

Lemma 14.3 (see [Rot83a; Prop4]) *Suppose that* H *is a matrix over* R *and let* $\varphi(\bar{v})$ *be the pp formula* $\exists\bar{w}\,(\bar{v}\,\bar{w})H=0$ *where* $l(\bar{v})=n$ *and* $l(\bar{w})=l$. *Let* $\theta(\bar{v},\bar{w})$ *be the* $\bigwedge$*-atomic formula* $(\bar{v}\,\bar{w})H=0$. *Then:*
(*a*) $\varphi(M)=\pi^M(n+l,n).\theta(M)$;
(*b*) $M.\theta(R)\leq\theta(M)$;
(*c*) $\pi(M.\theta(R))=M.\pi\theta(R)$ *and therefore* $M.\pi\theta(R)\leq\pi\theta(M)$ - *that is,* $M.\varphi(R)\leq\varphi(M)$ (14.2).

Proof Part (a) is obvious and (b) is an easy version of 14.2.

(c) A typical generator on the left-hand side of the equality has the form $(mr_1,\dots,mr_n)$ where there exist $r_{n+1},\dots,r_{n+l}$ in R such that $\theta(r_1,\dots,r_{n+l})$ holds. A typical generator on the right-hand side also has this form.

The other statements then follow immediately from (b) and (a). ▫

In the above notation, one may write "$\varphi=\pi\theta$".

To continue with this casting into matrix form, there is the following useful description of ML. Regard (as we have been doing) the elements of L as row vectors $(r_1,\dots,r_n)$. Then a typical generator of ML has the form $a(r_1,\dots,r_n)=(ar_1,\dots,ar_n)$ (matrix multiplication). A typical member of ML is a sum of such generators, so one has the next result.

Lemma 14.4 [Rot83a] *Suppose that* L *is a left submodule of* R^n *and let* M *be any module. Then* $ML=\Sigma\{M^tL^t : t\geqslant 1\}$, *where* M^t *is the set of* $1\times m$ *matrices with entries in* M, *and* L^t *is the set of* $m\times n$ *matrices over* R, *the rows of which are in* L.
Thus a typical element of ML *has the form* $\bar{a}X$ *where the entries of* $\bar{a}$ *come from* M *and where each row of the matrix* X *is in* L. ▫

A module M is said to be **flat** if the tensor functor $M\otimes_R-$ from ${}_R\mathcal{M}$ to Ab is exact. Since this functor necessarily is right exact, it is equivalent to require that $M\otimes-$ preserve monomorphisms. Every projective module is flat, and a finitely presented flat module necessarily is projective (see [St75; §I.11]). An element-wise criterion, well-known but related to our considerations, is derived next and then is used to give a characterisation of flatness in terms of the form of the pp-definable subgroups.

I retain the convention that tuples of elements of right (resp. left) modules are written as rows (resp. as columns), except that the reverse convention is used for free modules acting on modules.

Lemma 14.5 (see [St75; I.8.8]) *Let M_R be a right, and ${}_RL$ a left, R-module. Suppose that $\overline{m}\in M^n$ and $\overline{l}\in L^n$. Then $\overline{m}\otimes\overline{l}$ (i.e., $\Sigma m_i\otimes l_i$), as an element of $M\otimes L$, is 0 iff there is a matrix H with entries in R and a tuple $\overline{k}$ from L such that $\overline{m}H=0$ and $H\overline{k}=\overline{l}$; and this (by symmetry) occurs precisely if there is a matrix H' over R and a tuple $\overline{m}'$ from M such that $\overline{m}=\overline{m}'H$ and $H'\overline{l}=0$.*

The proof of 14.5 follows directly from the definition of $\otimes$ (cf. [St75]). It is then an easy exercise to establish the next result, from which the second corollary follows directly.

Corollary 14.6 [Ch60; Prop2.3], see [St75; I.10.7] *Let M be any module, Then M is flat iff whenever $\overline{m}\in M^n$ and $\overline{l}\in({}_RR)^n$ are such that $\overline{m}.\overline{l}\,(=\Sigma m_il_i)$ is 0, then there is $\overline{m}'$ in M and a matrix H over R with $\overline{m}=\overline{m}'H$ and $H\overline{l}=0$.*

Corollary 14.7 *Any pure submodule of a flat module is flat.*

Corollary 14.8 *Let M be any module. Then M is flat iff whenever $\theta(\overline{v})$ is a ($\bigwedge$-)atomic formula one has $\theta(M)=M.\theta(R)$.*

Proof Since for $\bigwedge$-atomic formulas θ' and θ'' one has $(\theta'\wedge\theta'')(M)=\theta'(M)\cap\theta''(M)$, it may as well be supposed that θ is an equation - say θ has the form $\overline{v}.\overline{l}=0$ for some column $\overline{l}$ over R. From 14.2/14.3, one has the direction $M.\theta(R)\subseteq\theta(M)$.

So suppose that M is flat and take $\overline{m}$ in $\theta(M)$; so $\overline{m}.\overline{l}=0$. By 14.6, there is a matrix H over R and there is $\overline{m}'$ in M with $\overline{m}=\overline{m}'H$ and $H\overline{l}=0$. From $H\overline{l}=0$ it follows that every row, $\overline{r}$, of H satisifes $\overline{r}.\overline{l}=0$. So, by 14.4, $\overline{m}=\overline{m}'H$ is in $M.\theta(R)$, as required.

If conversely the second condition is satisfied and if one has $\overline{m}\in M^n$ and $\overline{l}\in({}_RR)^n$ with $\overline{m}.\overline{l}=0$ then, setting $\theta(\overline{v})$ to be "$\overline{v}.\overline{l}=0$", one has $\overline{m}\in\theta(M)=M.\theta(R)$. So, by 14.4, there is a matrix H with rows in $\theta(R)$ - hence with $H\overline{l}=0$ - and there is $\overline{m}'$ in M, with $\overline{m}=\overline{m}'H$. So, by 14.6, M is indeed flat. □

So, in a flat module there are no relations between elements, beyond those "imposed by the ring". This yields the next result.

Theorem 14.9 [Rot83a: Prop4], [Zim77: 1.3] *The following conditions on a module M are equivalent:*

(i) M is flat;

(ii) $\varphi(M)=M.\varphi(R)$ for every pp formula $\varphi(v)$ in one free variable;

(iii) $\varphi(M)=M.\varphi(R)$ for every pp formula φ.

Proof The equivalence of (i) and (iii) is immediate from 14.8, on projecting and using 14.3. Of course (iii)⇒(ii). So it remains to show that (ii)⇒(i). It is shown that the condition (ii) has, as a consequence, that whenever $\theta(\overline{v})$ is atomic then $\theta(M)=M.\theta(R)$. Then we will finish by appealing to 14.8.

Therefore let $\theta(\overline{v})=\theta(v_1,\dots,v_n)$ be atomic. By 14.2/14.3 it need only be shown that $\theta(M)\subseteq M.\theta(R)$. If this is not the case, then choose θ for which this fails, θ having the minimum number, n say, of free variables.

Then, if $\overline{m}=(m_1,\dots,m_n)$ is in $\theta(M)\setminus M.\theta(R)$, one has $m_1\in\pi_1.\theta(M)$ where π_1 is projection to the first coordinate. The assumption (ii) implies that $\pi_1.\theta(M)=M.\pi_1\theta(R)=\pi_1(M.\theta(R))$ since the formula "$\pi_1\theta$" has just one free variable. Therefore, there is $m\in M$ and there is $\overline{r}=(r_1,\dots,r_n)$ in $\theta(R)$, such that $m_1=mr_1$.

Now, it cannot be that $\overline{m}-m\overline{r}$ is in $M.\theta(R)$. For if it were, $\overline{m}$ itself would be in $M.\theta(R)$. Notice that the first coordinate of $\overline{m}-m\overline{r}$ is 0. Let $\theta'(v_2,\dots,v_n)$ be $\theta(0,v_2,\dots,v_n)$.

By the minimality hypothesis, it must be that $\theta'(M)=M.\theta'(R)$. Now $\overline{m}-m\overline{r}$ is in the submodule $0\oplus\theta'(M)$ of M^n, so $\overline{m}-m\overline{r}$ lies in $0\oplus M.\theta'(R)$ - say $\overline{m}-m\overline{r}=(0,m'\overline{s})=(0,m's_2,\dots,m's_n)$ for some $(s_2,\dots,s_n)$ in $\theta'(R)$. Then $\theta(0,s_2,\dots,s_n)$ holds. Re-arranging the above equation, one obtains $\overline{m}=m\overline{r}+m'(0,\overline{s})\in M.\theta(R)$ - contrary to assumption, as required. ◻

Exercise 1 Deduce from **14.9** the following:

(a) flat modules are torsionfree in the sense that, if M is flat then, for every $m\in M$ and left regular element, c, of the ring, $mc=0$ implies $m=0$;

(b) the class of flat modules is closed under direct sums.

It follows from **14.9** that theories of flat modules can be no more complex than that of $R^{\aleph_0}$. For example, if R is commutative and with Krull dimension α then every flat module has elementary Krull dimension $\leqslant\alpha$. Of course, for projective modules this is just a consequence of the fact that any projective module is a direct summand of some direct sum of copies of R.

Considering what happens in the injective case, one might ask whether there is some direct connection between flat and projective modules analogous to that between absolutely pure and injective modules (Exercise 4.2/1; *viz.* that the latter are the pure-injective hulls of the former), from which this connection between R_R and flat modules would follow. The fact that $\mathbb{Q}$ is a flat non-projective $\mathbb{Z}$-module (exercise) shows that this naive analogy is invalid. A more reasonable question to ask is whether there is some duality (as opposed to direct analogy) involved; and, indeed, this is the case.

An exact sequence $0\longrightarrow K\longrightarrow N\longrightarrow M\longrightarrow 0$ is pure-exact if the embedding of K into N is pure. Dually to "pure-injective" say that M is **pure-projective** if every such pure-exact sequence ending in M is split. It was noted earlier (proof of 2.23(a)) that in a pure-exact sequence as above, if φ is pp then $\varphi(M)=\varphi(N)/[K\cap\varphi(N)\,(=\varphi(K))]$. In particular, the lattice of pp-definable subgroups of M is a quotient of that of N (and so has no greater complexity than that of N).

Exercise 2 Show that every finitely presented module is pure-projective.

The next result shows that flatness dualises absolute purity.

<u>Lemma 14.10</u> (see [St75; I.11.1]) *The module M is flat iff every exact sequence $0\longrightarrow K\longrightarrow N\longrightarrow M\longrightarrow 0$ is pure-exact.* ◻

In particular, applying this to a projective presentation of the flat module M, one sees again, by the proof of 2.23(a), that the complexity of the lattice of pp-definable subgroups of M is no more than that of R.

<u>Corollary 14.11</u> [Sab70: Prop2] *A module is projective iff it is flat and pure-projective.* ◻

This last corollary exactly dualises the fact that injective ≡ absolutely pure + pure-injective and generalises the fact that flat + finitely presented implies projective ([Laz64; Thm 1]).

What, if anything, is the dualisation of the existence of pure-injective hulls and the fact that every module is an elementary substructure of its pure-injective hull?.

From **14.9** one may deduce the following corollaries.

Corollary 14.12 [Rot83a: Prop15] *If* R_R *is totally transcendental (of pp-rank* α*) then every flat, in particular every projective, module is totally transcendental (of pp-rank* α*).* □

Corollary 14.13 *If* R *is left artinian of length* n *then every flat right module has Morley rank* $\leqslant n$. □

Corollary 14.14 *If* R *has elementary Krull dimension* $\leqslant\alpha$ *(in particular if* ${}_RR$ *has Krull dimension* $\leqslant\alpha$*) then every flat* R*-module has elementary Krull dimension* $\leqslant\alpha$. □

Corollary 14.15 *If the Krull dimension of* ${}_RR$ *is defined then, for every flat module* M*, the pure-injective hull,* $\overline{M}$*, has a decomposition as the pure-injective hull of a direct sum of indecomposable direct summands.* □

The corollaries **14.14** and **14.15** (the latter follows by **10.10** plus comments after **10.27**) improve on a result of Garavaglia [Gar80a; Example(h)]. One has similar results for m-dimension, width (see §10.2), or any other reasonable measure of complexity defined in terms of the lattice of pp-definable subgroups.

The first corollary, **14.12**, raises the question of which rings are, as modules over themselves, totally transcendental. That problem is addressed in the next section.

Next we consider those rings whose pp-definable subgroups are precisely the finitely generated left ideals; for then **14.9** becomes particularly informative. The ring R is said to be **left coherent** if it satisfies the equivalent conditions of the next theorem, for a proof of which I refer the reader to [St75; I.13.3]. A module is said to be **coherent** if all its finitely generated submodules are finitely presented.

Theorem 14.A [Ch60; Thm 2.13,Thm 2.2] *The following conditions on the ring* R *are equivalent:*

(i) ${}_RR$ *is a coherent left module: that is, every finitely generated left ideal is finitely presented;*

(ii) *every finitely presented left module is coherent;*

(iii) *every element of* R *has finitely generated left annihilator, and the intersection of any two finitely generated left ideals is again finitely generated*

(iv) *as (iii) but with* $({}_RR)^n$ *replacing* ${}_RR$ *and submodules replacing left ideals;*

(v) *every direct product of copies of the right module* R_R *is flat;*

(vi) *every direct product of flat right modules is flat.* □

The next result both characterises the class of left coherent rings and describes the right pp-definable subgroups of such rings (Garavaglia proved this [Gar80a; Lemma22] with the additional assumption that R is right perfect - cf. **14.19** below).

Theorem 14.16 [Rot83a: Prop7], [Zim77: 1.3] *The following conditions on the ring* R *are equivalent:*

(i) R *is left coherent;*

(ii) *the pp-definable subgroups of* R_R *are precisely the finitely generated left ideals;*

(iii) *the subgroups of* R^n *which are pp-definable in* R_R *are precisely the finitely generated submodules of* $({}_RR)^n$.

Proof (i)⇒(iii) Let $\varphi(v_1,\ldots,v_n)$ be pp: say it is $\exists w_1,\ldots,w_l \bigwedge_{j=1}^m \Sigma_i v_i r_{ij} + \Sigma_k w_k s_{kj} = 0$. Let $({}_RR)^{n+l} \xrightarrow{f} ({}_RR)^m$ be the morphism defined by sending $(r_1,\ldots,r_n,s_1,\ldots,s_l)$ to the m-tuple with j-th component $\Sigma_i r_i r_{ij} + \Sigma_k s_k s_{kj}$. By **14.A**, $\ker f$ is left finitely generated.

Hence $\varphi(R)$, being the image of $\ker f$ under projection to the first n coordinates, also is finitely generated, as required.

(iii)$\Rightarrow$(ii) This is trivial.

(ii)$\Rightarrow$(i) Left annihilators and finitely generated left ideals of R are right pp-definable (and hence, also, are their finite intersections). So this follows by 14.A(iii)$\Rightarrow$(i). ▫

Corollary 14.17 [Rot83a; Props4,7] *The ring R is left coherent iff, whenever M_R is a flat module, every pp-definable subgroup of M has the form ML for some finitely generated left ideal L of R. This condition will be satisfied exactly if, for every flat module M_R, every subgroup of M^n pp-definable in M has the form ML where L is a finitely generated submodule of $({}_RR)^n$.*

Proof $\Rightarrow$ This is immediate by 14.16 and 14.9.

$\Leftarrow$ This is immediate by 14.16, on taking $M=R$. ▫

From this one quickly derives (as is noted in [Rot83a]) some earlier results of Eklof and Sabbagh which were obtained without this explicit description of the pp-definable subgroups.

Theorem 14.18 [SE71; Thm 4] *The following conditions on a ring R are equivalent:*

(i) R is left coherent;

(ii) the class of flat modules is elementarily closed;

(iii) the class of flat modules is axiomatisable.

Proof [Rot83a: Prop8] That (iii) implies (ii) is immediate (a class $\mathcal{C}$ is elementarily closed if $M\in\mathcal{C}$ and $M'\equiv M$ implies $M'\in\mathcal{C}$: for example the class of finite structures is elementarily closed, though not elementary).

Since $R^{(\kappa)}\equiv R^{\kappa}$ for any κ (by 2.24) it follows, by 14.A(v)$\Rightarrow$(i) and the fact that every direct sum of flat modules is flat, that (ii) implies (i).

Now, for (i)$\Rightarrow$(iii), let $\varphi(v)$ be any pp formula with one free variable. By 14.16, $\varphi(R)$ has the form $\sum_{j=1}^{m} Rs_j$ for suitable $s_1,\dots,s_m$ in R. Let $\psi_\varphi(v)$ be the formula $\exists w_1,\dots,w_m\,(v=\sum_j w_j s_j)$, and note that $\varphi(R)=\psi_\varphi(R)$. But, more than this, for any module M one has $M.\varphi(R)=\psi_\varphi(M)$.

By 14.9, the module M is flat iff, for every pp formula $\varphi(v)$, one has $\varphi(M)=M.\varphi(R)=\psi_\varphi(M)$. That is, M is flat iff $M\models\varphi\leftrightarrow\psi_\varphi$ for every pp φ. Thus the class of flat modules is axiomatised by the set $\{\forall v\,(\varphi(v)\leftrightarrow\psi_\varphi(v)) : \varphi(v)$ is pp and $\psi_\varphi(v)$ is constructed as above$\}$. Hence this class is elementary. ▫

Exercise/Problem 3 Explain the following:

(i) R is left coherent iff the class of flat right modules is elementary (14.18);

(ii) R is right coherent iff the class of absolutely pure right modules is elementary (15.35 + 15.27).

Baudisch [Bd84] considers the extent to which tensor product preserves elementary equivalence between abelian groups. His work overlaps to some extent with some earlier unpublished work of Jackson [Ja73] Baudisch characterises those abelian groups A such that, whenever $A'\equiv A$, one has $A\otimes B\equiv A'\otimes B$ for all B, and also those abelian groups A such that $B\equiv B'$ implies $A\otimes B\equiv A\otimes B'$. He also shows that if R is semisimple artinian then, for every module A, if $B\equiv B'$ then $A\otimes B\equiv A\otimes B'$ (as abelian groups). In contrast, he gives an example of elementary equivalence not being preserved by tensor product, using modules over a boolean ring.

Sabbagh [Sab84] has shown that if P is a pure-projective module, then $M\equiv N$ implies $P\otimes M\equiv P\otimes N$ (as abelian groups): thus over any ring of finite representation type, tensor product preserves elementary equivalence. More generally, he shows that if the module P satisfies the condition: for all modules $\{M_i\}_i$, the natural morphism

$P\otimes\prod_i M_i \longrightarrow \prod_i P\otimes M_i$ is monic (*), then $P\otimes-$ preserves elementary equivalence. He shows that a module P satisfies condition (*) iff every countably generated submodule of P is contained in a countably generated pure submodule of P which is pure-projective (and so a countable module P satisfies (*) iff it is pure-projective).

14.2 Projective modules and totally transcendental rings

It has been seen (**14.18**) that the class of flat modules is elementary iff the ring R is left coherent. This suggests the question: under what conditions on R is the class of projective modules elementary? In the light of the discussion at the end of the previous section, this is to ask what completes the equation (or duality) concerning right modules:
right coherent : right noetherian = left coherent : ??
Another question, suggested by **14.12**, is: describe the totally transcendental rings algebraically. It turns out that these problems have the related solutions.

A ring R is said to be **right perfect** if it satisfies the equivalent conditions of the following theorem which is mainly due to Bass [Bas60] (see [Fai76; 22.29, 22.31A], for example).

Theorem 14.B *The following conditions on the ring R are equivalent:*
(*i*) *every right R-module has a projective cover;*
(*ii*) *R is left semi-artinian and R/J is semisimple artinian (where J is the Jacobson radical of R);*
(*ii′*) *J is right T-nilpotent and R/J is semisimple artinian;*
(*iii*) *R is left semi-artinian and R has no infinite set of orthogonal idempotents;*
(*iv*) *R has the dcc on finitely generated left ideals;*
(*v*) *every left R-module has the dcc on finitely generated submodules;*
(*vi*) *every flat right R-module is projective.* □

A **projective cover** of the module M is an epimorphism $P \twoheadrightarrow M$ from a projective module P, through which factors each epimorphism from a projective module to M (unlike the dual object of injective hull, projective covers do not exist in general). The ring R is said to be (right) **semi-artinian** if every non-zero module has a simple submodule. Also, J is **T-nilpotent** if, given any sequence $a_1, a_2, \dots, a_n, \dots$ of elements of J, there is $k\in\omega$ with $a_k a_{k-1}\dots a_1 = 0$.

Exercise 1 [Pr84; 3.17, 3.18] Suppose that R is right perfect. Show that R is actually left artinian iff there are no non-isolated irreducible types in the theory of R_R (cf. **11.38**).

By 3.1, if R is totally transcendental then R has the dcc on finitely generated left ideals and so is right perfect ([Sab75a; Prop8] for the countable case); in fact, by **14.23** below, it is even **semi-primary** - i.e., R/J is semisimple artinian and J is nilpotent. That the converse is false is shown by the next example.

Example 1 This is an example of a ring which is right artinian (so is semi-primary - in particular is right and left perfect - and is left totally transcendental) but which is not totally transcendental as a right module over itself. It is due to Zimmermann. I describe the example but, for more detail, the reader should see [Zim82].

Let T be a ring which is right artinian, has a derivation "′", has a ring endomorphism τ with $(\tau T)' = 0$, and is such that $T^{(i)} > T^{(i+1)}$ where $T^{(i)}$ denotes the i-th derivative of T (so $T^{(0)} = T$ and $T^{(i+1)} = (T^{(i)})'$). An example of such a ring is $T = K(x_i : i\in\omega)$ - the field of rational functions in the indeterminates x_i over a field of characteristic 0: for the derivation take the partial derivative $\delta/\delta x_0$: for the endomorphism take that defined by sending x_i to x_{i+1}.

Let N be the right T-module freely generated by x and y (say). Define a left T-module structure on N by setting $t(xt_1+yt_2)=xtt_1+y(t't_1+tt_2)$ $(t,t_1,t_2\in T)$.

Let the ring R be the **trivial extension** of T by the (exercise) bimodule N: $R=\{\begin{pmatrix} t & n \\ 0 & t \end{pmatrix}: t\in T,\ n\in N\}$. Then (exercise) R is right artinian.

Let M_R be the cyclic module R/xR (where x has been identified with $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$). As a T-module, M is the direct sum of $\bar{1}T$ and $\bar{y}T$ (where bar denotes image modulo xR). There is a left R-module structure on M given by $\begin{pmatrix} t & n \\ 0 & t \end{pmatrix}.(\bar{1}t_1+\bar{y}t_2)=\bar{1}\tau(t)t_1+\bar{y}\tau(t)t_2$ making M an (R,R)-bimodule (exercise: use $(\tau T)'=0$).

Then M_R is not totally transcendental (exercise: find an infinite descending chain of pp-definable subgroups), in fact M_R is not even pure-injective.

It follows (see [Zim82]) that the trivial extension S of R by M is not even right pure-injective. On the other hand S is right artinian (and so is left totally transcendental).

Zimmermann's paper [Zim82] contains a number of results on rings which are pure-injective or totally transcendental as modules over themselves (use of pp-definable subgroups is an essential tool in the proofs). The next result (see [Rot83a; p976] for references) shows that, in the presence of left coherence, a right artinian (and so, by **14**.B(ii), right perfect) ring is necessarily right totally transcendental.

Theorem 14.19 *Suppose that R is left coherent. Then R_R is totally transcendental iff R is right perfect*

Proof If R is left coherent then every pp-definable subgroup of R_R is a finitely generated left ideal (**14.16**). So, by **3.1**, R is t.t. iff it has the dcc on finitely generated left ideals - that is, iff (**14**.B) R is right perfect. ◻

Corollary 14.20 *If the ring R is left noetherian then R_R is totally transcendental iff R is left artinian.*

Proof This is clear, since left noetherian plus right perfect is equivalent (from the definitions) to left artinian. ◻

A related result is the following one.

Corollary 14.21 [Pr84; 3.16] *If R is any ring then the following conditions are equivalent:*

(i) R_R is totally transcendental and the pp-rank of R_R is finite;

(ii) R is left artinian.

Proof That (i) implies (ii) is clear by **5.13** and **5.18** (since finite pp-rank certainly implies that R is left noetherian). The converse follows by **14.16** and **14.20**, since (ii) implies that the lattice of finitely generated left ideals has finite length. ◻

Corollary 14.22 ([Gar80; Remark5 after Lemma4], [Sab75a; Prop8] for the countable case, also see [SE71; Prop4] and [Sab70; Thm3]) *If R is right perfect and left coherent then every flat (= projective) module is totally transcendental.*

Proof This is immediate by **14.19** and **3.7**. ◻

The fact that every projective module over a right perfect and left coherent ring is pure-injective (which follows from **14.22**) was already obtained by Sabbagh [Sab70: Thm3]. In fact, he then deduced the following result, which was later obtained also by Jensen, Zimmermann and Lenzing (see introduction to [Z-HZ78]). Zimmermann-Huisgen and Zimmermann give an example [Z-HZ78; Example 14] to show that a semiprimary (even commutative) ring with the acc on right annihilators need not be t.t. as a module over itself.

Theorem 14.23 [Sab70; Thm 3, Cor 1], [Lan76; §§2,3] *If the ring* R *is totally transcendental as a module over itself then it is semi-primary, with acc on right annihilators. In particular, every right perfect and left coherent ring is left perfect.*

Proof First, dcc on left annihilators is established: equivalently it is shown that if $\text{l-ann}(A)=\{r\in R : rA=0\}$ is any left annihilator of a subset A of R, then $\text{l-ann}(A)=\text{l-ann}(A_0)$ for some finite subset, A_0, of A. But this is clear: choose $a_1\in A$; if $\text{l-ann}(a_1)=\text{l-ann}(A)$ then we are finished; otherwise $\text{l-ann}(a_1)\supset\text{l-ann}(A)$ so choose $a_2\in A$ with $\text{l-ann}(a_1)\supset\text{l-ann}(a_1,a_2)$; and so on. Observe that $\text{l-ann}(a_1,\dots,a_n)$ is right pp-definable by the formula $\bigwedge_1^n va_i=0$. So **3.1** yields that the descending chain $\text{l-ann}(a_1)\supset\text{l-ann}(a_1,a_2)\supset\dots$ must stop at some point; and it stops at $\text{l-ann}(A)$.

Clearly dcc on left annihilators is equivalent to acc on right annihilators.

As remarked after **14.B**, we have that R is right perfect, so R/J is semisimple artinian. Therefore it remains to show that J is nilpotent.

Now, J is right T-nilpotent. To see that this is so, take a sequence $a_1,a_2,\dots$ of elements of J and consider the right pp-definable subgroups defined by the formulas "$a_na_{n-1}\dots a_1|v$". These form a descending chain which, by **3.1**, must stop. So, for some $n\in\omega$ there is $c\in R$ with $a_n\dots a_1=ca_{n+1}a_n\dots a_1$. Re-arrange this to get $(1-ca_{n+1})a_n\dots a_1=0$. Since ca_{n+1} is in the radical J, $1-ca_{n+1}$ is invertible. Hence $a_n\dots a_1=0$, as required.

By acc on right annihilators there is $n\in\omega$ with $\text{r-ann}(J^n)=\text{r-ann}(J^{n+1})$, where r-ann(-) denotes right annihilator. Suppose that J were not nilpotent. Then J^{n+1} would be non-zero; so there would be $a_1\in J$ with $J^na_1\neq 0$, hence with $J^{n+1}a_1\neq 0$. So there would be $a_2\in J$ with $J^na_2a_1\neq 0$; and so on. This would contradict right T-nilpotence, as required.

The second part follows by **14.19** and **14.B**. □

Corollary 14.24 [Sab70; Cor 2 to Thm 3] *If* R *is right perfect and left coherent, and if the module* R_R *is absolutely pure, then* R *is quasi-Frobenius.*

Proof Since R_R is absolutely pure and, by **14.22**, pure-injective, R is therefore right self-injective (i.e., R_R is injective). Moreover, by **14.23**, R has acc on right annihilators. It follows by, for example, [St75; XIV.3.5] that R is quasi-Frobenius. □

Theorem 14.25 [Ch60], [SE71; Thm 5] *The following conditions on the ring* R *are equivalent:*

(i) R *is right perfect and left coherent;*
(ii) the class of projective modules is elementarily closed;
(iii) the class of projective modules is elementary;
(iv) every product of projective modules is projective;
(v) R^κ *is projective for every cardinal* κ.

Proof The equivalence of (i), (iv) and (v) is a result of Chase (e.g., see [Fai76; 22.31B]).

That (i) implies (iii) follows from **14.18** and the fact that right perfect implies that flatness coincides with projectivity (**14.B**). Certainly (iii)⇒(ii). Finally, (ii) implies (iv), since R^κ is elementarily equivalent to $R^{(\kappa)}$, which is projective. □

So suppose that R is right perfect and left coherent (note that left artinian rings are included among such rings). Then, by **14.25**, there is a largest complete theory of projective modules: $T_{\text{proj}} = \text{Th}(\bigoplus\{P_T : T$ is a complete theory of projectives and P_T is a chosen model of $T\})$. By **14.22**, this theory is totally transcendental. So, by **3.14**, every projective is a direct sum of indecomposable projectives. How many indecomposable projectives are there? (in the dual, injective, case there may well be infinitely many).

In fact, it is well-known (see [SE71; Prop2] or, say, [Fai76; 22.23]) that if P is an indecomposable projective over a right perfect ring, then P is the projective cover of a simple module and has the form eR where e is a primitive idempotent of R (so the simple module corresponding to P is eR/eJ). Therefore, if R is also left coherent, T_{proj} is a totally transcendental theory which is finite-dimensional (there are only finitely many simple modules, since R/J is artinian).

I leave as an exercise, descriptions of categoricity, saturated models, and so on, but I do record the basic descriptions of projectives and a peculiar corollary. The following corollary is actually true without the assumption that the ring is coherent (see [SE71; Th1]) but (**14.22**) it is only over such rings that the projectives all are totally transcendental.

<u>Corollary 14.26</u> *Suppose that R is right perfect and left coherent. Then every projective module has an essentially unique decomposition $\bigoplus_{i=1}^{n} P_i^{(\kappa_i)}$ for suitable cardinals κ_i, where $R = \bigoplus_{i=1}^{n} P_i^{n_i}$ for certain $n_i \in \omega$, $n_i \geqslant 1$, where each P_i is indecomposable and where $P_i \not\simeq P_j$ if $i \neq j$.* □

<u>Corollary 14.27</u> *Suppose that R is right perfect and left coherent. Then the lattice of finitely generated left ideals of R (a sublattice of the lattice of all left ideals, since R is left coherent) has foundation rank $< \omega^n$, where n is the number of non-isomorphic indecomposable projectives.*

Proof Since R is right perfect, this lattice has the dcc, so it makes sense to talk about its length. By **5.13** this foundation rank is just the U-rank of T_{proj}. This theory is n-dimensional so the result follows by (say) **10.19** and **10.42**. □

Exercise 2

(a) If R is right perfect and left coherent then $R^{(\aleph_0)}$ is $\aleph_0$-saturated.

(b) If R is right perfect and left coherent and $T = T_{\text{proj}}$ as above, then every irreducible type in $S_1^T(0)$ is isolated (compare **11.38**).
Also, if all invariants of R are 1 or ∞ then $\text{End}M_0$, where M_0 is the prime model of $T_{\text{proj}} = \text{Th}(R)$, is the basic ring (i.e., modulo the radical, it is a product of division rings) Morita equivalent to R.

From the above, it is not difficult to characterise those rings over which the class of free modules is elementary.

<u>Theorem 14.28</u> [SE71; Thm 6] *The following conditions on the ring R are equivalent:*

(i) the class of free modules is elementary;

(ii) the class of free modules is elementarily closed;

(iii) R is left artinian and either is local (in the sense that R/J is a division ring) or is finite with R/J a simple ring.

Proof. (iii)⇒(i) In case (iii), R certainly is right perfect and left coherent. Moreover, in the first case R is the unique indecomposable projective, and in the second case R is a finite sum of copies of the unique indecomposable projective (so in either case we have unidimensionality of T_{proj}).

If R is infinite then, clearly, Th(R) is the theory of the free modules. If R is finite then choose a minimal pair φ_0/ψ_0 in R with $\mathrm{Inv}(R,\varphi_0,\psi_0)=k$ say. Then an axiomatisation for the free modules is given by $\{\mathrm{Inv}(-,\varphi,\psi)=1 : \mathrm{Inv}(R,\varphi,\psi)=1\} \cup \{$"$\mathrm{Inv}(-,\varphi_0,\psi_0)>nk \to \mathrm{Inv}(-,\varphi_0,\psi_0)>(n+1)k$" $: n\in\omega\}$ - for this axiomatises the members of $\mathcal{P}(\mathrm{Th}(R^{\aleph_0}))$ which are either infinite or are direct sums of an exact (finite) number of copies of R.

Clearly (i)$\Rightarrow$(ii). So we suppose that (ii) holds and deduce (iii). First, it is claimed that the class of projective modules is elementarily closed. Suppose that P is projective and that M is elementarily equivalent to P; choose Q such that $P\oplus Q$ is free. Then $M\oplus Q\equiv P\oplus Q$. By (ii), $M\oplus Q$ is free and hence its direct summand, M, is projective. So by **14.25**, R is right perfect and left coherent.

Next we see that, up to isomorphism, there is just one indecomposable projective. Suppose that $P_1,\dots,P_n$ $n\geqslant 2$ were the distinct indecomposable projectives. Then $R^{(\aleph_0)}\simeq \bigoplus_1^n P_i^{(\aleph_0)}$, and this is elementarily equivalent to $\bigoplus_1^{n-1} P_i^{(\aleph_0)}\oplus P_n^{(\aleph_1)}$. By (ii), this last module would be free, so isomorphic to $R^{(\kappa)}$ for some κ. Since each P_i occurs only finitely many times in the decomposition of R and since the direct-sum decomposition of P is unique, we would have a contradiction. Hence $n=1$ and R/J is a simple ring.

If R is finite then we have the desired conclusion.

Suppose then that R is infinite. We have just seen that $R\simeq P^n$ for some indecomposable projective P. Since the class of free modules is elementarily closed, in order to show that $n=1$ (i.e., that R is local) it will be enough to show that R and P are elementarily equivalent. One may show directly that each invariant of P is 1 or ∞. Alternatively, one may argue that since R is infinite, so must be P. So there is a, necessarily projective, proper elementary extension of P. This extension must be of the form $P^{(\kappa)}$ (by unidimensionality of T_{proj}) for some $\kappa\geqslant 2$. Hence $P\equiv P^2$ and so $R\equiv P^n\equiv P$, as required.

Finally, it must be shown that R is left artinian. If R is finite then this is so. Otherwise, let L be a minimal left ideal of R (we have seen already that R is right perfect). Then $L\simeq {}_R(R/J)$, by what has just been shown. Since R is left coherent, it follows (**14.A**) that J is left finitely generated. But then (by **14.23**, J is nilpotent) ${}_RR$ has a finite composition series with finitely generated factors, so R is left artinian. □

The proof above shows the following.

<u>Corollary 14.29</u> *If the class of free modules is elementary then it is unidimensional.* □

CHAPTER 15 TORSION AND TORSIONFREE CLASSES

As in Chapter 14, we deal here with not necessarily complete theories. In that chapter, our interest was in the flat and projective modules: here, we consider modules which are absolutely pure or injective within a certain class of modules.

A lot of what we do can be set within the context of "non-hereditary localisation"; so the first section sets out the basic notions of preradicals and torsion and torsionfree classes.

In the second section the axiomatisable torsion and torsionfree classes are characterised.

Let $\mathcal{K}$ be a universal Horn class of modules. We ask, in the third section, when $\mathcal{K}$ has a model-companion. This and related questions, for the case $\mathcal{K}=\mathcal{M}_R$, were the main topic of Eklof and Sabbagh's seminal paper [ES71]. They showed that the theory of modules has a model-companion iff the ring is right coherent and, in that case, the existentially complete modules are just the "fat" absolutely pure modules. We see that there is an analogous result for any universal Horn class of modules which has amalgamation. I don't assume the amalgamation property from the outset and I first derive a criterion for existence of a model-companion, in terms of the space of indecomposable pure-injectives in $\mathcal{K}$. We see that every universal Horn class of abelian groups has a model-companion.

Then, under the additional hypothesis that $\mathcal{K}$ has amalgamation, it is shown that $\mathcal{K}$ has a model-companion iff it is "coherent" in an appropriate sense: in particular, if the ring is right coherent, then every universal Horn class with amalgamation has a model-completion.

Throughout the section, we deal with the notions of absolute purity and injectivity relative to $\mathcal{K}$ and, when the class has amalgamation, with notions of "closed" submodules of free modules (cf. §1).

Universal Horn classes which are cogenerated by an injective module are particularly well-behaved. They are considered briefly in the last section. The class of localised modules is elementary iff the universal Horn class satisfies a coherence-like condition. This allows one to characterise locally finitely presented Grothendieck abelian categories as the "elementary" localisations of Module categories (the latter being the functor categories of the form $(\mathcal{S}^{op},\mathrm{Ab})$, where $\mathcal{S}$ is a small additive category).

15.1 Torsion, torsionfree classes and radicals

This section introduces the basic concepts of torsion theory and describes the connections between them (most proofs, being elementary, are omitted; they may be found in [St75; ChptVI], for example).

A **preradical** on $\mathcal{M}_R$ is simply a subfunctor of the identity functor on $\mathcal{M}_R$. Thus, τ is a preradical on $\mathcal{M}_R$ if, for each module M, τM is a submodule of M and if, whenever $M \xrightarrow{f} N$ is a morphism in $\mathcal{M}_R$, one has $f(\tau M) \leqslant \tau N$.

Given a preradical τ, one defines the classes:

$\mathcal{T}_\tau = \{ M \in \mathcal{M}_R : \tau M = 0\}$ – a typical **pretorsion** class;

$\mathcal{F}_\tau = \{ M \in \mathcal{M}_R : \tau M = 0\}$ – a typical **pretorsionfree** class.

Note that $(\mathcal{T}_\tau, \mathcal{F}_\tau) = 0$: if $M \in \mathcal{T}_\tau$ and $N \in \mathcal{F}_\tau$, then the only morphism from M to N is the zero morphism.

Examples 1

(i) Suppose that R is a commutative domain. Define τ by $\tau M = \{m \in M : mr = 0$ for some non-zero element r of $R\}$. This yields the usual notion of torsion and torsionfree (for abelian groups in particular).

(ii) For any ring R, define τ by $\tau M = \{m \in M : mc = 0$ for some right regular $c \in R\}$, where $c \in R$ is said to be **right regular** if, for every $r \in R$, $cr = 0$ implies $r = 0$. This is the direct generalisation of (i).

(iii) For any ring R, define $\tau M = \mathrm{soc} M$ - the sum of the simple submodules of M. Then $\mathcal{T}_\tau$ consists of the semisimple modules, and $M \in \mathcal{F}_\tau$ iff M has no simple submodule.

(iv) Let R be any ring. Define τM to be the sum of all submodules of M which have finite length. Then $M \in \mathcal{T}_\tau$ iff every finitely generated submodule of M has finite length ("locally finite" in the terminology of **[Aus74a]**); $\mathcal{F}_\tau$ consists of those modules with no non-zero artinian (equivalently, no simple) submodule. In particular, the torsionfree classes in (iii) and (iv) coincide, although the torsion classes need not (take R to be any artinian non-semisimple ring). The torsion class in (iv) is the closure of that in (iii) under arbitrary extensions (cf. **15.3**(b) below).

(v) Let R be any ring, E any module. Define τ by $\tau M = \bigcap \{\ker f : f \in (M,E)\}$. Thus $M \in \mathcal{T}_\tau$ iff $(M,E) = 0$, and $M \in \mathcal{F}_\tau$ iff M embeds in some power of E. The argument for this latter point is often used and proceeds as follows. Define $M \longrightarrow M^{(M,E)}$ by $m \longmapsto (fm)_{f \in (M,E)}$. Since $\tau M = 0$, this is an embedding. The converse is obvious, on considering the projections.

(vi) Let R be commutative and suppose that p is a 1-type. Then F_p (cf. §12.1) may be regarded as a functor from $\mathcal{M}_R$ to $\mathcal{M}_R$ and, as such, it is a preradical. One has $M \in \mathcal{T}_{F_p}$ iff $p(M) = M$; and $M \in \mathcal{F}_{F_p}$ iff $p(M) = 0$.

(vii) Let R be a commutative domain. Define τM to be the divisible part of M: $\tau M = \{m \in M : \forall r \in R\,(r \neq 0 \to \exists m' \in M\,(m'r = m))\}$. This defines a preradical τ, with $\mathcal{T}_\tau$ consisting of the divisible modules and $\mathcal{F}_\tau$ being the class of "reduced" modules.

These examples should give some idea of the diversity encompassed by the notion of a preradical. The reader may well have observed that in some of these examples the preradical satisfies notable properties beyond the defining one. Two important such properties are the following. The preradical τ is a **radical** if, for every module M, one has $\tau(M/\tau M) = 0$ (thus, if one removes the τ-torsion part of a module by factoring it out, then one obtains a τ-torsionfree module). The preradical τ is **idempotent** if, for every module M, one has $\tau(\tau M) = \tau M$ (so, being torsion is independent of context) - for instance, if φ is Λ-atomic then τ_{F_φ} is idempotent. An important strengthening of the latter condition is introduced in the next section, after **15.9**.

Exercise 1 Referring to the examples above, show that (i) and (ii) describe radicals but that (iii) does not; show that (i) and (ii) are idempotent but that (vi) need not be, nor need be (v) unless, say, E is injective.

Exercise/Problem 2 What conditions on a pp formula φ (say, over a commutative ring) ensures that F_φ is idempotent, respectively, a radical?

Next, I state a series of results. As remarked already, their proofs are straightforward and so are omitted (refer to [St75]). The symbol $\mathcal{C}$ will always refer to a subclass of the class, $\mathcal{M}_R$, of R-modules.

<u>Lemma 15.1</u> (see [St75; VI.1.4])

(*a*) *The following are equivalent conditions on* $\mathcal{C}$:

(*i*) $\mathcal{C}$ *is closed under quotients and coproducts;*

(*ii*) $\mathcal{C}$ *is a pretorsion class;*

(*iii*) *there is a unique idempotent preradical* τ *with* $\mathcal{C} = \mathcal{T}_\tau$.

(*b*) *The following conditions on* $\mathcal{C}$ *are equivalent:*

(*i*) $\mathcal{C}$ *is closed under subobjects and products;*

(*ii*) $\mathcal{C}$ *is a pretorsionfree class;*

(*iii*) *there is a unique radical* τ *with* $\mathcal{C} = \mathcal{F}_\tau$. □

Given preradicals τ and σ on $\mathcal{M}_R$, write $\tau \leqslant \sigma$ if $\tau M \leqslant \sigma M$ for every module M. From this relation one deduces $\mathcal{T}_\tau \subseteq \mathcal{T}_\sigma$ and $\mathcal{F}_\tau \supseteq \mathcal{F}_\sigma$.

Given a preradical τ, the idempotent preradical which defines the same torsion class (see 15.1) is denoted $\hat{\tau}$. Thus $\hat{\tau}$ is the largest idempotent preradical below τ. Analogously, denote by $\bar{\tau}$ the radical which defines the same torsionfree class as τ. Then $\bar{\tau}$ is the smallest radical above τ. In particular, one has $\hat{\tau} \leqslant \tau \leqslant \bar{\tau}$. The next result follows easily.

<u>Lemma 15.2</u> *Suppose that* τ *and* σ *are preradicals on* $\mathcal{M}_R$.

(*a*) *If* τ *and* σ *are idempotent, then* $\tau \leqslant \sigma$ *iff* $\mathcal{T}_\tau \subseteq \mathcal{T}_\sigma$.

(*b*) *If* τ *and* σ *are radicals, then* $\tau \leqslant \sigma$ *iff* $\mathcal{F}_\tau \supseteq \mathcal{F}_\sigma$. □

The next lemma complements 15.1.

<u>Lemma 15.3</u> (see [St75; VI.2.1, 2.2])

(*a*) *A radical* τ *is idempotent iff* $\mathcal{F}_\tau$ *is closed under extensions.*

(*b*) *An idempotent preradical is a radical iff* $\mathcal{T}_\tau$ *is closed under extensions.*

(*c*) *A class* $\mathcal{C} \subseteq \mathcal{M}_R$ *has the form* $\mathcal{T}_\tau$ *for some idempotent radical* τ *iff* $\mathcal{C}$ *is closed under quotients, coproducts and extensions.*

(*d*) *A class* $\mathcal{C} \subseteq \mathcal{M}_R$ *has the form* $\mathcal{F}_\tau$ *for some idempotent radical* τ *iff* $\mathcal{C}$ *is closed under subobjects, products and extensions.* □

A class $\mathcal{C}$ is said to be **closed under extensions** if, whenever $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is an exact sequence, with A and C in $\mathcal{C}$, then B is in $\mathcal{C}$.

A class as in 15.3(c), respectively 15.3(d), is termed a **torsion class**, respectively a **torsionfree class**. Given an idempotent radical τ, the pair $\langle \mathcal{T}_\tau, \mathcal{F}_\tau \rangle$ is called a **torsion theory**.

<u>Lemma 15.4</u> (see [St75; §VI.2])

(*a*) *If* τ *is an idempotent radical, then (with the obvious notation)* $\mathcal{T}_\tau = \{M : (M, \mathcal{F}_\tau) = 0\}$ *and* $\mathcal{F}_\tau = \{M : (\mathcal{T}_\tau, M) = 0\}$.

(*b*) *If* $\langle \mathcal{T}, \mathcal{F} \rangle$ *is a pair of subclasses of* $\mathcal{M}_R$ *such that* $(\mathcal{T}, \mathcal{F}) = 0$ *and* $\mathcal{T}, \mathcal{F}$ *are maximal with respect to this property, then* $\langle \mathcal{T}, \mathcal{F} \rangle$ *is a torsion theory.*

(*c*) *The correspondence between idempotent radicals and torsion theories is bijective (and note that each of* τ, $\mathcal{T}$ *or* $\mathcal{F}$ *determines everything else).* □

Every class, $\mathcal{C}$, of modules cogenerates a torsion theory in the following sense: one sets $\mathcal{T}_\mathcal{C} = \{M : (M, \mathcal{C}) = 0\}$ (and so $\mathcal{F}_\mathcal{C} = \{M : \mathcal{T}_\mathcal{C} = 0\}$).

Also, given a class $\mathcal{C}$, set $\mathrm{cog}\mathcal{C} = \{M : M$ embeds in a product of modules from $\mathcal{C}\}$; if $\mathcal{C} = \{N\}$ then write $\mathrm{cog}N$ for $\mathrm{cog}\{N\}$. Clearly $\mathrm{cog}\mathcal{C}$ is a pretorsionfree class, but it need not be a torsionfree class, so may be smaller than "$\mathcal{C}$" in the last paragraph (for an example, take $\mathcal{C}$ to be the class of simple modules). It is this latter notion of cogeneration which fits in well with the concerns of this chapter.

At least if $\mathcal{C}$ is a proper class, $\mathcal{F}_\mathcal{C}$ need not be of the form $\mathcal{F}_C$ for any single module C: one may take for $\mathcal{C}$ the class of cotorsion-free abelian groups. See Dugas and Göbel [DuGö85] for references and more on this theme. Also see [DFS87], where it is shown that the torsionfree classes cogenerated by a single element are precisely those cogenerated by a pure-injective (cf. the next result). (Analogous results for torsion classes are in [DH83].)

Say that a subclass, $\mathfrak{D}$, of $\mathcal{M}_R$, which is closed under submodules, has **rank** $\leqslant \kappa$ if, for any module M, one has $M \in \mathfrak{D}$ iff every $<\kappa$-generated submodule of M lies in $\mathfrak{D}$. The example above shows that a torsionfree class need not have a rank.

<u>Lemma 15.5</u> *Suppose that* $\mathcal{C}$ *is closed under submodules, products and pure-injective hulls. Then* $\mathrm{cog}\mathcal{C}$ *has a cogenerator and, in fact,* $\mathrm{rank}\,\mathcal{C} \leqslant (|R| + \aleph_0)^+$.

Proof For every module, D, in $\mathrm{cog}\mathcal{C}$ with no more than $|R| + \aleph_0$ generators, choose some pure-injective C' in $\prod \mathcal{C}$ into which D embeds. Let C be the product of all the C' thus obtained.

Suppose that M is such that every submodule with no more than $|R| + \aleph_0$ generators is in $\mathrm{cog}\mathcal{C}$. Let $a \in M$. Then there is M', pure in M, containing a and $\leqslant |R| + \aleph_0$-generated. By assumption, there is an embedding of M' into C. Since M' is pure in M and C is pure-injective, this embedding extends to a morphism $M \xrightarrow{f} C$: note that $fa \neq 0$. Therefore, by the usual argument (cf. Ex 1(iv) above), M embeds in $C^{(M,C)}$ and so $M \in \mathrm{cog}\mathcal{C}$. Thus the rank of $\mathrm{cog}\mathcal{C}$ is as claimed, Also, the argument has shown that $\mathrm{cog}C = \mathrm{cog}\mathcal{C}$. □

An example which shows that one may not replace $|(|R| + \aleph_0)^+$ by $|R| + \aleph_0$ in the above is given by taking $\mathcal{C}$ to be the class cogenerated by the abelian group $\mathbb{Z}_{(2)}$. Then every finitely generated submodule of $\mathbb{Q}$, being free, is in $\mathcal{C}$, but $\mathbb{Q}$ is not in $\mathcal{C}$.

Say that the class $\mathfrak{D}$, closed under submodules, is **locally defined** if $\mathrm{rank}\,\mathfrak{D} \leqslant \aleph_0$. So, in particular, a universal Horn class (see §2) of modules is locally defined.

I now digress briefly to indicate how a locally defined class of modules may be described by specifying a class of submodules of $R^{(\aleph_0)}$. This may enable some of the material in §2 for universal Horn classes to be carried through for more general classes. It could also give an alternative approach to "localisation at pure-injectives" (see §12.3).

Define a **closed system**, K, on R to be a set of the form $K = \bigcup_{k \geqslant 1} K_n$, where K_n is a set of submodules of R^n, subject to the following conditions:

(i) $R^n \in K_n$ for each $n \geqslant 1$;

(ii) if $I \in K_n$ and $r_1, \ldots, r_m \in R^n$, then $(I : (r_1, \ldots, r_m)) \in K_m$, where $(I : (r_1, \ldots, r_m)) = \{(s_1, \ldots, s_m) \in R^m : \sum_{i=1}^m r_i s_i \in I\}$.

Say that the members of K are **closed**.

Let $\mathfrak{D}$ be a non-empty class, closed under submodules. Define $\mathrm{K}(\mathfrak{D})$ by $\mathrm{K}(\mathfrak{D}) = \{I \leqslant R^n : n \geqslant 1 \text{ and } R^n/I \in \mathfrak{D}\}$. Given a closed system K, define a class $\mathrm{D}(K) = \{M :$ for all $n \geqslant 1$ and for all $I \leqslant R^n$, if R^n/I embeds in M then $I \in K\}$.

<u>Lemma 15.6</u> *With notation as above:*

(*a*) $\mathfrak{D} = \mathrm{D}(K)$ *is a locally defined class closed under submodules, and* $\mathrm{KD}(K) = K$;

(*b*) $K = \mathrm{K}(\mathfrak{D})$ *is a closed system, and* $\mathrm{DK}(\mathfrak{D}) = \mathfrak{D}$.

Proof (a) That $\mathrm{D}(K)$ is closed under submodules will follow if it is shown that for every $I \in K_n$, every finitely generated submodule of R^n/I has the form R^m/J for some m and some $J \in K_m$.

So let $r_1, \ldots, r_m \in R^n$: I claim that $R^m/(I:(r_1, \ldots, r_m))$ is isomorphic to $\sum_{i=1}^m \bar{r}_i R = (\sum_{i=1}^m r_i R + I)/I$. To see this, define $f: R^m \longrightarrow R^n/I$ by sending $(s_1, \ldots, s_m)$ to $(\sum r_i s_i + I)/I$. This induces a morphism $R^m/(I:(r_1, \ldots, r_m)) \longrightarrow R^n/I$ since each element of $(I:(r_1, \ldots, r_m))$ is sent into I. Furthermore, by definition of $(I:(r_1, \ldots, r_m))$, this induced morphism is monic. Clearly the morphism is epi. Thus the claim is established.

The remainder of the proof is trivial. □

<u>Corollary 15.7</u> *There is an inclusion-preserving bijection between locally defined classes, closed under submodules, and closed systems.* □

One may go on, and note that this establishes a bijection between locally defined pretorsionfree classes and closed systems which are closed under arbitrary intersections. If such a pretorsionfree class is axiomatisable (I do not know whether it is worthwhile putting this in terms of the corresponding closed system), then it is a universal Horn class. Such classes are considered in the next section.

15.2 Universal Horn classes, varieties and torsion classes

A class is said to be **universal** if it may be axiomatised by a set of universal sentences. A class is universal iff it is closed under submodules and ultraproducts. We will be concerned with classes which are also closed under products. For these, one has the following characterisation, where one says that a class is **universal Horn** (or a **quasivariety**) if it may be axiomatised by sentences of the form $\forall \bar{v}\, \theta(\bar{v})$ and $\forall \bar{v}\, (\varphi(\bar{v}) \rightarrow \theta(\bar{v}))$ where θ is atomic and φ is $\bigwedge$-atomic.

<u>Lemma 15.8</u> (see e.g. [Co81; 2.8, 4.4]) *The following conditions on a class $\mathcal{C}$ (of modules) are equivalent:*
(i) $\mathcal{C}$ is a universal Horn class;
(ii) $\mathcal{C}$ is a universal class closed under products;
(iii) $\mathcal{C}$ is closed under submodules, products and ultraproducts;
(iv) $\mathcal{C}$ is an elementary pretorsionfree class.

Proof For the equivalence of (i), (ii) and (iii), see the references given. The equivalence of (iii) and (iv) is obvious from **15.1**(b). □

It is a consequence of **15.8** and **15.1** that, if $\mathcal{C}$ is a universal Horn class of modules, then there is a unique radical τ on $\mathcal{M}_R$ with $\mathcal{C} = \mathcal{F}_\tau$. This radical is given explicitly by: $\tau M = \bigcap\{M' : M' \leqslant M$ and $M/M' \in \mathcal{C}\}$. Such radicals - i.e., those for which $\mathcal{F}_\tau$ is an elementary class - are characterised in the next result. A preradical τ is of **finite type** if τ commutes with filtered colimits: that is, if $\tau \varinjlim_\lambda M_\lambda = \varinjlim_\lambda \tau M_\lambda$.

<u>Proposition 15.9</u> [Ek75; 1.2], [Pr79; Thm7], [Kom80; Thm1] *If τ is a radical on $\mathcal{M}_R$ then the following conditions are equivalent:*
(i) $\mathcal{F}_\tau$ is a universal Horn class;
(ii) $\mathcal{F}_\tau$ is closed under filtered colimits;
(iii) τ is of finite type.

Proof The equivalence of (i) and (ii) is essentially [Ek75; 1.2]. I repeat the proof here. Suppose then that $\mathcal{F}_\tau$ is a universal Horn class, so is closed under subobjects and reduced products [CK73; 6.2.2]. The standard construction of the filtered colimit of a filtered family $(M_\lambda)_\lambda$, produces the colimit as a subobject of the reduced product $\prod_\lambda M_\lambda/\mathfrak{D}$, where $\mathfrak{D}$ is the filter on Λ generated by the sets of the form $\{\mu \in \Lambda : \mu \geqslant \lambda\}$ for $\lambda \in \Lambda$. Thus (i) implies (ii).

For the converse, suppose that $\mathcal{F}_\tau$ is closed under filtered colimits. Then, since any ultraproduct may be obtained as the colimit of a filtered family of products of its components, one sees, from **15.8**, that (ii) implies (i).

The implication (iii)$\Rightarrow$(ii) is easy (see proof of **15.10** below), so the proof is finished by showing that (iii) holds if $\mathcal{F}_\tau$ is closed under filtered colimits. Therefore, let $(M_\lambda)_\lambda$ be a filtered family of modules, and consider the correspondingly filtered family, $(\tau M_\lambda)_\lambda$, of torsion submodules. Let $M = \varinjlim M_\lambda$ and set $N = \varinjlim \tau M_\lambda \leqslant M$. It must be shown that $\tau M = N$.

Now, the universal property of colimits certainly gives the inclusion $N \leqslant \tau M$. Also, for each $\lambda \in \Lambda$, one has the short exact sequence $0 \longrightarrow \tau M_\lambda \longrightarrow M_\lambda \longrightarrow M_\lambda/\tau M_\lambda \longrightarrow 0$ with $M_\lambda/\tau M_\lambda$ in $\mathcal{F}_\tau$, since τ is a radical. Consider the colimit of this filtered family of short exact sequences: $0 \longrightarrow \varinjlim \tau M_\lambda \longrightarrow \varinjlim M_\lambda \longrightarrow \varinjlim (M_\lambda/\tau M_\lambda) \longrightarrow 0$ - this sequence is exact since $\mathcal{M}_R$ is Grothendieck [Pop73; 2.8.6]. By hypothesis, $\varinjlim(M_\lambda/\tau M_\lambda)$ lies in $\mathcal{F}_\tau$: that is, M/N lies in $\mathcal{F}_\tau$: hence $\tau M \leqslant N$. Therefore $\tau M = N$, as required. □

If τ is a left exact radical, then by $\mathcal{U}_\tau$ one denotes the set of right ideals I such that $R/I \in \mathcal{T}_\tau$ - the set of right ideals τ-**dense** in R_R. From $\mathcal{U}_\tau$ one may recover τ (see [St75; §VI.4]). The radical τ is said to be **left exact** if it is so as a functor: that is, iff $\mathcal{T}_\tau$ is closed under submodules; equivalently, iff $\mathcal{F}_\tau$ is closed under injective hulls. Then one says that the torsion theory $(\mathcal{T}_\tau, \mathcal{F}_\tau)$ is **hereditary**. Another equivalent is that τ is hereditary iff, for every inclusion $M \leqslant N$, one has $\tau(M) = M \cap \tau(N)$.

In the case that we are dealing with a hereditary torsion theory, an equivalent condition to those in **15.9** is that the filter $\mathcal{U}$ of τ-dense right ideals be "cofinally finitely generated", in the sense that every member of $\mathcal{U}$ contains a finitely generated member of $\mathcal{U}$ ([Gol75], [St75]).

Example 1 Let R be any domain, and define the idempotent radical τ by $\tau M = \{m \in M : mr = 0 \text{ for some non-zero } r \in R\}$. If R is uniform (for example, if R is commutative) then τ yields the Goldie torsion theory. In any case, $\mathcal{F}_\tau$ is an elementary class, being axiomatised by $\{\forall v(vr = 0 \to v = 0) : r \in R,\ r \neq 0\}$. Hence τ is of finite type.

<u>Corollary 15.10</u> [Pr79; Cor8] *If the preradical τ is of finite type, then so is the smallest radical $\bar{\tau}$ above it.*

Proof By 15.1, $\mathcal{F}_\tau = \mathcal{F}_{\bar{\tau}}$, so it will be enough to deduce, from the assumption that τ is of finite type, that this class is closed under filtered colimits. So let $M = \varinjlim M_\lambda$ be a filtered colimit of modules, M_λ, in $\mathcal{F}_\tau$. Then $\tau M = \tau \varinjlim M_\lambda = \varinjlim \tau M_\lambda$ (by hypothesis), and this equals $\varinjlim 0 = 0$. Hence $M \in \mathcal{F}_\tau = \mathcal{F}_{\bar{\tau}}$, as required. □

<u>Corollary 15.11</u> [Pr79; Cor9] *If τ is a radical, then there is a largest radical, $\tilde{\tau}$, of finite type below τ.*

Proof Let $\mathcal{K}$ be the intersection of all univeral Horn classes containing $\mathcal{F}_\tau$. Then $\mathcal{K}$ is itself a universal Horn class so, by **15.9**, has the form $\mathcal{F}_{\tilde{\tau}}$ for some radical, $\tilde{\tau}$, of finite type. Since $\mathcal{F}_{\tilde{\tau}} \supseteq \mathcal{F}_\tau$, **15.2** gives $\tilde{\tau} \leqslant \tau$ and, clearly, from the choice of $\mathcal{K}$, $\tilde{\tau}$ is the largest such radical. □

Now, I consider **varieties**: those classes defined by sentences of the form $\forall \bar{v}\, \theta(\bar{v})$, where θ is $(\bigwedge)$-atomic. One has the following well-known characterisation of varieties.

<u>**Lemma 15.12**</u> (see [Co81; 3.1]) *A class $\mathcal{C}$ (of modules) is a variety iff it is closed under subobjects, products and quotients.* □

In particular, by **15.8** and **15.1**, every variety of modules has the form $\mathcal{F}_\tau$ for some unique radical τ which, by **15.9**, must be of finite type. Those first two results also give the following.

<u>Proposition 15.13</u> [Pr79; Prop10] *Let $\mathcal{C}$ be a pretorsionfree class. Then $\mathcal{C}$ is a variety iff $\mathcal{C} = \mathcal{T}_{\tau'}$ for some (idempotent) preradical τ'.* □

The subclass $\mathcal{C}$ of $\mathcal{M}_R$ is a TTF (torsion, torsionfree) **class** if there are idempotent radicals τ and τ' on $\mathcal{M}_R$ such that $\mathcal{C} = \mathcal{F}_\tau = \mathcal{T}_{\tau'}$. The triple $(\mathcal{T}_\tau, \mathcal{C}, \mathcal{F}_{\tau'})$ is then termed a **TTF-theory**. The following results describe the TTF-classes. The first characterises the hereditary torsion classes which are elementary.

<u>Proposition 15.14</u> [Pr78; 4.4] *Let τ' be a left exact radical on $\mathcal{M}_R$. Then the following conditions are equivalent:*

(i) $\mathcal{T}_{\tau'}$ *is closed under products;*

(ii) $\mathcal{T}_{\tau'}$ *is an elementary class;*

(iii) $\mathcal{U}_{\tau'} = \{I \leqslant R_R : A \leqslant I\}$ *for some idempotent ideal, A, of R.*

Proof (i)⇒(ii) Since $\mathcal{T}_{\tau'}$ is closed under quotients it is, since closed under products, also closed under ultraproducts. Since τ' is left exact, $\mathcal{T}_{\tau'}$ is closed under (elementary) submodules. Hence $\mathcal{T}_{\tau'}$ is indeed an elementary class.

(ii)⇒(iii) I show that $\mathcal{U}_{\tau'}$ is closed under arbitrary intersections: from that it follows easily (e.g., see [St75; VI.6.12]) that $\mathcal{U}_{\tau'}$ is as described. So assume inductively that $\mathcal{U}_{\tau'}$ is closed under intersections of sets of cardinality $<\kappa$, where one may suppose [St75; p.144] that $\kappa \geqslant \aleph_0$.

Let $\mathcal{I} = \{I_\alpha : \alpha < \kappa\}$ be a set of right ideals in $\mathcal{U}_{\tau'}$, and set $I = \bigcap \mathcal{I}$. If there is a subset $\mathcal{I}' \subseteq \mathcal{I}$, of cardinality strictly less than κ, such that $\bigcap \mathcal{I}' = I$, then the induction hypothesis gives $I \in \mathcal{U}_{\tau'}$. So suppose that there is no such subset.

For $\alpha < \kappa$, set $J_\alpha = \bigcap \{I_\beta : \beta < \alpha\}$. Let $\mathfrak{D}$ be any uniform ultrafilter on κ. Set $M = \prod\{R/J_\alpha : \alpha < \kappa\}/\mathfrak{D}$. Then, by (ii) and the induction hypothesis, $M \in \mathcal{T}_{\tau'}$. So, in particular, there is some $K \in \mathcal{U}_{\tau'}$ with $\bar{1}.K = 0$, where $\bar{1} = (1 + J_\alpha)_\alpha/\mathfrak{D}$. This implies that for each $k \in K$ and $\alpha < \kappa$, there is $\alpha' < \kappa$ with $\alpha \leqslant \alpha'$ and $k \in J_{\alpha'}$ (that is, the set of α' with $k \in J_{\alpha'}$ must be cofinal in κ since, by uniformity of $\mathfrak{D}$, this set has cardinality κ). But $\alpha \leqslant \alpha'$ implies $J_{\alpha'} \leqslant J_\alpha$, and so $K \subseteq \bigcap\{J_\alpha : \alpha < \kappa\} = I$. Then $K \in \mathcal{U}_{\tau'}$ implies $I \in \mathcal{U}_{\tau'}$, as required.

(iii)⇒(i) Let $\{M_i : i \in I\} \subseteq \mathcal{T}_{\tau'}$. It must be shown that $M = \prod_i M_i$ is in $\mathcal{T}_{\tau'}$. Let A be as in (iii). Then, for each $i \in I$ and $a_i \in M_i$, $a_i A = 0$. So, for any $\bar{a} = (\bar{a}_i)_i \in M$, one has $\bar{a}A = 0$. Thus $M \in \mathcal{T}_{\tau'}$, as required. ▫

<u>Proposition 15.15</u> [Pr79; Thm 12] *The following are equivalent for a subclass* $\mathcal{C}$ *of* $\mathcal{M}_R$:

(*i*) $\mathcal{C}$ *is a variety closed under extensions*;

(*ii*) $\mathcal{C}$ *is a universal class, closed under quotients, coproducts and extensions*;

(*iii*) $\mathcal{C}$ *is a* TTF-*class*;

(*iv*) $\mathcal{C}$ *is an axiomatisable hereditary torsion class.*

Proof (i)⇒(ii) This is immediate from the definitions.

Suppose that (ii) holds, so, by **15.1**, there is an idempotent preradical τ' with $\mathcal{C} = \mathcal{T}_{\tau'}$. Since $\mathcal{C}$ is closed under submodules, τ' is left exact and also, by **15.3**, τ' is a radical. Since $\mathcal{C}$ is elementary, and since direct sums are elementarily equivalent to direct products (**2.24**), $\mathcal{C}$ is closed under products so, by **15.1**, $\mathcal{C} = \mathcal{F}_\tau$ for some (idempotent) radical τ. Hence $\mathcal{C}$ is a TTF class.

That (iii) implies (iv) follows by **15.14**.

Finally, (iv)⇒(i) is immediate by **15.1**, **15.3** and **15.12**. ▫

One has the following corollary.

<u>Corollary 15.16</u> [Pr79; Cor 13] *Let* $(\mathcal{T}, \mathcal{C}, \mathcal{F})$ *be a* TTF-*theory. Then* $(\mathcal{T}, \mathcal{C})$ *is a torsion theory of finite type.*

Proof This is immediate by **15.9** and **15.15**. ▫

Exercise 1 It is a rather strong condition to impose on a torsion class that it be axiomatisable. It may be shown that, under quite wide circumstances, if $\mathcal{T}$ is an axiomatisable class and contains all the simple modules then it is all of $\mathcal{M}_R$. Specifically, one has the following.

(i) [Pr79; Prop 14] If every two-sided idempotent ideal of R is finitely generated on one side and if $(\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory on $\mathcal{M}_R$ such that every simple module is in $\mathcal{T}$, then $\mathcal{T}$ is an axiomatisable class iff $\mathcal{T} = \mathcal{M}_R$.

(ii) [Pr79; Prop15] If $(\mathcal{T}, \mathcal{F})$ is a stable (i.e., $\mathcal{T}$ is closed under injective hulls) hereditary torsion theory on $\mathcal{M}_R$ such that every simple R-module lies in $\mathcal{T}$, then $\mathcal{T}$ is an axiomatisable class iff $\mathcal{T} = \mathcal{M}_R$.

Exercise 2 Let $(\mathcal{G}, \mathcal{F}_{\mathcal{G}})$ be the Goldie torsion theory (see [St75; §VI.6]).

(i) [Pr78; 4.2] $\mathcal{F}_{\mathcal{G}}$ is an axiomatisable torsion class iff $R/\tau_{\mathcal{G}}R$ has finite uniform dimension.

(ii) [Pr79; Prop16] The Goldie torsion class $\mathcal{G}$ is axiomatisable iff the socle of R, soc$R$, is $\mathcal{G}$-dense in R (that is, iff $R/\mathrm{soc}R \in \mathcal{G}$).

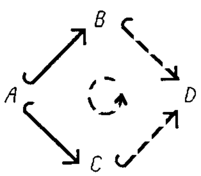
15.3 Model-companions and model-completions of universal Horn classes

Throughout this section, $\mathcal{K}$ will be a universal Horn class of modules - that is, an axiomatisable class of modules which is closed under submodules and products. Then (9.37) $\mathcal{K}$ has the form Mod$(T_\forall)$ for some complete theory T of modules, which is, in general, by no means unique. ($T_\forall$ is the set of all universal sentences in (the deductive closure of) T.) For example, the abelian groups $\mathbb{Z}_4$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ cogenerate, in the sense of 9.37(a), the same universal Horn class. Recall (§1) that I put $\mathcal{K} = \mathrm{cog}C$ if $\mathcal{K} = \{M \in \mathcal{M}_R : M$ embeds in some power of $C\}$ and then say that C **cogenerates** $\mathcal{K}$.

This section deals with the problem of determining when $\mathcal{K}$ has a model-companion (or model-completion) and describing the models. So, we will be concerned with questions concerning "complete" or "injective" members of $\mathcal{K}$. It was with such questions, with $\mathcal{K}$ being the entire class of R-modules, that Eklof and Sabbagh [ES71] initiated the recent model-theoretic study of modules. My own investigations began with the replacement of the class $\mathcal{M}_R$ by a hereditary torsionfree class ([Pr78]-[Pr80]). It should be noted that this still is a "quantifier-free" case, and pp-elimination of quantifiers does not come into play.

The approach that I take in this section is that of [Pr81a].

Let $\mathcal{K}$ be a universal Horn class (of modules). I recall some definitions and results - but beware that they have been specialised (for the general ones, see [Mac77] for instance). A structure M in $\mathcal{K}$ is **existentially complete** (e.c.) in $\mathcal{K}$ if every embedding of M into a member of $\mathcal{K}$ is an existential one: an embedding $M \hookrightarrow M'$ is **existential** if every finite system of equations and inequations with parameters from M and a solution in M' already has a solution in M. If M is thus embedded in M' then one says that M is **existentially closed** in M'. Every member of $\mathcal{K}$ embeds in an e.c. member of $\mathcal{K}$ (because $\mathcal{K}$ is an $\forall\exists$-class). It may be the case that the class, $\mathrm{EC}_{\mathcal{K}}$, of e.c. members of $\mathcal{K}$ is an axiomatisable one. In that case it follows that the theory of $\mathrm{EC}_{\mathcal{K}}$ is complete (since modules have the joint embedding property): it is called the **model-companion** of (the theory of) $\mathcal{K}$. If $\mathcal{K}$ does have a model-companion T, then T has elimination of quantifiers iff $\mathcal{K}$ has the amalgamation property, and then one says that $\mathcal{K}$ has **model-completion** T.



A class $\mathcal{C}$ has the **amalgamation** property, AP, if any diagram of embeddings as shown, with A, B and C in $\mathcal{C}$, has a completion as shown, with D in $\mathcal{C}$.

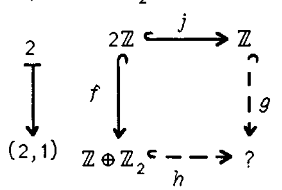
In the case of modules one has, because of pp-elimination of quantifiers, the following property, stronger than usual.

<u>Lemma 15.17</u> *If M and N are existentially complete in $\mathcal{K}$ and if M is embedded in N, then this embedding is an elementary one.*

Proof Since M is e.c., its embedding into N is existential so, in particular, is pure. By, say, [Che76; III.13], **2.15** and **2.26**, every two e.c. modules are elementarily equivalent. So, by **2.26**, the result follows. ▫

Eventually, I will have to make the additional assumption that $\mathcal{K}$ has the amalgamation property, but it is possible to avoid this for a while. Let me begin with an example of a universal Horn class of abelian groups which fails to have amalgamation.

Example 1 [Fis77] Take $\mathcal{K}$ to be the universal Horn class of abelian groups cogenerated by $\mathbb{Z}_2 \oplus \mathbb{Q}$, and consider the diagram shown, where j is the natural inclusion, and f takes $2 \in 2\mathbb{Z}$ to $(2,1) \in \mathbb{Z} \oplus \mathbb{Z}_2$. Then there is no monic completion of the square in $\mathcal{K}$.



For, if g, h were as shown then, since 2 divides $gj(2_{\mathbb{Z}})$, it would have to be that $h(2,1) = h(2,0) + h(0,1)$ were also divisible by 2. So $h(0,1)$ would be divisible by 2 and an element of order 2. But $\mathcal{K}$ has no element of order 4 – contradiction.

There are two points involved in investigating the e.c. structures of $\mathcal{K}$: what are the e.c. structures? (in terms of some structural description); do they form an elementary class? Let us begin by dealing with the first question.

Let A be any e.c. member of $\mathcal{K}$; then every embedding $A \hookrightarrow B$ is existential so is, in particular, pure. If A is actually pure-injective, then every such embedding splits. Thus one is lead to consider notions of absolute purity and injectivity, relative to $\mathcal{K}$ (unfortunately, this means that we have "relatively absolutely pure" modules). (cf. [BH61], [Skl78], [St67,68]).

Say that $A \in \mathcal{K}$ is **absolutely pure** in $\mathcal{K}$ if every embedding $A \hookrightarrow B$, with $B \in \mathcal{K}$, is pure: write $A \in \mathrm{Abs}(\mathcal{K})$. Say that $A \in \mathcal{K}$ is injective in $\mathcal{K}$ if every embedding $A \hookrightarrow B$, with $B \in \mathcal{K}$, is split: write $A \in \mathrm{Inj}(\mathcal{K})$.

<u>Proposition 15.18</u> [Pr81a; 2.2] *Let $\mathcal{K}$ be a universal Horn class of modules, and let A be a member of $\mathcal{K}$.*
(*a*) $\mathrm{Inj}(\mathcal{K}) \subseteq \mathrm{Abs}(\mathcal{K})$.
(*b*) $A \in \mathrm{Abs}(\mathcal{K})$ *iff* $\overline{A} \in \mathrm{Inj}(\mathcal{K})$.
(*c*) $\mathrm{Inj}(\mathcal{K})$ *consists exactly of the pure-injective members of* $\mathrm{Abs}(\mathcal{K})$.

Proof Part (a) is immediate from the definitions. For part (b)$\Rightarrow$, suppose that $A \in \mathrm{Abs}(\mathcal{K})$ and let $\overline{A} \hookrightarrow B \in \mathcal{K}$ be an embedding. Since the induced embedding of A into B is pure, so is strictly pp-type preserving, it follows by **4.14** that the embedding of $\overline{A}$ into B is pure, so is split, as required. Part (c) now follows. For (b)$\Leftarrow$, the following lemma is needed (if $\mathcal{K}$ has AP then (b)$\Leftarrow$ is easy).

<u>Lemma 15.19</u> *Suppose that $j: A \longrightarrow M$ is an embedding. Then there is an elementary extension, M', of M and an embedding of the pure-injective hull $\overline{A}$ of A into M', extending j.*

Proof Let $\bar{a}\frown\bar{b}$ be an enumeration of $A\frown(\overline{A}\setminus A)$. Let $p(\bar{v},\bar{w})$ be the pp-type of $\bar{a}\frown\bar{b}$ and let $q(\bar{v})$ be the type of $j\bar{a}$ in M. Consider $\Psi(\bar{w}) \equiv p(j\bar{a},\bar{w}) \cup q(j\bar{a}) \cup \{w_i \neq 0\}$, where $\bar{w} = (w_i)_i$. I claim that this set is finitely satisfied in M.

Otherwise, there is $\varphi(\bar{v},\bar{w}) \in p$, $\theta(\bar{v}) \in q$ and $b_i \in \overline{A}\setminus A$, such that $M \models \forall \bar{w}\,(\varphi(j\bar{a},\bar{w}) \wedge \theta(j\bar{a}) \rightarrow w_i = 0)$. By **4.10**, we may assume that φ includes a formula linking b_i to A; so $A \models \varphi(\bar{a},\bar{w}) \rightarrow w_i \neq 0$. Now, $\overline{A} \models \varphi(\bar{a},\bar{b})$ so, since A is pure in $\overline{A}$, there is $\overline{d}$ in A with $A \models \varphi(\overline{a},\overline{d})$. By choice of φ, $d_i \neq 0$, where d_i is the element corresponding to b_i.

Then, since j is monic, $M \models \varphi(j\overline{a},j\overline{d}) \wedge \theta(j\overline{a}) \wedge d_i \neq 0$ - contradiction, as required.

So let $\overline{c}$ in $M' \succ M$ realise $\Psi(\overline{w})$. Since $\mathrm{pp}^M(j\overline{a}^\frown\overline{c}) \supseteq \mathrm{pp}^{\overline{A}}(\overline{a}^\frown\overline{b})$, there is an extension of j to a morphism from $\overline{A}$ to M'. Since $c_i \neq 0$ for each i, this extension is monic, as required. □

So, returning to **15.18**(b), suppose that $\overline{A} \in \mathrm{Inj}(\mathcal{K})$ and let $j: A \longrightarrow M \in \mathcal{K}$ be an embedding. By the lemma, there is an extension to an embedding $f: \overline{A} \longrightarrow M'$, where $M \prec M' \in \mathcal{K}$. Since $\overline{A}$ is injective in $\mathcal{K}$, f is pure and so the composition fj is pure. Certainly then, the co-restriction of fj to M is pure, as required. □

Note that $\mathrm{Inj}\,\mathcal{K}$ is closed under direct products and direct summands.

<u>Lemma 15.20</u> *Let $\mathcal{K}$ be a universal Horn class and let C be existentially complete in $\mathcal{K}$. Then $C^{\aleph_0}$ is existentially complete in $\mathcal{K}$.*

Proof Let $C^{\aleph_0} \hookrightarrow A \in \mathcal{K}$ be an embedding, and let $\sigma \equiv \exists\overline{v}\,(\theta(\overline{v},\overline{c}) \wedge \bigwedge \neg\chi_i(\overline{v},\overline{c}))$ be satisfied in A, where $\overline{c}$ is in $C^{\aleph_0}$, θ is $\bigwedge$-atomic and the χ_i are atomic. It will be enough to show that this is satisfied in $C^{\aleph_0}$, since a typical existential sentence is a disjunction of such sentences. It may be assumed that $\theta(\overline{v},\overline{0}) \geqslant \chi_i(\overline{v},\overline{0})$ for each i.

Suppose that $C^{\aleph_0}$ did not satisfy σ: then, since all indices in $C^{\aleph_0}$ are "1" or "∞", it follows by Neumann's Lemma that, in $C^{\aleph_0}$, $\theta(\overline{v},\overline{0})$ is equivalent to one of the $\chi_i(\overline{v},\overline{0})$ - say to $\chi_j(\overline{v},\overline{0})$. Therefore the same is true in C. But then we have the suituation that C is embedded (*via* $C^{\aleph_0}$) in A, where the latter satisfies the existential sentence $\exists\overline{v}\,(\theta(\overline{v},\overline{0}) \wedge \neg\chi_j(\overline{v},\overline{0}))$, which is not satisfied in C. This contradicts C being existentially complete. □

<u>Corollary 15.21</u> *Let $\mathcal{K}$ be a universal Horn class, and let C be existentially complete in $\mathcal{K}$. Then $C \equiv C^{\aleph_0}$ and C is absolutely pure in $\mathcal{K}$.*

Proof Consider the embedding of C into $C^{\aleph_0}$. By **15.20**, $C^{\aleph_0}$ is e.c. so, by **15.17**, this is an elementary embedding. □

It follows that the existentially complete members of $\mathcal{K}$ are the "fat" members of $\mathrm{Abs}(\mathcal{K})$ in the sense of **[Sab71]**. This may be put in topological terms as follows. Define the following subset of $\mathcal{I}(\mathcal{K})$ (the space of indecomposables in $\mathcal{K}$): $\mathcal{U}(\mathrm{Inj}(\mathcal{K})) = \{N \in \mathcal{I}(\mathcal{K}) : N \in \mathrm{Inj}(\mathcal{K})\}$ - the subset of injective-in-$\mathcal{K}$ points of $\mathcal{I}(\mathcal{K})$. For any member A of $\mathcal{K}$, $\mathcal{U}(A)$ denotes the set of indecomposable summands of $\overline{A}$. Note that if A is a discrete pure-injective then $\mathrm{cl}\,\mathcal{U}(\overline{A}) = \mathcal{I}(\overline{A}) = \mathcal{I}(A)$, where "cl" denotes topological closure.

<u>Lemma 15.22</u> *Let $\mathcal{K}$ be a universal Horn class of modules, and let C be in $\mathcal{K}$. Then C is existentially complete in $\mathcal{K}$ iff $\overline{C}$ is existentially complete in $\mathcal{K}$.*

Proof Suppose first that C is e.c. Let $\overline{C}$ be embedded in $A \in \mathcal{K}$, and suppose that the latter satisfies the formula $\exists\overline{v}\,\chi(\overline{v},\overline{d})$ where χ is quantifier-free and $\overline{d}$ is in $\overline{C}$. Let $p(\overline{w})$ be the pp-type in $\overline{C}$ of $\overline{d}$ over C. Consider any $\varphi(\overline{w},\overline{c})$ in p: one has that A satisfies $\exists\overline{w}\,(\varphi(\overline{w},\overline{c}) \wedge \exists\overline{v}\,\chi(\overline{v},\overline{w}))$ so, since C is e.c., this formula is satisfied also in C. Therefore the set of formulas $p(\overline{w}) \wedge \exists\overline{v}\,\chi(\overline{v},\overline{w})$ is finitely satisfied in C. By **4.5**, $\exists\overline{v}\,\chi(\overline{v},\overline{w})$ is in $p(\overline{w})$. Therefore $\overline{C}$ satisfies $\exists\overline{v}\,\chi(\overline{v},\overline{d})$, as required.

For the converse, suppose that $\overline{C}$ is e.c., and let C be embedded in $A \in \mathcal{K}$. By **15.19**, the embedding of C into A extends to an *embedding* of $\overline{C}$ into an elementary extension, A', of A. If σ is an existential sentence with parameters from C which is satisfied in A, then it is satisfied in A' and so in $\overline{C}$. But C is an elementary substructure of $\overline{C}$ (**2.27**), and so the result follows. Alternatively, we may use that fact that an existential substructure of an e.c. structure is e.c. (see [Che76; §III.3] or [Mac77]). □

Lemma 15.23 *Let $\mathcal{K}$ be a universal Horn class of modules and let C be existentially complete in $\mathcal{K}$. Suppose that N is injective in $\mathcal{K}$. Then $C \oplus N$ is existentially complete in $\mathcal{K}$.*

Proof By **15.22** it may be supposed that C is pure-injective. Let $C \oplus N$ be embedded in $A \in \mathcal{K}$. Since, by **15.21**, $C \oplus N$ is injective in $\mathcal{K}$, A may be decomposed as $C \oplus N \oplus B$. Suppose that A satisfies the existential statement $\exists \overline{v}(\theta(\overline{v},\overline{c},\overline{n}) \wedge \bigwedge_i \neg\chi_i(\overline{v},\overline{c},\overline{n}))$ where θ is $\bigwedge$-atomic and the χ_i are atomic, $\overline{c}$ is in C and $\overline{n}$ is in N. Let $\overline{a}$ in A be such that $\theta(\overline{a},\overline{c},\overline{n}) \wedge \bigwedge \neg\chi_i(\overline{a},\overline{c},\overline{n})$ holds: decompose $\overline{a}$ as $(\overline{c}',\overline{m},\overline{b}) \in C \oplus N \oplus B$.

Projecting to $C \oplus N$, one deduces $C \oplus N \models \theta((\overline{c}',\overline{m}),\overline{c},\overline{n})$ (abusing notation a little): projecting to B, one obtains $B \models \theta(\overline{b},\overline{0},\overline{0})$. For each i, let $\varepsilon\chi_i(\overline{v})$ be $\chi_i(\overline{v},\overline{0},\overline{0})$ or $\neg\chi_i(\overline{v},\overline{0},\overline{0})$ according as $\chi_i(\overline{b},\overline{0},\overline{0})$ holds or not. So B satisfies the existential parameter-free sentence $\exists\overline{v}(\theta(\overline{v},\overline{0},\overline{0}) \wedge \bigwedge_i \varepsilon\chi_i(\overline{v}))$. Therefore A, and so C, satisfies it also: let $\overline{d}$ witness $\overline{v}$ in C. Since (**15.21**) $C \equiv C^{\aleph_0}$ and by Neumann's Lemma, $\overline{d}$ may be chosen so that, for each i such that $\neg\chi_i(\overline{c}',\overline{c},\overline{0})$ holds, $(-\overline{d},\overline{0})$ does not lie in the same coset of $\chi_i(\overline{v},\overline{w},\overline{0})$ as $(\overline{c}',\overline{c})$.

We have $\theta((\overline{c}'+\overline{d},\overline{m}),\overline{c},\overline{n})$: we want also to have $\neg\chi_i((\overline{c}'+\overline{d},\overline{m}),\overline{c},\overline{n})$, for each i. If $\chi_i((\overline{c}',\overline{m}),\overline{c},\overline{n})$ holds, then since $\neg\chi_i(\overline{b},\overline{0},\overline{0})$ must hold, so does $\neg\chi_i(\overline{d},\overline{0},\overline{0})$, and hence we do have $\neg\chi_i((\overline{c}'+\overline{d},\overline{m}),\overline{c},\overline{n})$. If, on the other hand, $\neg\chi_i((\overline{c}',\overline{m}),\overline{c},\overline{n})$ holds then, although we do not know whether or not $\chi_i(\overline{d},\overline{0},\overline{0})$ holds, we did choose $\overline{d}$ so that $(-\overline{d},\overline{0})$ does not lie in the same coset of $\chi_i(\overline{v},\overline{w},\overline{0})$ as $(\overline{c}',\overline{c})$: therefore $\neg\chi_i(\overline{c}'+\overline{d},\overline{c},\overline{0})$, and hence $\neg\chi_i((\overline{c}'+\overline{d},\overline{m}),\overline{c},\overline{n})$, holds.

Thus we have shown that the original sentence holds in $C \oplus N$, and so this module is indeed existentially complete. □

Lemma 15.24 *Let $\mathcal{K}$ be a universal Horn class of modules and let $A \equiv A^{\aleph_0}$ be absolutely pure in $\mathcal{K}$. If $\mathrm{cl}\,\mathcal{U}(A) \supseteq \mathcal{U}(\mathrm{Inj}(\mathcal{K}))$ then A is existentially complete in $\mathcal{K}$. The converse holds if $\overline{A}$ is discrete.*

Proof Suppose that A is not e.c. in $\mathcal{K}$. Then there is an embedding $A \longrightarrow B \in \mathcal{K}$ and an existential formula $\exists\overline{v}(\theta(\overline{v},\overline{a}) \wedge \bigwedge_{i=1}^{n} \neg\gamma_i(\overline{v},\overline{a}))$ where $\overline{a}$ is in A, θ is $\bigwedge$-atomic and the γ_i are atomic, such that this formula is satisfied in B but not in A. Since any member of $\mathcal{K}$ embeds in an e.c. member of $\mathcal{K}$, B may be taken to be existentially complete. It may be supposed that $\theta(\overline{v},\overline{0}) \geqslant \gamma_i(\overline{v},\overline{0})$, for each i. By Neumann's Lemma and since $A \equiv A^{\aleph_0}$, the only way in which this formula can fail in A is to have $\theta(\overline{v},\overline{0})$ equivalent in A to some $\gamma_i(\overline{v},\overline{0})$. Therefore the neighbourhood $(\theta(\overline{v},\overline{0})/\gamma_i(\overline{v},\overline{0}))$ does not intersect $\mathrm{cl}\,\mathcal{U}(A)$.

For each $b \in B$, let p_b be a type (in $\mathrm{Th}(B^{\aleph_0})$) maximal with respect to extending $\mathrm{pp}^B(b)$ and not increasing the $\bigwedge$-atomic type (i.e., omitting all "$vr=0$" where $br \neq 0$). By **4.33**, p_b is irreducible. Let c_b realise p_b: then $N(c_b)$ is injective in $\mathcal{K}$. For, any embedding $N(c_b) \hookrightarrow H \in \mathcal{K}$ does not increase the annihilator of c_b so, by maximality of p_b and **4.14**, it is pure.

Since $\mathrm{pp}(c_b) \geqslant \mathrm{pp}(b)$, there is a morphism $B \longrightarrow N(c_b)$ taking b to c_b. Let $B \longrightarrow \prod_B N(c_b)$ be the morphism induced by all of these. By construction, this is an embedding; since B is e.c. in $\mathcal{K}$, it is pure. So, for some $b \in B$, the component $N(c_b)$ lies in $(\theta(\overline{v},\overline{0})/\gamma_i(\overline{v},\overline{0}))$. But then the injective $N(c_b)$ does not lie in $\mathrm{cl}\,\mathcal{U}(A)$, and the first statement follows.

For the second, suppose that A is existentially complete in $\mathcal{K}$ and that $\overline{A}$ is discrete. Suppose that there is $N \in \mathcal{U}(\mathrm{Inj}(\mathcal{K}))\setminus\mathrm{cl}\,\mathcal{U}(A)$. Then there would be some neighbourhood

(φ/ψ) of N, isolating N from $\mathrm{cl}\,\mathcal{U}(A) = \mathcal{I}(A)$. Now, by 15.23, the module $A \oplus N$ is existentially complete in $\mathcal{K}$, and so the canonical embedding of A into it should be an elementary one (15.17). But $A \oplus N$ satisfies $\exists v(\varphi(v) \wedge \neg\psi(v))$, whereas A does not satisfy this - contradiction, as required. □

Corollary 15.25 *Let $\mathcal{K}$ be a universal Horn class of modules and let A be in $\mathcal{K}$. If $A \equiv A^{\aleph_0}$, A is absolutely pure in $\mathcal{K}$ and* $\mathrm{cl}\,\mathcal{U}(A) \supseteq \mathcal{U}(\mathrm{Inj}(\mathcal{K}))$, *then A is existentially complete in $\mathcal{K}$. The converse holds if $\overline{A}$ is discrete.* □

Theorem 15.26 *Let $\mathcal{K}$ be a universal Horn class of modules. Then $\mathcal{K}$ has a model-companion iff* $\mathcal{U}(\mathrm{Inj}(\mathcal{K}))$ *is a closed subset of* $\mathcal{I}(\mathcal{K})$.

Proof $\Rightarrow$ Let C be e.c. in $\mathcal{K}$. Suppose that $N \in \mathrm{cl}\,\mathcal{U}(\mathrm{Inj}(\mathcal{K}))$: by 15.23, $C \oplus N$ is e.c. in $\mathcal{K}$. By 15.21, $C \oplus N$ is absolutely pure in $\mathcal{K}$, so N must be injective in $\mathcal{K}$, as required.

$\Leftarrow$ Let $T = T^{\aleph_0}$ be the complete theory such that $\mathcal{I}(T) = \mathcal{U}(\mathrm{Inj}(\mathcal{K}))$ (4.67). Let C be any e.c. member of $\mathcal{K}$. By the proof of 15.24, C purely embeds in a product of members of $\mathcal{U}(\mathrm{Inj}(\mathcal{K}))$: by 15.17, this is an elementary embedding. So, by hypothesis, C is a model of T. Conversely, if A is a model of T then A satisfies the conditions of 15.25 and so A is e.c., as required. □

In this case, 16.7 gives a strong restriction on the members of $\mathcal{I}(\mathrm{Inj}(\mathcal{K})) = \mathcal{U}(\mathrm{Inj}(\mathcal{K}))$.

Corollary 15.27 [ES71; 4.8], [Wh76], [Pr81a] *Let $\mathcal{K}$ be a universal Horn class of modules. Then $\mathcal{K}$ has a model-companion iff* $\mathrm{Abs}(\mathcal{K})$ *is axiomatisable.*

Proof $\Rightarrow$ By 15.26, $\mathcal{U}(\mathrm{Inj}(\mathcal{K}))$ is a closed subset of $\mathcal{I}(\mathcal{K})$, so the set of axioms $\{\mathrm{Inv}(-,\varphi,\psi) = 1 : (\varphi/\psi) \cap \mathcal{I}(\mathrm{Inj}) = \emptyset\}$ serves to define $\mathrm{Abs}(\mathcal{K})$.

$\Leftarrow$ The axioms for $\mathrm{Abs}(\mathcal{K})$ define, in $\mathcal{I}(\mathcal{K})$, exactly the subset $\mathcal{U}(\mathrm{Inj}(\mathcal{K}))$, so the result follows by 15.26. □

This may be regarded in terms of cogeneration. By 15.21, in order to determine the e.c. members of $\mathcal{K}$, it will be essentially enough to describe the members of $\mathrm{Inj}(\mathcal{K})$ which are existentially complete. Therefore we consider some notions of the "extent" of $N \in \mathrm{Inj}(\mathcal{K})$ in $\mathcal{K}$. We have already met one in §9.4: N elementarily cogenerates $\mathcal{K}$ if every $A \in \mathcal{K}$ *purely* embeds in some power of N. The other notion is: N (algebraically) cogenerates $\mathcal{K}$ if every $A \in \mathcal{K}$ embeds in some power of N. We know already (9.37) that $\mathcal{K}$ has an elementary cogenerator: the proof of the first part of 15.24 shows that $\mathcal{K}$ has an algebraic cogenerator which is injective in $\mathcal{K}$. Set $E = \prod\{N_\lambda^{\aleph_0} : N_\lambda \in \mathcal{I}(\mathrm{Inj})\}$.

Lemma 15.28 *Let $\mathcal{K}$ be a universal Horn class, and let E be as defined above. Then E is an algebraic cogenerator which is injective in $\mathcal{K}$. Furthermore, any algebraic cogenerator C which also is injective in $\mathcal{K}$ satisfies $C^{\aleph_0} \equiv E$.*

Proof By the proof of 15.24, E is an algebraic cogenerator which is injective in $\mathcal{K}$. (Of course, E is injective in $\mathcal{K}$, since it is a product of injectives in $\mathcal{K}$.)

Now let C and D be any two algebraic cogenerators which are injective in $\mathcal{K}$. Then one has embeddings $C \hookrightarrow D^{\kappa} \hookrightarrow (C^{\mu})^{\kappa}$, for suitable powers of C and D. These embeddings are pure, since C and D are injective in $\mathcal{K}$, so by 2.25, $C^{\aleph_0} \equiv D^{\aleph_0}$, as required. □

Theorem 15.29 [Pr81a; 2.4], cf. [Pr79a; Prop 1] *Let $\mathcal{K}$ be a universal Horn class of modules, and let E be defined as above. Then $\mathcal{K}$ has a model-companion iff E is an elementary cogenerator.*

Proof $\Leftarrow$ By 15.26, it is enough to show that $\mathcal{U}(\mathrm{Inj}(\mathcal{K}))$ is closed. So let N_0 be in $\mathrm{cl}(\mathcal{U}(\mathrm{Inj}(\mathcal{K}))) = \mathrm{cl}\,\mathcal{U}(E)$. Since E is an elementary cogenerator, it follows that N_0 is a factor of some power of E. Such a module must be injective since E is, as required.

⇒ If $\mathcal{K}$ has a model-companion then, by **15.27**, Abs($\mathcal{K}$) is axiomatisable. Therefore if $N = \overline{N}|M \equiv E$ then N is in Inj($\mathcal{K}$), and hence is a factor of some power of E, as required. ◻

As an application of these ideas, I show that every universal Horn class of abelian groups has a model-companion (in fact, any inductive class of abelian groups with the joint embedding property has a model companion [Sab71; Cor 6]).

Let $\mathcal{K}$ be a universal Horn class of abelian groups and consider the set $\mathcal{U}$(Inj($\mathcal{K}$)) of indecomposables which are injective in $\mathcal{K}$.

(a) Let p be a prime and suppose that some member of $\mathcal{K}$ contains an element of order p. Since the pp-types extending this $\bigwedge$-atomic type are linearly ordered, there is exactly one of $\mathbb{Z}_p, \mathbb{Z}_{p^2}, ..., \mathbb{Z}_{p^\infty}$ in Inj($\mathcal{K}$).

(b) If some member of $\mathcal{K}$ contains an element of infinite order, then $\mathbb{Q}$ is in Inj($\mathcal{K}$) and no $\overline{\mathbb{Z}(p)}$ is in Inj($\mathcal{K}$) ($\mathcal{K}$ is an elementary class, so if $\overline{\mathbb{Z}(p)}$ is in $\mathcal{K}$ then so is $\mathbb{Q}$).

(c) So $\mathcal{U}$(Inj($\mathcal{K}$)) consists of, for each prime p, either nothing, $\mathbb{Z}_{p^n}$ or $\mathbb{Z}_{p^\infty}$, possibly together with $\mathbb{Q}$ (and if there are infinitely many points, $\mathbb{Q}$ must occur).

(d) What are the possible cluster points for such a set X? There can be no finite cluster point, since such points are isolated. There can be no cluster point of the form $\overline{\mathbb{Z}(p)}$ - for consider the neighbourhood $(v = v/p|v)$ of $\overline{\mathbb{Z}(p)}$. This neighbourhood can contain at most one point of X, necessarily of the form $\mathbb{Z}_{p^n}$: then the open set $(p^n|v / p^{n+1}|v)$ isolates $\overline{\mathbb{Z}(p)}$ from X. So any cluster point already is in X.

Therefore, by **15.26**, one has the following result.

<u>Proposition 15.30</u> [Sab71; Cor 6] *Let $\mathcal{K}$ be any universal Horn class of abelian groups. Then $\mathcal{K}$ has a model-companion.* ◻

For instance, if $\mathcal{K}$ is as in Fisher's example (before **15.18**), then the model-companion of $\mathcal{K}$ is the theory of $\mathbb{Q} \oplus \mathbb{Z}_2{}^{\aleph_0}$.

Problem 1 Investigate the situation for arbitrary universal classes of abelian groups.

Of course, a universal Horn class of abelian groups need not have a model-completion, since it need not have the amalgamation property. The example of Fisher quoted at the beginning of this section shows that the universal theory of $\mathbb{Q} \oplus \mathbb{Z}_2{}^{\aleph_0}$ does not have a model-completion.

There follows an example of a universal Horn class of modules with no model-companion. The ring involved is quite "large" from the point of view of finiteness conditions (cf. **15.38**).

Example 2 Let K be a field, and let R be the ring $K[x_i\ (i \in \omega) : x_i x_j = 0\ (i,j \in \omega)]$ (Ex 2.1/6(vii)). Consider the universal Horn class, $\mathcal{K}$, cogenerated by the (models of the) theory of R_R. Now (since $J \simeq (R/J)^{(\aleph_0)}$), $\mathcal{I}(\mathcal{K})$ consists of the two indecomposables: R_R and R/J. Observe that R/J embeds in R, so $\mathcal{U}$(Inj($\mathcal{K}$)) = $\{R\}$. On the other hand, $\{R\}$ is not a closed subset of $\mathcal{I}(\mathcal{K})$, for the types of the x_i $(i \in \omega)$ cluster to a type which says "$vJ = 0$" and "$x_i \nmid v$" for all i - that is, the type of $1 + J$ in R/J. So, by **15.26**, $\mathcal{K}$ does not have a model-companion.

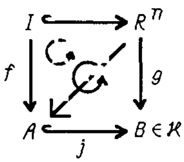
One may note that $\mathcal{K}$ does not have amalgamation: one may embed R/J into R by taking $1 + J$ to x_0 or to x_1, and then there is no way of amalgamating these embeddings in $\mathcal{K}$, since the theory of $R^{\aleph_0}$ says that the only element divisible by x_0 and x_1 is the zero element. One may also observe that every module in $\mathcal{K}$ is totally transcendental.

For contrast, consider the next example which, although not having amalgamation, does have a model-companion.

Example 3 Take K to be an infinite field and consider the ring $R = K[x_1, x_2 : x_i x_j = 0\ (i,j \in \{1,2\})]$ (Ex 2.1/6(vi)). Let $\mathcal{K}$ be the universal Horn class cogenerated by the theory of $R^{\aleph_0}$. As in the previous example, the space of indecomposables consists of R and R/J but now there are no non-isolated irreducible types in $\mathrm{Th}(R^{\aleph_0})$ (note that the Morley rank is 3), so the model-companion is just this theory. As in the previous example, amalgamation fails.

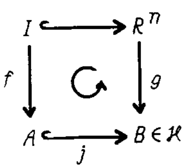
We have a criterion (15.26) for existence of a model-companion, in terms of the space $\mathcal{I}(\mathcal{K})$. The criterion given in 15.27 is that the class of absolutely pure modules be axiomatisable. In the case originally treated ([ES71]), it was this second result that was used, and a criterion in terms of the lattice of right ideals of the ring was found. In this original case, where $\mathcal{K}$ is the class of all R-modules, Eklof and Sabbagh showed that there is a model-companion (iff there is a model-completion) iff the ring R is right coherent.

Now, $\mathcal{M}_R$ has the amalgamation property. It has been seen that not every universal Horn class has this property, and it appears that a nice criterion for companionability, in terms of the lattice of right ideals (or a non-trivial one in terms of the lattice of pp-types) is not available (in [Pr81a; Erratta] I tried to obtain such a criterion, but it is not now clear to me whether the alleged criterion is useful or even correct). For that reason, when I develop a criterion for companionability in terms of the right ideal lattice of the ring, I will eventually need to impose the condition that $\mathcal{K}$ has amalgamation. The main point is, of course, to discover under what conditions the property of being absolutely pure in $\mathcal{K}$ is an elementary one.



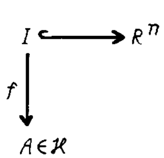
Lemma 15.31 (see [ES71; §3]) *Let $\mathcal{K}$ be a universal Horn class and let $A \in \mathcal{K}$. Then $A \in \mathrm{Abs}(\mathcal{K})$ iff for every diagram as shown, where I is a finitely generated submodule of R^n and $B \in \mathcal{K}$, there is a factorisation $R \longrightarrow A$ as shown.*

Proof Consider a diagram as shown.



Let $\bar{a}$ generate I, and let $\bar{e}$ be canonical generators for R^n. Then there is a matrix equation ($\bigwedge$-atomic formula) $\bar{a} = \bar{e}H$ for some matrix, H, over R. Therefore one has $jf\bar{a} = g\bar{e}.H$, and so B satisfies the pp formula $\exists \bar{w}\,(jf\bar{a} = \bar{w}H)$. Therefore, if A is absolutely pure in $\mathcal{K}$, then this is also satisfied in A: suppose that $\bar{c}$ in A witnesses $\bar{w}$. Then one may define a morphism $R^n \longrightarrow A$ by sending $\bar{e}$ to $\bar{c}$, and this is a factorisation of the required form.

For the converse, let $A \hookrightarrow B \in \mathcal{K}$ be an inclusion. Suppose that B satisfies the $\bigwedge$-atomic formula $\theta(\bar{a},\bar{b})$ where $\bar{a}$ is in A and $\bar{b}$ is in B. Since A is a submodule, it may be supposed that $\theta(\bar{a},\bar{b})$ has the form $\bigwedge_j t_j(\bar{b}) = a_j$, where the t_j are terms. A morphism is defined from R^n to B by sending canonical generators, $\bar{e}$, of R^n to $\bar{b}$ (n is the length of $\bar{b}$). Then $t_j(\bar{e})$ is sent to a_j. Let I be the submodule of R^n generated by the $t_j(\bar{e})$. Thus we obtain the outer square of a diagram as in the statment of the lemma. Let h be the morphism from R^n to A as shown. Then, for each j, one has $t_j(h\bar{e}) = a_j$ and so A satisfies $\theta(\bar{a},h\bar{e})$. Therefore $A \hookrightarrow B$ is pure, as required. ▫



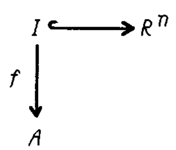
This is the key to axiomatising the property of being absolutely pure in $\mathcal{K}$, but one problem has yet to be disposed of: how does one tell whether a diagram such as that opposite may be completed by a member of $\mathcal{K}$? At last, amalgamation has to be brought into the picture.

Following §1, we say that a submodule M of the module M' is **$\mathcal{K}$-closed** in M' if M'/M is a member of $\mathcal{K}$. We particularly have in mind submodules of the free modules R^n. Since $\mathcal{K}$ is closed under products, there is an associated notion of **$\mathcal{K}$-closure**: $\mathrm{cl}_{\mathcal{K}}(M,M')$ is the least submodule of M' which contains M and is $\mathcal{K}$-closed in M'. So $M'/\mathrm{cl}_{\mathcal{K}}(M,M')$ is the "largest" quotient of M'/M which is in $\mathcal{K}$.

Let I be a (finitely generated) submodule of R^n: denote by $0_{\mathcal{K}}$ the $\mathcal{K}$-closure of the zero submodule in R^n, and consider $\mathrm{pk}\,I = I \cap 0_{\mathcal{K}}$. ("pk" for pushout kernel).

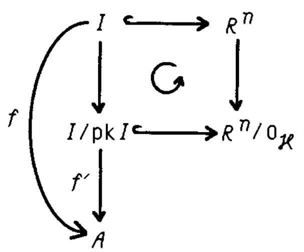
Note that $\mathrm{pk}\,I$ need not coincide with the $\mathcal{K}$-closure of 0 in I (that may be strictly smaller): consider the case where $\mathcal{K}$ is the class of abelian groups of exponent 2, and take I to be $2\mathbb{Z}$ - so $\mathrm{pk}\,2\mathbb{Z} = 2\mathbb{Z} > \mathrm{cl}_{\mathcal{K}}(0,2\mathbb{Z}) = 4\mathbb{Z}$ (the point is that the radical corresponding to $\mathcal{K}$ need not be left exact). Of course, $\mathrm{pk}\,I$ is $\mathcal{K}$-closed in I since $I/\mathrm{pk}\,I$ embeds in $R^n/0_{\mathcal{K}}$.

Observe that if a diagram of the sort above is completable in $\mathcal{K}$, then $\ker f \geq \mathrm{pk}\,I$. The converse is true if $\mathcal{K}$ has the amalgamation property and A is absolutely pure in $\mathcal{K}$.

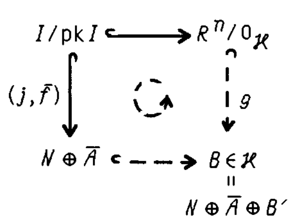


Proposition 15.32 [Pr81a; 2.5] *Let $\mathcal{K}$ be a universal Horn class of modules with the amalgamation property. Let* $A \in \mathrm{Abs}(\mathcal{K})$. *Consider the diagram opposite, where I is finitely generated. The diagram may be completed in $\mathcal{K}$ as shown iff* $\ker f \geq \mathrm{pk}\,I$.

Proof The direction "$\Rightarrow$" always holds.



For the converse, suppose that $\ker f \geq \mathrm{pk}\,I$. One has a factorisation as shown opposite.
Since $A \in \mathrm{Abs}(\mathcal{K})$, $\overline{A} \in \mathrm{Inj}(\mathcal{K})$ (15.18). Choose an embedding, j, of $I/\mathrm{pk}\,I$ into some $N \in \mathrm{Inj}(\mathcal{K})$. Consider the diagram below where $\bar{f}$ is the composition of f' and an embedding of A into its pure-injective hull $\overline{A}$.



The morphisms are embeddings so, by the amalgamation property, there is a completion in $\mathcal{K}$ as shown. Since (by 15.18) $N \oplus \overline{A}$ is injective in $\mathcal{K}$, there is a decomposition $B = N \oplus \overline{A} \oplus B'$ for some B'. Let $\bar{e}$ be the image of canonical generators of R^n in $R^n/0_{\mathcal{K}}$. Then there is a matrix H, such that $\bar{a} = \bar{e}H$ generates $I/\mathrm{pk}\,I$.
One has $g\bar{e}.H = g.\bar{e}H = (j,\bar{f}).\bar{a} = (j\bar{a}, \bar{f}\bar{a}, \bar{0}) \in N \oplus \overline{A} \oplus B'$. Set $g\bar{e} = (\bar{n}, \bar{a}', \bar{b}) \in N \oplus \overline{A} \oplus B'$: so $\bar{a}'H = \bar{f}\bar{a}$. Therefore, if $h: R^n/0_{\mathcal{K}} \longrightarrow \overline{A}$ is defined by taking $\bar{e}$ to $\bar{a}'$, then one has $h\bar{e}.H = \bar{f}\bar{a}$.
Composing this with the canonical projections in the first diagram, a diagram of the required sort is obtained. □

As in §8.3 and §14.1, we think of the elements of free modules being written as column vectors when appropriate.

How then, can we say that the kernel of a morphism contains $\mathrm{pk}\,I$? For in order to axiomatise $\mathrm{Abs}(\mathcal{K})$, it will be enough to be able to say that every such morphism has an extension to R^n. (The point is to make existence of a square as in the diagram a "local" property, at least for the modules which are absolutely pure in $\mathcal{K}$: without amalgamation, it is not clear that this can be done.)

Let $\mathcal{K}$ be a universal Horn class of modules: say that $\mathcal{K}$ is **coherent** if, for every finitely generated submodule I of any R^n, $I/\mathrm{pk}I$ is $\mathcal{K}$-**finitely presented** in the sense that if $K \hookrightarrow R^m \twoheadrightarrow I/\mathrm{pk}I$ is a presentation of $I/\mathrm{pk}I$, then K is $\mathcal{K}$-**finitely generated in** R^m in the sense that, for some finite set, $\bar{b}$, of elements of K, one has $K = \mathrm{cl}_{\mathcal{K}}(\bar{b}R, R^m)$ (note that it is not required that K be the $\mathcal{K}$-closure *in* K of some finite set).

So, for example, the class of all R-modules is coherent iff the ring R is right coherent in the usual sense.

<u>Theorem 15.33</u> *Let $\mathcal{K}$ be a universal Horn class of modules with amalgamation. If $\mathcal{K}$ is coherent then* $\mathrm{Abs}(\mathcal{K})$ *is an axiomatisable subclass of $\mathcal{K}$.*

Proof By **15.31** and **15.32**, in order to say that a module A is absolutely pure in $\mathcal{K}$, it is sufficient to say that, given any $n \geq 1$, any finitely generated submodule I of R^n and any morphism $f: I \longrightarrow A$ with $\ker f \geq \mathrm{pk}I$, then there is an extension of this morphism to one from R^n to A.

Let $R^m \twoheadrightarrow I$ be epi and set $\bar{a} = (a_1, \ldots, a_m)$ to be the image of the canonical generators $\bar{e}'$ of R^m. Let $\bar{a} = \bar{e}H$, where $\bar{e}$ is a canonical set of generators for R^n. Since $\mathcal{K}$ is coherent, if $K \hookrightarrow R^m \twoheadrightarrow I/\mathrm{pk}I$ is exact, where $R^m \twoheadrightarrow I/\mathrm{pk}I$ is induced from $R^m \twoheadrightarrow I$, then there are $c_1, \ldots, c_k$ in K such that $K = \mathrm{cl}_{\mathcal{K}}(\sum_i c_i R, R^m)$. Therefore, a morphism f from I to $A(\in\mathcal{K})$ with $\ker f \geq \mathrm{pk}I$ is just an m-tuple of elements of A which annihilates the matrix C, where C has for its j-th column the element c_j, written as an $m\times 1$ vector. So our axiom corresponding to this choice of I and f should begin: "$\forall \bar{v}(\bar{v}C = 0 \rightarrow \ldots)$".

Then one must express that there is a lifting of f to some $g: R^n \longrightarrow A$ with $g\bar{a} = f\bar{a}$ - that is, with $f\bar{a} = g(\bar{e}H) = g\bar{e}.H$. In other words, it must be said that $f\bar{a}$ ("$\bar{v}$") is "divisible" by H. Therefore the required axiom is: "$\forall \bar{v}(\bar{v}C = 0 \rightarrow \exists \bar{w}(\bar{v} = \bar{w}H))$".

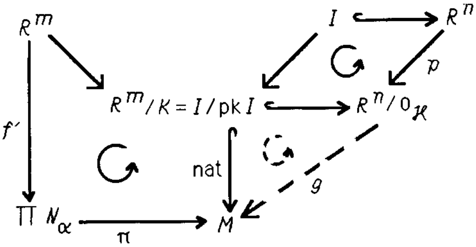
The set of all these (take any $n \geq 1$, any $n\times m$ matrix H, and let C be a matrix whose columns $\mathcal{K}$-generate, in R^m, the left-hand term in the presentation of that submodule of R^n which is generated by the columns of H) axiomatises $\mathrm{Abs}(\mathcal{K})$. ◻

In fact the converse also holds, and so one obtains the following criterion.

<u>Theorem 15.34</u> cf. [Wh76] *Let $\mathcal{K}$ be a universal Horn class of modules with amalgamation. Then $\mathcal{K}$ has a model-companion iff $\mathcal{K}$ is a coherent class.*

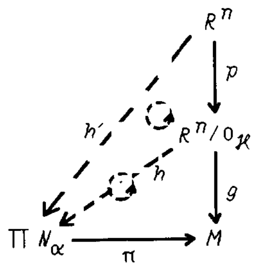
Proof [ES71; 3.12] By **15.27** and **15.33**, it remains to show that if $\mathcal{K}$ is not coherent then $\mathrm{Abs}(\mathcal{K})$ is not an elementary class. So suppose that $\mathcal{K}$ is not coherent. Then there is some finitely generated $I \leq R^n$ and exact sequence $K \hookrightarrow R^m \twoheadrightarrow I/I \cap 0_{\mathcal{K}}$, with K not $\mathcal{K}$-finitely generated in R^m. Suppose that K is the $\mathcal{K}$-closure in R^m of the (module generated by the) set $\bar{b} = (b_\alpha)_{\alpha<\kappa}$: $K = \mathrm{cl}_{\mathcal{K}}(\bar{b}, R^m)$, where the cardinality, κ, of the "generating set" has been minimised. Our assumption is that $\kappa \geq \aleph_0$.

For each $\alpha<\kappa$ set $K_\alpha = \mathrm{cl}_{\mathcal{K}}((b_\beta)_{\beta<\alpha}, R^m)$. The definitions plus minimality of κ imply $K_\alpha < K$ and also that the sequence of K_α's is eventually increasing, with limit K. For each α, choose $N_\alpha \in \mathrm{Inj}(\mathcal{K})$ containing R^m/K_α. Let $\mathfrak{D}$ be any uniform ultrafilter on κ, and let M be the ultraproduct $\prod\{N_\alpha : \alpha<\kappa\}/\mathfrak{D}$: observe that the kernel of the natural projection $\pi: \prod N_\alpha \twoheadrightarrow M$ is $\varinjlim_\alpha \prod\{N_\beta : \beta<\alpha\}$. If $\mathrm{Abs}(\mathcal{K})$ were axiomatisable, then M would be absolutely pure in $\mathcal{K}$: we shall see that this is not the case.



Consider the composite of natural morphisms $f: R^m \twoheadrightarrow \prod_{\alpha<\kappa} R^m/K_\alpha \hookrightarrow \prod_{\alpha<\kappa} N_\alpha \twoheadrightarrow M$. Also, denote by f' the composite of the first two morphisms: so $f = \pi f'$. If $c \in K$ then there is $\alpha<\kappa$ with $c \in K_\alpha$, so c is in the kernel of the final projection π, hence is in $\ker f$. The converse is equally clear; so $\ker f = K$.

In particular, $I/\mathrm{pk}I \simeq R^m/K$ embeds naturally in M. Since $\mathcal{K}$ has amalgamation, if $M \in \mathrm{Abs}(\mathcal{K})$ then there is, by **15.31** and **15.32**, a lifting $g: R^n/0_{\mathcal{K}} \longrightarrow M$ as shown.



By projectivity of R^n, gp lifts to a morphism $h': R^n \longrightarrow \prod N_\alpha$ such that $gp\bar{e} = \pi h'\bar{e}$, where $\bar{e}$ is a canonical generator of R^n and $p: R^n \longrightarrow R^n/0_{\mathcal{K}}$ is the canonical projection. Since $\prod N_\alpha \in \mathcal{K}$, there is a factorisation $h' = hp$ as shown. So $\pi h = g$.
Let $\bar{e}'$ be a canonical generator for R^m, $\bar{a}$ its image in $I/\mathrm{pk}I$, $\bar{d} = hp\bar{e}$ and suppose that the matrix H is such that $\bar{a} = p\bar{e}.H$. Then $\pi\bar{d}.H = \pi h.p\bar{e}.H = \pi h\bar{a} = g\bar{a} = \pi f'\bar{e}'$. So $\bar{d}H - f'\bar{e}' \in \ker\pi = \varinjlim_\alpha \prod\{N_\beta : \beta<\alpha\}$. Hence $\bar{d}H - f'\bar{e}'$ lies in $\prod\{N_\beta : \beta<\alpha\}$ for some $\alpha<\kappa$. Let $\gamma \geqslant \alpha$: then one has $(\bar{d}H)_\gamma = (f\bar{e}')_\gamma$, where the subscript "$\gamma$" denotes the γ-th coordinate in $\prod_{\beta<\alpha} N_\beta$.

Now, $b_\gamma = \bar{e}'X_\gamma$ for some matrix X_γ; so $\bar{a}X_\gamma$, being the image of $\bar{e}'X_\gamma$ in R^m/K, is zero. Hence $\bar{e}H.X_\gamma = 0$. Consider the α-th coordinate of $f'b_\gamma$: $(f'b_\gamma)_\alpha = (f'\bar{e}'X_\gamma)_\alpha = (f'\bar{e}')_\alpha X_\gamma = (\bar{d}H)_\alpha X_\gamma = (hp\bar{e}H)_\alpha X_\gamma = (h.\bar{e}HX_\gamma)_\alpha = (h0)_\alpha = 0$. So, for all $\gamma \geqslant \alpha$, the α-th coordinate of $f'b_\gamma$ is zero - that is, the image of b_γ in R^m/K_α is 0. In other words, $b_\gamma \in \Sigma\{b_\beta R : \beta<\alpha\}$ - contradicting the choice of the b_γ's. □

Working with more general structures, Wheeler obtains a result very similar to this. His result does not immediately apply to give **15.34**, since his definition of "coherent" is apparently stronger than mine, but I would find it surprising if his result could not be made to yield **15.34** as a corollary.

The first corollary is the global case, $\mathcal{K} = \mathcal{M}_R$, considered by Elkof and Sabbagh [ES71]. The third is the relativisation to torsionfree classes cogenerated by injective modules, as considered in [Pr78]. The third corollary is immediate since coherence of R guarantees that $\mathcal{K}$ is coherent.

<u>Corollary 15.35</u> [ES71; 4.1, 4.8] *Let R be any ring. Then the class of right R-modules has a model-companion iff it has a model-completion iff R is right coherent.* □

Example 4 The commutative local ring $R = K[X_i (i \in \omega)]/\langle X_i : i \in \omega\rangle^2$ (cf. Ex 2.1/6(vii)) is not coherent: the sole simple module $S = R/J$ is not finitely presented, yet embeds in R: indeed R is perfect (has the dcc on finitely generated right ideals) so (**14.B**) every non-zero module has a simple submodule - necessarily isomorphic to S. Therefore the only indecomposable injective is the injective hull, $E(S)$, of S: indeed, every injective module is the injective hull of a direct sum of copies of $E(S)$.

Since R is not coherent, 15.33 implies that the class of absolutely pure modules is not axiomatisable. Let us find a module which is elementarily equivalent to an injective module but is not absolutely pure.

Let a be a generator for S. Since it is consistent that a is divisible by x_i (for $ax_i = 0$), there is an element b_i of the injective hull of S such that $b_i x_i = a$. Since $x_i x_j = 0$ and R is commutative, it must be that $b_i x_j = 0$ for $j \neq i$. Therefore, in $E(S)$, the chain of pp-definable subgroups $\mathrm{ann}\, x_0 > \mathrm{ann}\{x_0, x_1\} > \ldots$ is strictly descending. Thus the type which says "$vx_i = 0$ $(i \in \omega)$ and $v \neq 0$" is consistent with the theory of $E(S)$ so is realised, by c say, in some elementary extension of $E(S)$. Now c cannot be divisible by any non-invertible element of R (since $J^2 = 0$) and so the hull of c is simply a copy of S - which is certainly not absolutely pure (so neither is the elementary extension of $E(S)$), as required.

Corollary 15.36 *Let R be right coherent. Then the existentially complete R-modules are precisely the absolutely pure modules, M, which satisfy $M \equiv M^{\aleph_0}$ and which are "fat" in the sense that $\mathcal{U}(M)$ is a dense subset of $\mathcal{U}(\mathrm{Inj})$ (the closed subset of $\mathcal{I}_R$ consisting of all indecomposable injectives). If R is actually right noetherian, then all these modules are injective.* □

Corollary 15.37 [Pr78; 2.13] *Let E be any injective R-module such that the (hereditary) torsionfree class $\mathcal{F}_E$ cogenerated by E is axiomatisable (i.e., 15.9, is of finite type). Then $\mathcal{F}_E$ has a model-completion iff R is E-coherent, in the sense that every finitely generated submodule of a free module has the form R^m/K, where K has a finitely generated submodule which is E-dense in it.* □

Corollary 15.38 *Let R be a right coherent ring and let $\mathcal{K}$ be any universal Horn class of R-modules with the amalgamation property. Then $\mathcal{K}$ has a model-completion.* □

In particular, this applies if R is right noetherian or von Neumann regular.

Corollary 15.39 *Let $\mathcal{K}$ be a universal Horn class of modules, every member of which is totally transcendental. Suppose also that $\mathcal{K}$ has amalgamation. Then $\mathcal{K}$ has a model-completion.*

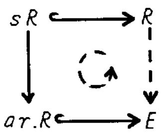
Proof One checks the condition for $\mathcal{K}$ to be coherent: so let $I \simeq R^m/K$ be a finitely generated submodule of some R^n. If there were an infinite increasing sequence $\mathrm{cl}_{\mathcal{K}}(0, R^m) < \mathrm{cl}_{\mathcal{K}}(a_1, R^m) < \mathrm{cl}_{\mathcal{K}}(\{a_1, a_2\}, R^m) < \ldots$ of $\mathcal{K}$-closed submodules of R^m (contained in K) then one would obtain, in a suitably large member, M, of $\mathcal{K}$, a strictly decreasing sequence of pp-definable subgroups: $M > \mathrm{ann}_M(a_1) > \mathrm{ann}_M(a_1, a_2) > \ldots$, and this would contradict the t.t. condition on M (by 3.1). □

Compare this with the examples after 15.30.

Another possible approach to some of the results above would be to use the fact that the model-completion, if it exists, has elimination of quantifiers (see 16.2).

There is a direct derivation of the result of Eklof and Sabbagh with the following short proof (refer to §16.1 for any points which are not obvious).

First, following [Gar80a; Thm 5] and [Zg84; 5.8], we see that, in injective modules, pp formulas are equivalent to annihilator conditions. Consider a special case.



Suppose that $\varphi(v)$ is the pp formula $\exists w\,(vr = ws)$. Set $A = r\text{-}\mathrm{ann}_R\, s = \{t \in R : st = 0\}$. Let E be any injective module. The claim is that, for any $a \in E$, $E \models \varphi(a)$ iff $a.rA = 0$. The implication "$\Rightarrow$" is easy since, if $ar = bs$ and $t \in A$, then $art = bst = 0$. For the converse, assume that $ar.A = 0$. Then the morphism from sR to $ar.R$, defined by sending s to ar is well-defined. By injectivity of E, the diagram may be completed as shown.
Then the image of $1 \in R$ is a witness for "w" and so $E \models \varphi(a)$.

The proof just given adapts immediately to the general case, and one obtains the following result.

Proposition 15.40 [Gar80a; Thm 5], [Zg84; 5.8] (also cf. [ES71; 3.2]) *Let $\varphi(\bar{v})$ be a pp formula. Then there is a submodule, I, of R^n such that, for any injective module E and any $\bar{a} \in E$, one has $E \models \varphi(\bar{a})$ iff $\bar{a}I = 0$.*
If $\varphi(\bar{v})$ is written in the form $\exists \bar{w}\,(\bar{v}U = \bar{w}S)$, where U is an $n \times k$ matrix (so $l(\bar{v}) = n$) and S is an $m \times k$ matrix over R, then the submodule I is $U.\mathrm{ann}\,S$.

Proof Recall our convention of treating tuples of elements of modules as row vectors and elements of free modules (in their role as "operators") as column vectors. Let A be the right annihilator of S "in R": $A = r\text{-}\mathrm{ann}\,S = \{\bar{r} \in R^k : S\bar{r} = 0\}$. The claim is that $E \models \varphi(\bar{a})$ iff $\bar{a}.UA = 0$, where UA is (well-)defined to be the set of all $U\bar{r}$ with $\bar{r} \in A$: so UA is a submodule of R^n.

The proof is just as in the special case. One replaces "sR" there by the submodule SR of R^m generated by the (k) columns of S; $\bar{a}U.R$ is the submodule of E generated by the (k) elements $\bar{a}.\kappa_i(U)$, where $\kappa_i(U)$ is the i-th column of U, and the morphism from SR to $\bar{a}UR$ is defined by taking the i-th column of S to $\bar{a}.\kappa_i(U)$ (this is well-defined if $\bar{a}U.\mathrm{r\text{-}ann}\,S = 0$). □

Denote $U.\mathrm{r\text{-}ann}\,S$, as above, by I_φ.

Exercise 1 ([Zg84; 5.8]) Show that I_φ is pp-definable in the left module ${}_RR^n$: show that every pp-definable submodule of ${}_RR^n$ is of this form [Hint: the first point is easy and the second involves only writing pp formulas in an appropriate way].

Exercise 2 ([Zg84; 5.8]) Show that $I_{\varphi \wedge \psi} = I_\varphi + I_\psi$ and $I_{\varphi + \psi} = I_\varphi \cap I_\psi$.

So, in any injective module, every pp formula reduces to a quantifier-free condition. This does not mean that every injective module has complete elimination of quantifiers (indeed that is not so) - for there is no reason to suppose that the submodule I_φ is finitely generated. In fact, the pp formula φ is equivalent, in every injective module, to a quantifier-free formula iff the submodule I_φ is finitely generated (to see this, use that there exists an injective module into which every cyclic module embeds).

Recall that the ring R is right coherent iff, whenever I and J are finitely generated right ideals then so is $I \cap J$ and, also, each right ideal of the form $\mathrm{r\text{-}ann}\,s$ $(s \in R)$ is finitely generated. It is not difficult to see that these conditions are equivalent to: whenever I and J are finitely generated submodules of R^n $(n \in \omega)$, then so is $I \cap J$ and, for every n-tuple $\bar{s}$ of elements of R, the right annihilator of $\bar{s}$ is a finitely generated submodule of R^n.

Therefore, if R is right coherent, $I_\varphi = U.\mathrm{r\text{-}ann}\,S$ is finitely generated. For $\mathrm{r\text{-}ann}\,S$ is the (finite) intersection of the $\mathrm{r\text{-}ann}\,\rho_j(S)$, where $\rho_j(S)$ is the j-th row of S, and so it is finitely generated.

Theorem 15.41 [ES71; §§3,4] *Suppose that R is right coherent. Then every injective module has complete elimination of quantifiers (indeed has* elim-Q$^+$ *- by*

16.5). *Hence every module elementarily equivalent to an injective one is absolutely pure.*

Proof The first statement has just been justified. For the second, suppose that M is elementarily equivalent to an injective module E. Let $\bar{a}$ be in M. Then $M \oplus E(\bar{a}'R)$ is elementarily equivalent to the injective module $E \oplus E(\bar{a}'R)$, where $\bar{a}'$ has been taken so that $\bar{a}'R \simeq \bar{a}R$. Since $\bar{a}$ and $\bar{a}'$ have the same quantifier-free type and $E \oplus E(\bar{a}'R)$ has elimination of quantifiers, they have the same type. Thus $\bar{a}$ has the maximal pp-type consistent with its quantifier-free type and so every embedding of M into a module preserves its pp-type. That is, M is absolutely pure. □

Now suppose, conversely, that R is such that every injective module has complete elimination of quantifiers. Let E^* be an injective module which embeds every cyclic module. Take $s \in R$, and consider the pp formula $\exists w\,(v = ws)$. By **15.40**, there is a right ideal I of R such that, for every $a \in E$, one has $E \models \varphi(a)$ iff $aI = 0$: also by **15.40**, $I = \text{r-ann}\, s$. By assumption (and 16.5(a)) there is a finitely generated right ideal J such that, for every $a \in E$, one has $E \models \varphi(a)$ iff $aJ = 0$. Since every cyclic module embeds in E, it follows that r-ann $s = I = J$ is finitely generated.

Also, take any two finitely generated right ideals I and J. Since these are finitely generated, "$vI = 0$" and "$vJ = 0$" are pp formulas. In E^*, the sum of the subgroups that they define is defined by the pp formula "$v(I \cap J) = 0$". By an argument like that in the previous paragraph, one concludes that $I \cap J$ is finitely generated. Thus the conditions for R to be coherent are satisfied, so one has the following.

Theorem 15.42 cf. [ES71; §4] *The ring R is right coherent iff every injective module has complete elimination of quantifiers.* □

The result, **15.35**, of Eklof and Sabbagh, now follows easily. For, if a module is existentially complete in $\mathcal{M}_R$, it must be absolutely pure: then apply **15.41** and **15.27** (the latter result, in this special case, may be given a short proof) observing that, if there is an injective module without elimination of quantifiers, then the class of absolutely pure modules cannot be axiomatisable.

15.4 Elementary localisation

The main result in this section is the characterisation of when the modules resulting from a localisation form an elementary subclass of the original category of modules. Throughout the section, τ is a left exact torsion radical, so $\langle \mathcal{T}_\tau, \mathcal{F}_\tau \rangle$ is a hereditary torsion theory (cf. §2). There is a localisation functor, Q_τ, which takes a module M first to its largest torsionfree quotient $M/\tau M$, and then takes this to its τ-injective hull $E_\tau(M/\tau M)$. Given a torsionfree module F, its **τ-injective hull**, $E_\tau(F)$, is the module defined by the condition $E_\tau(F)/F = \tau(E(F)/F)$. A module is **$\tau$-injective** if it is injective over every embedding of a τ-dense right ideal into R, and $E_\tau(F)$ is the smallest τ-injective module containing F. One denotes by $\mathcal{M}_R{}^\tau$ the image of this functor: it is the full subcategory of $\mathcal{M}_R$ whose objects are the τ-torsionfree τ-injective modules, but it has a further structure. The module $Q_\tau(R)$ actually has the structure of a ring, and every module of the form $Q_\tau(M)$ has a natural $Q_\tau(R)$-module structure, which extends its structure as an R-module. In many "classical" cases the "localised" category $\mathcal{M}_R{}^\tau$ is equivalent to the full category of modules over $Q_\tau(R)$: but this is not invariably so (see the Gabriel-Popescu Theorem below).

The work described in this section has (like the previous one) its origins in the paper [ES71] of Eklof and Sabbagh. The reader is directed to the references cited for further details and results.

Example 1 Let R be a commutative domain, and let τ be the usual torsion radical (see Ex15.1/1): so the process $M \mapsto M/\tau M$ factors out all torsion elements of M. Given a torsionfree module M and a non-zero element, e, of its injective hull $E(M)$, one has $eR \cap M \neq 0$: thus there is a non-zero element r of R, such that r annihilates the image of e in $E(M)/M$ and so, by definition, $E_\tau(M)=E(M)$. The elements of $E(M)$, where M is τ-torsionfree, may be seen to be "fractions" mr^{-1} (under the usual equivalence relation), where $m \in M$ and r is a non-zero element R. In particular, $E_\tau(R)$ is isomorphic to the usual field of quotients, K, of R.

It follows that the localised category, $\mathcal{M}_R{}^\tau$, is equivalent to the category of vectorspaces over K.

For more on this see [St75] (Chpt.II for the classical case of modules of fractions).

One may ask, therefore, when $\mathcal{M}_R{}^\tau$ is an axiomatisable subclass of $\mathcal{M}_R$. First we see that if $\mathcal{M}_R{}^\tau$ is elementary then τ is of finite type (equivalently, by 15.9, $\mathcal{F}_\tau$ is axiomatisable). For if $\mathcal{F}_\tau$ is not axiomatisable, then, since it is closed under submodules, there is some ultraproduct $M = \prod_i M_i/\mathcal{D}$ of torsionfree modules, such that M is not torsionfree. Each M_i may be embedded in a member, F_i, of $\mathcal{M}_R{}^\tau$. Consider the corresponding ultraproduct $F = \prod_i F_i/\mathcal{D}$: this embeds M and so is not itself torsionfree, so certainly is not in $\mathcal{M}_R{}^\tau$. Thus $\mathcal{M}_R{}^\tau$ is not an elementary class.

Say that the torsion radical τ on $\mathcal{M}_R$ is **elementary** if it is of finite type and if every finitely generated τ-dense right ideal I of R is **τ-finitely presented**, in the sense that if $K \hookrightarrow R^m \twoheadrightarrow I$ is a presentation of I, then K has a τ-dense finitely generated submodule (because τ is left exact, the definition is less subtle than that given in §15.3 for a universal Horn class to be coherent and, indeed, this is strictly weaker than the condition for $\mathcal{F}_\tau$ to be a coherent class).

<u>Theorem 15.43</u> [Pr78; 2.20] *Let τ be a left exact torsion radical on $\mathcal{M}_R$. Then the category, $\mathcal{M}_R{}^\tau$, of localised modules is an elementary subclass of $\mathcal{M}_R$ iff τ is elementary.*

Proof I outline the proof (see the comments later). For the direction "$\Leftarrow$", one notes that if a τ-dense right ideal I, finitely generated by $\bar{a}$ say, is τ-finitely presented, then one may write down a formula, $\psi_I(\bar{v})$, such that, if M is torsionfree and if $\bar{c}$ is a sequence matching $\bar{a}$, then there is a morphism taking $\bar{a}$ to $\bar{c}$ iff M satisfies $\psi_I(\bar{c})$. Then one proceeds as in the proof of 15.33.

For the converse, one produces an ultraproduct more or less as in the proof of 15.34. □

<u>Corollary 15.44</u> *Suppose that the ring R is right coherent, and let τ be any left exact torsion radical on $\mathcal{M}_R$. Then the localised class, $\mathcal{M}_R{}^\tau$, of modules is an elementary subclass of $\mathcal{M}_R$.* □

Example 2 The class of $\mathbb{Q}$-vectorspaces is an elementary subclass of the class of abelian groups ($\mathbb{Z}$-modules). Thus 15.43 relates to the question, attributed by Poizat to Sabbagh: for which rings R is the category of R-modules axiomatisable in a finite language?

If one compares 15.34 (for the case $\mathcal{K}=\text{cog}E$, where E is injective) and 15.43 then it should come as no surprise that they are corollaries of a more general theorem. I now state that theorem (I did not give it in the first place since it looks less interesting than its corollaries). Let τ' be a left exact radical on $\mathcal{M}_R$: a module M is **$(\aleph_0, \tau')$-injective** if it is injective over all embeddings of a finitely generated τ'-dense right ideal into R; it is **τ'-injective** if it is injective over all embeddings of a τ'-dense right ideal into R.

Theorem 15.45 [Pr78; 2.5], also see [Kom80; Thm 2] *Let* τ *and* τ' *be left exact radicals on* $\mathcal{M}_R$. *Then the following conditions are equivalent:*
(*i*) *the class of* $(\aleph_0,\tau')$*-injectives in* $\mathcal{F}_\tau$ *is elementary;*
(*ii*) *every finitely generated* τ'*-dense right ideal in* R *is* τ*-finitely presented.* □

Actually, this theorem is proved in [Pr78a] in rather more generality: the generating set $\{R\}$ for $\mathcal{M}_R$ may be replaced by, for example, the set of all finitely generated free modules, and definitions are modified accordingly.

In order to derive 15.34 for the case of a hereditary torsionfree class, take τ' to be the identity functor on $\mathcal{M}_R$ (so every right ideal is τ'-dense). One may note that the resulting notion of $(\aleph_0,\tau')$-injective ("$\aleph_0$-injective") is apparently weaker than absolutely pure, but one may work instead with this weaker concept (alternatively, take the generating set for $\mathcal{M}_R$ to be the set of finitely generated free modules: then $(\aleph_0,\tau')$-injective becomes precisely "absolutely pure").

In order to derive 15.43, one puts $\tau=\tau'$ and notes that, if τ is of finite type, then the $(\aleph_0,\tau)$-injectives in $\mathcal{F}_\tau$ are actually τ-injective.

It does not seem unlikely that there is a generalisation of 15.45 which covers the case of an arbitrary universal Horn class of modules with amalgamation.

I end this section and chapter with a discussion of logic in locally finitely presented Grothendieck categories and an application of 15.43.

From now on, let $\mathcal{C}$ denote a Grothendieck abelian category. An object C of $\mathcal{C}$ is **finitely generated** if it cannot be expressed as a directed sum of proper subobjects. A finitely generated object C is **finitely presented** if, whenever $0\longrightarrow K\longrightarrow B\longrightarrow C\longrightarrow 0$ is an exact sequence in $\mathcal{C}$ with B finitely generated, then K is finitely generated. The category $\mathcal{C}$ is **locally finitely presented** if it has a generating set, $\mathcal{G}$, of finitely presented objects (so, for every object C of $\mathcal{C}$, there is an epi from a direct sum of objects of $\mathcal{G}$ to C).

Examples of such categories are Module categories: those of the form $\mathcal{M}_{\mathcal{S}}=(\mathcal{S}^{op},\mathbf{Ab})$ where $\mathcal{S}$ is a small pre-additive category, such as a ring.

In such a category, $\mathcal{C}$, one can develop first-order logic in a natural way (see [Pr78a]). In particular, one may talk about axiomatisable subclasses of such categories. Fix a generating set $\mathcal{G}$ of finitely presented objects. For an object C of $\mathcal{C}$ and a "generator" $G\in\mathcal{G}$, an **element** of C of sort G is a morphism from G to C. For example, an element of a module M is "really just" a morphism from R to M (the "element" is the image of $1\in R$).

One key to the development of first-order logic in such a context is the fact that a finitely generated object is finitely presented iff it is projective over all epis of the form $\prod_i C_i \twoheadrightarrow \prod_i C_i/\mathcal{D}$, where $\mathcal{D}$ is a filter on the index set I and the right-hand side is a reduced product.

We also need a generalisation of the Gabriel-Popescu Theorem ([St75; X 4.1]), which says that if $\mathcal{C}$ is a Grothendieck abelian category then there is a ring, R, and a left exact radical, τ on $\mathcal{M}_R$, such that $\mathcal{C}$ is equivalent to the localised category $\mathcal{M}_R{}^{\tau}$. The generalisation starts with a Grothendieck abelian category with generating set $\mathcal{G}$: the result (see [Pr80; 1.1]) is that there is a left exact radical, τ on $\mathcal{M}_{\mathcal{G}}=(\mathcal{G}^{op},\mathbf{Ab})$, such that $\mathcal{C}$ is equivalent to $\mathcal{M}_{\mathcal{G}}{}^{\tau}$.

One may ask what connections exist between finiteness conditions on $\mathcal{C}$ and finiteness conditions on τ, where τ is such that $\mathcal{C}\simeq\mathcal{M}_{\mathcal{G}}{}^{\tau}$. One would expect a more natural relationship than that one obtains by representing $\mathcal{C}$ as $\mathcal{M}_R{}^{\tau'}$, unless $\mathcal{C}$ happens to be a module category.

It turns out that $\mathcal{C}$ is locally finitely presented iff $\mathcal{C}$ is an "elementary" localisation of a Module category (the definition of "elementary" in this context is the natural generalisation of that given above). The exact statement is below: it is independent of the generating set chosen for $\mathcal{C}$, so long as one confines oneself to generating sets of finitely presented objects. A "Giraud subcategory" of a Grothendieck abelian category, $\mathcal{C}$, is one which has the form $\mathcal{C}^{\tau}$ for some left exact radical τ on $\mathcal{C}$: there are various characterisations of such subcategories [St75; ChptX].

<u>Theorem 15.46</u> [Pr80; 2.3] *Let $\mathcal{C}$ be a Grothendieck abelian category. Then the following conditions are equivalent:*

(i) *$\mathcal{C}$ is locally finitely presented;*

(ii) *$\mathcal{C} \simeq \mathcal{M}_{\mathcal{G}}{}^{\tau}$ for some generating set , $\mathcal{G}$, of finitely presented objects and some elementary left exact radical, τ, on $\mathcal{M}_{\mathcal{G}}$;*

(iii) *$\mathcal{C}$ is an axiomatisable Giraud subcategory of a Module category.* □

CHAPTER 16 ELIMINATION OF QUANTIFIERS

A complete theory of modules has (complete) elimination of quantifiers if every definable subset of a model may be defined without the use of quantifiers. Thus, in a module with elimination of quantifiers, the (pp-)type of every element is determined by its annihilator - so our study comes close to being "purely algebraic".

A good deal of initial work in the model theory of modules was done in a context where one has elimination of quantifiers, and that comparatively "algebraic" case has proved to be quite a reliable guide as to what to expect when we do have to take account of quantifiers. The first section begins by delineating some of the consequences of elimination of quantifiers. One soon discovers that elimination of quantifiers is just a little weaker than one would like: so we introduce and work with the stronger condition (denoted elim-Q^+) that every pp formula is actually a conjunction of atomic formulas (elimination of quantifiers guarantees only that it is a boolean combination of atomic formulas). In any theory with elim-Q^+ the indecomposable pure-injectives are "small" in the sense that they are uniform (any two non-zero submodules intersect non-trivially).

A ring is regular iff all of its modules have complete elimination of quantifiers. That is proved in the second section, and the consequences are pursued. The usual spectrum of algebraic finiteness conditions is much shortened for regular rings: we see that the same goes for our model-theoretic finiteness conditions. All our work on regular rings is aided by the fact that there is a simple canonical form for the invariants of §2.4.

In the latter part of the section, attention is restricted to commutative regular rings. It is shown that the largest complete theory of modules has m-dimension iff it has width iff its boolean algebra of idempotents is superatomic. It is also seen that this largest theory has a prime model iff the boolean algebra of idempotents is atomic.

In a final section, I briefly discuss continuous pure-injectives.

16.1 Complete elimination of quantifiers and its consequences

The complete theory T is said to have **(complete) elimination of quantifiers**, or **elim-Q** for short, if every formula is equivalent, modulo T, to a boolean combination of atomic formulas: that is, if for every formula $\chi(\bar{v})$ there is a boolean combination, $\theta(\bar{v})$, of atomic formulas such that the sentence $\forall\bar{v}(\chi(\bar{v})\leftrightarrow\theta(\bar{v}))$ is in T. One may say that if a theory has complete elimination of quantifiers then the study of its models becomes relatively algebraic, since every definable set may be defined without using quantifiers. The following are two useful criteria for a complete theory to have elim-Q.

<u>Theorem 16.1</u> (see [Poi85; §5.c]) *Let T be a complete theory. Each of the following conditions is necessary and sufficient for T to have complete elimination of quantifiers.*

(*a*) *If A is a substructure of a model of T and if $f:A\hookrightarrow M$ and $g:A\hookrightarrow M$ are two embeddings of A into a $|A|^+$-saturated model, M, of T, then there is an automorphism of M taking fA isomorphically to gA.*

(*b*) *If $\bar{a}$ and $\bar{b}$ are (finite) tuples of elements in a model of T such that $\bar{a}$ and $\bar{b}$ have the same quantifier-free type, then $\bar{a}$ and $\bar{b}$ have the same type.* □

It follows from (b) that every type p is equivalent to the set, p', of quantifier-free formulas in it. Therefore, the map which takes a pp-type p to the set p_0 of atomic formulas in it, is 1-1 (for p may be recovered from $p_0\wedge\neg p_0^-$). So we obtain the following.

Lemma 16.2 *Suppose that* T *is a complete theory of modules which has* elim-Q. *Then the map* $\pi : \mathrm{P}^T \longrightarrow \mathrm{Latt}(R)$, *which takes a pp-type* p *to its "intersection" with the ring, is one-one.* □

As in Chapter 15, we say that a right ideal I is T-**closed** if there is an element of a model of T with annihilator equal to I. So, **16.2** says that, if T has Elim-Q, then the map $p \longmapsto p \cap R$ defines an isomorphism from the poset of pp-types for T to the poset of T-closed right ideals of R (if closed under intersection, these are lattices). We say that the T-closed right ideal I is T-**finitely generated** if there is a finitely generated right ideal I_0 contained in it such that there is no T-closed right ideal I' with $I_0 \leqslant I' < I$.

Then the map $p \longmapsto p \cap R$ takes T-finitely generated pp-types to T-finitely generated right ideals. For, suppose $p \in \mathrm{P}^T$ and φ is pp with $T \vdash p \leftrightarrow \varphi$. By elimination of quantifiers, φ is equivalent to a formula of the sort $\bigvee_k (\psi_k \wedge \bigwedge_i \neg\theta_{ik})$ $(*)$, where the ψ_k and θ_{ik} are $\bigwedge$-atomic and where we may suppose that $\theta_{ik} \to \psi_k$ for all i,k. If ψ_k is $\bigwedge_j v r_j = 0$, let I_k be the right ideal $\Sigma_j r_j R$: so, if $a \in M \models T$, then $M \models \psi_k(a)$ iff $aI_k = 0$. Similarly, define J_{ik} corresponding to θ_{ik}.

Now. let a be an element with pp-type p in some model of T. Then $\varphi(a)$ holds, so we have, for some k, $aI_k = 0$ and $aJ_{ik} \neq 0$ (for all i). Let $I = \mathrm{ann}\, a = $ "p"$\cap R$. Let I' be a T-closed right ideal with $I_k \leqslant I' \leqslant I$ and let b be an element with annihilator I' in some model of T. Then $bI_k = 0$: also, since $aJ_{ik} \neq 0$, certainly $bJ_{ik} \neq 0$. Therefore, by $(*)$, b satisfies φ, equivalently, p. Therefore, since $aI = 0$ we have $bI = 0$ and so $I' = I$. Thus I is T-finitely generated, as claimed.

Examples 1 I now give some examples of theories of modules with elim-Q.

(i) Suppose that R is a right noetherian ring, let E be an injective module over R and let T be its theory. Then the criterion of **16.1** is satisfied since every model of T is injective (recall that E is t.t.), so partial isomorphisms extend. A case of particular interest is obtained when E is an existentially complete model of the theory of R-modules. This example was considered in detail by Eklof and Sabbagh [ES71]: some of their results were relativised and extended to Σ-injective modules in [Pr78, 79a, 82]. In [Bou79], Bouscaren considered forking and ranks (for right coherent rings) in terms of the right ideals corresponding to pp-types. Most of what she proved in that special case is a consequence of **16.2**, so holds for any theory $T = T^{\aleph_0}$ with elim-Q (and provides a model for what happens in the non-elim-Q case, since $\mathrm{P}(R)$ generalises $\mathrm{Latt}(R)$).

(ii) Let R be a (von Neumann) regular ring. Then every theory of modules over R has elim-Q: indeed this property characterises these rings. This follows easily from the results of §15.3 (see especially **15.38**), and is given another proof below (**16.16**).
In consequence, model theory and algebra over such rings "coincide": see §2 for more on this case.

(iii) Let T be a complete theory of abelian groups. By **16.1**(b), T has elim-Q iff whenever two elements of a model have the same order then they have the same type (and so, in particular, must be divisible by precisely the same prime powers). First note that $\mathbb{Q}$ is in the closure of each $\mathbb{Z}_{p^\infty}$ and each $\overline{\mathbb{Z}_{(p)}}$. If $\mathbb{Q}$ occurs, then no $\overline{\mathbb{Z}_{(p)}}$ does so (otherwise we have two torsionfree elements with different divisibilities); also, if there is a torsion element, then (take its sum with an element of $\mathbb{Q}$) it must be completely divisible. Therefore one obtains the following.

Theorem 16.3 [Fe8?] *The complete theories* T *of abelian groups which have elimination of quantifiers are precisely those which satisfy the condition that* $\mathcal{I}(T)$

is either a subset of $\{\mathbb{Q}\}\cup\{\mathbb{Z}_{p^\infty}: p$ *prime*$\}$ *or is a finite set of the form* $\{\mathbb{Z}_{p_1{}^{n_1}},\ldots,\mathbb{Z}_{p_k{}^{n_k}}\}$ *where the primes* $p_1,\ldots,p_k$ *are distinct.* ▫

This result has been generalised to Dedekind domains by Ruyer [Ru84; Thm] and Weispfenning [Wei85; Thm 4] (see comments in [Wei85]). One may prove it by the same kind of argument after localising (cf. §2.$\mathbb{Z}$, especially **2.$\mathbb{Z}$11**). Weispfenning [Wei85; Thm 4*] delineates conditions necessary and sufficient for the elimination of quantifiers to be (primitive) recursive.

Theorem 16.4 *Let* R *be a Dedekind domain. The complete theories,* T, *of* R-*modules with elimination of quantifiers are exactly those which satisfy the condition that* $\mathcal{I}(T)$ *is a subset of* $\{K\}\cup\{E(R/P): P$ *prime*$\}$ *or is of the form* $\{R/P_1{}^{n_1},\ldots,R/P_k{}^{n_k}\}$ *where the* P_i *are distinct primes. Here,* K *is the field of quotients of* R. ▫

Exercise 1 [Wei85; Thm 3] Let R be a commutative domain and suppose that M has elimination of quantifiers. Show that M either is divisible or is annihilated by a non-zero element of R (note that M being torsion is insufficient).

If a complete theory of modules has elim-Q, then every formula is equivalent to a boolean combination of atomic formulas. Does it follow that every pp formula is equivalent to a positive boolean combination, or even to a conjuction, of atomic formulas? The answer, at least to the second question, is in general "no" (Ex 2 below). That is unfortunate, since this seems to be a more useful property than straight elim-Q. Let us say that a theory has **elim-Q$^+$** if every pp formula is equivalent to a conjunction of atomic formulas (that is, the condition on a tuple that it be the initial segment of a solution vector for a given system of linear equations is equivalent to the condition that it be a solution vector for a certain other system of linear equations). I show first that if $T=T^{\aleph_0}$ has elim-Q then it has elim-Q$^+$.

Proposition 16.5 [Gar80; Lemma 8], [Pr81a; 3.2] *Let* T *be a complete theory of modules with elimination of quantifiers.*

(*a*) *If* $T=T^{\aleph_0}$ *then* T *has* elim-Q$^+$.

(*b*) *If* T *has* elim-Q$^+$ *then* $T^{\aleph_0}$ *has* elim-Q$^+$ (*indeed,* $T^{\aleph_0}$ *has the "same" elimination of quantifiers as* T).

(*c*) *If* T *has* elim-Q$^+$ *and if* T' *is a component theory* (*cf.* §2.6) *of* T *then* T' *has* elim-Q$^+$.

Proof (a) Suppose that the pp formula φ is equivalent to $\bigvee_k(\psi_k\wedge\bigwedge_i\neg\theta_{ik})$ where ψ_k is $\bigwedge$-atomic and the θ_{ik} are atomic. Fix k: then $\psi_k\wedge\bigwedge_i\neg\theta_{ik}$ implies φ; so $\psi_k\le\varphi\cup\bigcup_i\theta_{ik}$. By Neumann's Lemma and the fact that $T=T^{\aleph_0}$, either $\psi_k\leqslant\varphi$ or $\psi_k\le\theta_{ik}$ for some i. In the latter case, the k-th conjunct is vacuous and so may be dropped.

Therefore, it may be assumed that $\varphi\geqslant\psi_k$ for each k. So $\varphi=\bigcup_k\psi_k$. Again by Neumann's Lemma, it follows that $\varphi=\psi_k$ for some k. That is, φ is equivalent to a $\bigwedge$-atomic formula, as required.

(b) Let M be any model of T; so $M^{\aleph_0}$ is a model of $T^{\aleph_0}$ and has the "same" lattice of pp-definable subgroups as M. Let φ be pp. Then there exists a $\bigwedge$-atomic formula ψ such that $\varphi(M)=\psi(M)$. It follows by **2.10** that $\varphi(M^{\aleph_0})=\psi(M^{\aleph_0})$, as required.

(c) Let M' be a model of T'. Then there exists a module M'' such that $M'\oplus M''$ is a model of T. Let φ be pp; so there exists a $\bigwedge$-atomic formula ψ with $\varphi(M)=\psi(M)$. It follows that $\varphi(M')=\psi(M')$, as required. ▫

As a consequence, the question of whether elim-Q implies elim-Q$^+$ reduces to that of whether or not T having elim-Q implies that $T^{\aleph_0}$ has elim-Q. The following example shows that elim-Q$^+$ is genuinely stronger than elim-Q. This example has elim-Q, but there is a pp formula which, although equivalent to a disjunction of atomic formulas, is not equivalent to a conjunction of atomic formulas. I do not know (whether there is) an example where negations are necessary.

Example 2 Let R be the ring $\mathbb{Z}_2[x,y:x^2=y^2=0]$. The Jacobson radical, $xR+yR$, is defined by the pp-formula $\exists u,w(v=ux+wy)$. It is also defined by the quantifier-free formula $vx=0 \vee v(x+y)=0 \vee vy=0$ (the disjuncts separately define xR, $(x+y)R$ and yR). One sees that every ideal of R may be defined by a quantifier-free formula. Therefore R has elim-Q. But it is easy to check that $xR+yR$ cannot be defined by a conjunction of atomic formulas. So R does not have elim-Q$^+$. If one feels happier with an infinite example, then it is only necessary to consider $R \oplus (R/J)^{\aleph_0}$.

Now I describe some consequences of elim-Q$^+$. Let us suppose that T has elim-Q$^+$. Recall that elements $\bar{a}$ and $\bar{b}$ are said to be linked if there is a pp formula φ such that $\varphi(\bar{a},\bar{b}) \wedge \neg\varphi(\bar{a},\bar{0})$ holds. Elim-Q$^+$ means that φ may be taken to be a conjunction of atomic formulas. On considering what this means, one obtains the following.

<u>Proposition 16.6</u> [Pr81a; 3.1] *Suppose that the complete theory* T *has* elim-Q$^+$. *Let* $\bar{a}, \bar{b}$ *be in a model of* T. *Then* $\bar{a}$ *is linked to* $\bar{b}$ *iff* $\bar{a}R \cap \bar{b}R \neq 0$ ($\bar{a}R$ *denotes the submodule generated by* $\bar{a}$). □

It follows that pp-essential embeddings (see §4.1) are just essential embeddings. Recall that a module M is said to be uniform if each pair of non-zero submodules has non-empty intersection. For example, an injective module is uniform iff it is indecomposable. The next corollary partly generalises this; only partly because, although every injective "locally" has elim-Q$^{(+)}$, the theory of an injective need not have elim-Q (see 15.42).

<u>Corollary 16.7</u> [Pr81a; 3.5] also [Gar80; Lemma9] *Suppose that* T *is a complete theory of modules with* elim-Q$^+$ (*for example, suppose that* T *has* elim-Q *and that* $T=T^{\aleph_0}$). *Then every indecomposable pure-injective direct summand of a model of* T *is a uniform module.* □

Note that **16.7** implies that elimination of quantifiers has both a local and a global aspect, in that only certain indecomposable pure-injectives can possibly occur but, even then, only certain combinations of them have elim-Q.

Garavaglia asked whether, if T is a t.t. theory with elim-Q, the indecomposable summands of models of T must be "small" in the sense of being uniform. He established this for commutative noetherian rings (see below) and conjectured that it was true for any theory with elim-Q over a right noetherian ring (such a theory necessarily being t.t. by **16.2**). The above result shows that the t.t/noetherian assumption is irrelevant, except perhaps in the case that T has elim-Q without having elim-Q$^+$. I give his rather pretty proof for this possibility not covered by 16.7.

<u>Proposition 16.8</u> [Gar80; Lemma9] *Suppose that* R *is a commutative ring and that* N *is an indecomposable module over it with* Elim-Q. *Then* N *is uniform.*

Proof First, choose non-zero elements, a and b, which lie in minimal non-zero pp-definable subgroups, $\varphi(N)$ and $\varphi'(N)$, of N: we will see that $aR \cap bR \neq 0$.

Suppose otherwise: then $aR \cap (a+b)R = 0$. For if $ar = as+bs$ then $a(r-s)=bs$ and so, by assumption, $bs=0$. But, by **4.59**, $\varphi(N)$ and $\varphi'(N)$ are isomorphic as R-modules and, by minimality, all non-zero elements in each have the same annihilator. So $as=0$. Hence $ar = as+bs = 0$.

It follows that (a,b) and $(a,a+b)$ have the same quantifier-free type (for $(a,b)\mapsto(a,a+b)$ defines a module isomorphism from $aR\oplus bR$ to $aR\oplus(a+b)R$). So, by elimination of quantifiers, they have the same type. Now, by **4.11**, there is a pp formula θ linking a and b: $\theta(a,b)\wedge\neg\theta(a,0)$. It may be assumed that $\exists v\,\theta(v,w)$ defines $\varphi'(N)$ and hence $\theta(0,w)$ defines the zero subgroup. By what has just been shown, $\theta(a,a+b)$ holds. This gives $\theta(0,a)$: so $a=0$ - contradiction.

Now, let c be any non-zero element of N. Choose a non-zero multiple cr of c whose pp-type, generated by $\varphi''(v)$ say, has minimum pp-rank among such multiples. Let b be as above and take ψ such that $\psi(cr,b)\wedge\neg\psi(cr,\ 0)$ holds and such that $\psi(v,w)$ proves $\varphi''(v)\wedge\varphi'(w)$. By **9.7**, $\{s\in R : crs\in\psi(N,0)\}=(\psi(N,0):cr)=(\psi(0,N):b)=\mathrm{ann}\,b$. Therefore, if $s\in R$ is such that $bs=0$ then $crs=0$, and conversely.

Thus cr and b have the same annihilator so, by Elim-Q, have the same pp-type. Thus every non-zero element has a non-zero multiple with critical type so, by the first part, every two non-zero elements of N have a common non-zero multiple, as required.

Note that, since R is commutative (so the pp-definable subgroups are submodules), it follows that there is unique minimal non-zero pp-definable subgroup. □

I finish this section by describing forking and ranks in the case of a complete theory $T=T^{\aleph_0}$ with elim-Q. Such descriptions were given by Bouscaren in her thesis [Bou79] (for a summary see [Bou80]) for the special case of the model-completion of the theory of modules over a right coherent ring and, in more detail, for modules over commutative regular rings. In fact, her proofs apply almost word-for-word to the general case. I just state the results, since the proofs are straightforward from **16.2** and the results of §5.2. It should be said that the shape of many results in Chapter 5 was suggested by Bouscaren's work.

Suppose for the remainder of this section that $T=T^{\aleph_0}$ is a complete theory of modules with elim-Q. Let π be the canonical projection from $\mathrm{P}(R)$ to $\mathrm{Latt}(R)$ - the image is the set of right ideals which are T-closed in the sense of Chapter 15 (namely, I is T-closed if R/I embeds in a model of T). Since the image is $\bigcap$-complete, every right ideal I has a T-closure $\mathrm{cl}_T I$ which is the smallest T-closed right ideal containing I.

Following [Bou79], one associates to any type $p(v)\in S_1(A)$ the triple (K_p,I_p,f_p) where $K_p=\{r\in R : \text{"}vr=0\text{"}\in p\}$, $I_p=\{r\in R : \text{"}vr=y\text{"}$ is represented in $p\}$ and where f_p is the morphism from I_p to A which associates to $r\in I_p$ the (unique) element $a\in A$ such that "$vr=a$" is in p. One sees that from (K_p,I_p,f_p) (and A) one may recover p. Notice that K_p is just $R\cap p^+$.

As for which triples may occur in this way: every T-closed right ideal I may occur as K_p, and may occur with any right ideal $J\geqslant I$ as I_p (let c realise the type of an element with annihilator I and take $A=cJ$). Moreover, given $p\in S_1(A)$ and $J\geqslant I_p$ there is q extending p with $I_q=J$ (use **5.11**). The proof of the next result follows [Bou79; 1.2,1.4] using **5.3** (cf. [Bou79; 1.3]).

<u>Theorem 16.9</u> (cf. [Bou79; 1.2 1.4]) *Suppose that* $T=T^{\aleph_0}$ *has elimination of quantifiers. Let* $p\in S(A)$ *and* $q\in S(B)$, *where* $A\subseteq B$, *be types. Then* q *is an extension of* p *iff* $K_p=K_q$, $I_p\leq I_q$ *and* $f_q\restriction I_p=f_p$. *Furthermore,* q *is a non-forking extension of* p *iff* $\mathrm{cl}_T(I_q)=\mathrm{cl}_T(I_p)$. □

One may check that the fundamental order is isomorphic to the poset with elements the set of pairs (K,I) where K and I are T-closed and $K\leqslant I$, where the ordering is given by $(K,I)\leqslant(K',I')$ iff $K=K'$ and $I\leqslant I'$ (e.g., [Kuc87; 2.4]) The next result follows from **16.9** and **5.13**.

Theorem 16.10 (cf. [Bou79; §1.3]) *Suppose that* $T=T^{\aleph_0}$ *has elimination of quantifiers. Let* p *be any type. Then* p *has* U-*rank iff the ring* R *has the ascending chain condition on* T*-closed right ideals containing* I_p. *In particular,* T *is superstable (iff it is t.t.) iff* R *has acc on* T*-closed right ideals.* □

Theorem 16.11 (cf. [Bou79; Prop4]) *Suppose that* $T=T^{\aleph_0}$ *has elimination of quantifiers. Let* p *be any type. Then* p *has Morley rank iff* R *has acc on* T*-closed right ideals above* I_p *and* I_p *is* T*-finitely generated (i.e. is the* T*-closure of a finite set of elements).* □

For the proof of 16.11 one should use 5.15.

Example 3 Let R be commutative regular. Then (15.42 or 16.15) the theory T^* has elim-Q$^+$. Suppose that R has an infinitely generated maximal ideal I and let p be the type of the image of $1\in R$ in R/I. Then $I_p=I$ so by, 16.10, p has U-rank 1. On the other hand (by 16.11) p does not have Morley rank.

It is rather remarkable that a significant amount of the above goes through for "modules" over a semigroup S ("S-systems"). There appears to be no analogue of the pp-elimination of quantifiers for such structures; moreover there is the added complication that not every right congruence on the underlying semigroup S is generated by a right ideal of S. Nevertheless, Gould has carried over to S-systems a significant part of Bouscaren's work on modules [Gou87], [Gou87a]. There are, however, some notable differences.

For instance, in place of (K_p,I_p,f_p) as above, one has (ρ_p,I_p,f_p), where ρ_p is a right congrunce, I_p a right ideal which is a union of ρ_p-classes and f_p is a morphism from I_p to A (with notation as before). Knowing the kernel of $f_p(=\rho_p\cap(I_p\times I_p))$ instead of ρ_p is not enough to determine p.

Gould shows that if S is a right coherent monoid then the theory of S-systems has a model-companion T_S. She shows that T_S is superstable iff S has the acc on right ideals ("weakly noetherian"). But, in contrast with the modules case, this is not equivalent to T_S being totally transcendental: for that, one requires a certain further condition on congruences. She shows that T_S is totally transcendental with $\mathrm{MR}(p)=\mathrm{UR}(p)$ for all types p iff S has the acc on right congrunces (noetherian") and satisfies a still stronger condition (which, under many circumstances, forces S to be finite; for example, if S is a group satisfying these equivalent conditions then it must be finite).

Proposition 16.12 [Gar80; Lemma7] *Suppose that* R *is right noetherian and that* T *has* elim-Q. *Then* T *is totally transcendental.*

Proof This is immediate from 16.2 and 3.1. □

Other results of this sort may be proved by the same kind of argument.

Theorem 16.13 (cf. 10.30) *Suppose that* T *has* elim-Q. *Then*:

(*a*) *If* R *has Krull dimension (in the sense of* [RG67]) *then* T *has m-dimension and, in particular,* T *has continuous part zero. Indeed, the m-dimension of* T *is the m-dimension of the lattice of* T*-finitely generated* T*-closed right ideals of* R.

(*b*) *If the lattice of finitely generated right ideals of* R *has width (in the sense of* §10.2) *then* T *has width and so has no continuous pure-injectives.*

Proof This follows by 16.2 and the discussion which follows it, together with 10.9. □

At least if T is closed under products (so every right ideal has a T-closure), if the lattice generated by the finitely generated right ideals has a given finiteness condition then so does the lattice of T-finitely generated T-closed right ideals.

It is enough, in the above, to look at the finitely generated right ideals, since elementary Krull dimension and width are defined in terms of pp formulas rather than pp-types and, by elimination of quantifiers (and the discussion after **16.2** when we do not have Elim-Q^+), pp formulas correspond to finitely generated right ideals. Observe that there is a difference: if R is regular, but not semisimple artinian, then the Krull dimension of Latt(R) is "∞", but that of Latt^f(R) may be strictly less than "∞" (cf. **16.26** below).

16.2 Modules over regular rings

A ring R is said to be (von Neumann) **regular** if every finitely generated right ideal may be generated by an idempotent element (and hence is a direct summand of R). There are many characterisations of these rings.

Theorem 16.A ([Goo79; Chpt1]) *The following are equivalent conditions on a ring* R:
(*i*) R *is regular*;
(*ii*) *every finitely generated left ideal is generated by an idempotent*;
(*iii*) *for every* $r \in R$ *there exists* $x \in R$ *such that* $r = rxr$;
(*iv*) *every (right or left) module is absolutely pure*;
(*v*) *every (right or left) module is flat.* □

In particular, regularity is a two-sided property. It also follows that a regular ring is (right and left) coherent. Condition (iv) says that every embedding between modules is pure, so the following are some of the many corollaries that may be deduced.

Corollary 16.14 *Suppose that* R *is a regular ring.*
(*a*) *An inclusion,* $M \leqslant N$, *is an elementary embedding iff* $M \equiv N$.
(*b*) *Every pure-injective module is injective; so the hull of an element or set,* $\bar{a}$, *is the injective hull of the module it generates.* □

Corollary 16.15 *Suppose that* R *is a regular ring.*
(*a*) *Every theory of* R*-modules has* elim-Q^+.
(*b*) *The morphism* $\pi : \mathrm{P} \longrightarrow \mathrm{Latt}(R)$, *which takes a pp-type* p *to the right ideal* $K_p = \{r \in R : \text{"}vr = 0\text{"} \in p\}$, *is an isomorphism of lattices which takes* P^f *onto* Latt^f(R). □

In **16.15**, part (b) follows from (a) which, in turn, is an immediate consequence of **16.1**, **16**.A(iv) and **2.17**. Actually, property (a) characterises regular rings (also cf. §15.3).

The next result has been discovered by a number of people: as well as the references to 16.17, there is an unpublished note by Hodges; also see [ES71; 3.25] and [Sab71; Cor4].

Theorem 16.16 *The ring* R *is regular iff* T^* *has complete elimination of quantifiers.*

Proof One direction is **16.15**. Suppose, for the converse, that T^* has elim-Q^+. Let $M \leqslant N$ be any inclusion of modules and let M^* be any model of T^*. Then $M^* \oplus M$ and $M^* \oplus N$ both are models of T^* so, by the hypothesis, the embedding $M^* \oplus M \leqslant M^* \oplus N$ is an elementary one. In particular, the embedding $M \leqslant N$ is pure. So regularity of R follows by **16**.A(iv). □

Exercise 1 [Wei85] Show that R is regular iff every 2-generated module has complete elimination of quantifiers. Show that every singly generated module having elimination of quantifiers is not enough.

What of groups of n-tuples are pp-definable in a module M over a regular ring R? If $\varphi=\varphi(v_1,\ldots,v_n)$ is pp then, by **14.9** and **16**.A(v), one has $\varphi(M)=M\varphi(R)$. Applying **14.16**, we see that $\varphi(R)$ is a typical finitely generated submodule of R^n. Now, it is not difficult to show that every finitely generated submodule of R^n is a direct summand [Goo79; 1.11] - that is, is generated by an idempotent in the endomorphism ring of R^n. Thinking of such an endomorphism as an $n\times n$ matrix, we deduce the following (there are two ways of expressing the result, since such a matrix determines both a kernel and an image - so we can define a subgroup by an annihilator condition or by a divisibility condition).

Proposition 16.17 [Rot83a; Prop 11], [Zim77; 1.3], [Wei85; Thm 1] (also see [Gar79; Lemma 8] for the commutative case, and [GJ81; 2.3]) *Let R be any ring. Then R is regular iff the following equivalent conditions are satisfied:*

(i) if φ is a pp formula in n free variables then there is an $n\times n$ idempotent matrix H, with entries from R, such that, for any module M, one has that $\varphi(M)$ is the solution set to the matrix equation $\bar{v}H=0$;

(ii) if φ is a pp formula in n free variables then there is an $n\times n$ idempotent matrix H with entries from R such that, for any module M, the subgroup $\varphi(M)$ is the solution set to the matrix equation $\bar{v}H=\bar{v}$. □

Considering the special case of formulas in one free variable, one obtains that, for any module M, every pp-definable subgroup of M has the form Me for some idempotent e of R; alternatively, has the form $\mathrm{ann}_M e'$ for some idempotent e' of R.

Corollary 16.18 [Rot83a; Lemma 20] *Suppose that R is regular. Then every invariant* Inv(-,φ,ψ, *is equivalent to one of the form* Inv(-,"$v=ve$","$v=0$"); *alternatively, to one of the form* Inv(-,"$ve=0$","$v=0$"), *where e is an idempotent in R.*

Proof By **16.17** every invariant is equivalent to one of the form Inv(-,"$v=ve$","$v=vf$") for some idempotents e, f of R, and it may be supposed that $Re\geqslant Rf$. Now, $Re=Rf\oplus Re(1-f)$ and $e(1-f)$ is idempotent (since R is non-commutative (= not necessarily commutative!) these points are not absolutely trivial: use the fact that $f\in Re$ implies $f=fe$). Thus, if M is any module then $Me/Mf\simeq Me(1-f)$. Thus the first form follows. The second form is obtained on noting that $Me=\mathrm{ann}_M(1-e)$. □

Weispfenning [Wei85] considers the extent to which the replacement of a pp formula by an equivalent quantifier-free formula is effective. He shows [Wei85; Thm 1*] that if R is a (primitive) recursive regular ring, then the assignment is (primitive) recursive. Moreover, if M is an R-module, then the following are equivalent: the complete theory of M is (primitive recursively) decidable; the complete theory of M allows (primitive) recursive elimination of quantifiers (in the above sense); the set of pairs, (e,n), with e an idempotent of R and $n\in\omega$ and is such that $|Me|\geqslant n$, is (primitive) recursive.

As a consequence of the above, one has that T^* is just the theory of $R^{\aleph_0}$.

Corollary 16.19 *Suppose that R is regular. Then $T^*=\mathrm{Th}(R^{\aleph_0})$.* □

Another consequence is the characterisation of those rings over which the theory of non-zero modules is model-complete (that is, those rings over which every embedding between non-zero modules is an elementary one); of course the zero module must be excluded since it cannot be elementarily equivalent to a non-zero module. The result was proved by Tyukavkin. Both

Rothmaler [Rot84; Thm 19] and Weispfenning [Wei85; Thm 2] gave simpler proofs (similar to that below), using the elimination of quantifiers.

<u>Corollary 16.20</u> [Tyu82; Thm 1] *The following conditions on the ring R are equivalent.*

(i) The theory of R-modules is model-complete.

(ii) R is an infinite simple regular ring.

(iii) The theory of left R-modules is model-complete.

Proof (i)⇒(ii) Since every embedding between non-zero R-modules is elementary and hence pure, regularity follows by 16.A(iv).

That R must be infinite is clear, since otherwise R would not be elementarily equivalent to R^2.

Finally, R must be simple since, if I is a non-zero ideal of R, then the module R/I satisfies the sentence $\forall v(vr=0)$, where r is some chosen non-zero element of I. Since R does not satisfy this sentence it follows that $I=R$, as required.

(ii)⇒(i) Let R be simple regular. Take any non-zero module M; then $\mathrm{ann}M=0$. Hence R embeds in a suitable power of M (take elements, m_λ, in M such that $0=\bigcap_\Lambda \mathrm{ann}(m_\lambda)$ and map $1\in R$ to $(m_\lambda)_\lambda \in M^\Lambda$). Hence $R^{\aleph_0}$ embeds in some power of M. Since R is regular, this embedding is a pure one and we conclude, by 16.19, that $M^{\aleph_0}$ is a model of T^*: we must show that M is a model of T^* (this shows completeness; then 2.25 shows model-completeness).

If not, then, by 16.18, there is a non-zero idempotent e of R with Me finite. Since R is simple, M is faithful, so this implies that Re is finite. But then the socle of R would be a non-zero ideal, hence equal to R: therefore ($1\in R$!) R would be finite. So, if R is infinite, $M\equiv M^{\aleph_0}$ and so every non-zero module is a model of the (complete!) theory T^*.

The proof is completed by the observation that condition (ii) is right/left symmetric. ▫

It is not known whether the theory of non-zero R-modules being complete is enough to imply that R is an infinite simple regular ring (and hence that the theory of modules is model-complete). Rothmaler in [Rot84] notes that, in the presence of coherence on either side, the condition is indeed enough (by 16.A and 14.18 (left coherent) and 15.41 (right coherent)).

Therefore the infinite simple regular non-artinian rings are quite pathological, in the sense that every indecomposable pure-injective is elementarily equivalent to every other, yet there may be $2^{\aleph_0}$ of them.

Example 1 Let U be an $\aleph_0$-dimensional vector space over the countable field K. Let R be the endomorphism ring of U_K modulo its socle (the socle consists of those endomorphisms with finite-dimensional image). Then R is a simple regular ring which is injective as a module over itself [Goo79; 9.12] and is clearly not semisimple artinian. Indeed, since its socle is zero, it has no indecomposable summands (it is actually isomorphic to each of its finitely generated non-zero right ideals) and hence is a continuous pure-injective.

Therefore, by 10.9, T^* has width ∞. By the result above, all indecomposble pure-injectives are elementarily equivalent, yet there are $2^{\aleph_0}$ of them. To see the last point, it will be enough to show that R has $2^{\aleph_0}$ non-isomorphic simple modules.

It is left as an exercise to show that R has $2^{\aleph_0}$ maximal right ideals. Now, if I and I' are maximal ideals, then $R/I \simeq R/I'$ iff there is an element $s\in R$ such that $I=\{r\in R: sr\in I'\}$ (exercise; see [LM73]). Now use a counting argument (this is the only reason for my assuming that K is countable).

In this section I tend to confine attention to T^*, or at least to theories T with $\mathcal{I}(T)=\mathcal{I}(T^*)$. This is justified because if M is a faithful module over a regular ring then, by

16.18, Th($M^{\aleph_0}$) = T^*. If M is not faithful then R may be replaced by $R/\mathrm{ann}_R M$, which is still regular (by 16.A(iii)).

By 16.15, P(R) "is" the lattice of right ideals of R, so consideration of finiteness conditions is simplified. It turns out that just about any decent finiteness condition on a regular ring forces it to be artinian (rather, the finiteness conditions which are appropriate to regular rings are very different from those seen in other parts of ring theory).

Theorem 16.B *Let R be a regular ring. Then the following conditions on R are equivalent:*
(i) R is semisimple artinian;
(ii) R has the dcc on finitely generated right (or left) ideals;
(iii) R is (right or left) noetherian;
(iv) R has Krull dimension;
(v) R has no infinite set of pairwise orthogonal idempotents. □

The equivalences may be found in [Goo79] (for (iv)⇒(v) one may use that a module which contains an infinite direct sum of submodules cannot have Krull dimension: see [GR73; 1.4]).

Proposition 16.21 [Zim77; 4.3], [Pa77; 3.2.10], [Rot83a; Thm 23] ([Gar79; Thm 4] for the commutative case) *Let M be a module over the regular ring R. Then M is totally transcendental iff $R/\mathrm{ann}_R M$ is semisimple artinian*

Proof If M is t.t. then, by 3.1 and 16.15(b), $\overline{R} = R/\mathrm{ann}_R M$ has dcc on finitely generated right ideals and so, by 16.B, is artinian. The converse is clear. □

The next result follows from 16.15 and 16.18. An example of a superstable non totally transcendental module over a regular ring is the subring of $\prod\{\mathbb{Z}_p : p \text{ prime}\}$ consisting of the "eventually constant" sequences - see Ex 16.2/2 below.

Proposition 16.22 [Rot83a; Thm 24] ([Gar79; Thm 5] for the commutative case) *Let M be a module over the regular ring R. Then M is superstable iff for every descending chain $Me_0 \geqslant Me_1 \geqslant \ldots$, with the e_i idempotents of R, the indices $[Me_i : Me_{i+1}]$ eventually are finite; this happens iff, for every orthogonal set $\{e_0, e_1, \ldots, e_n, \ldots\}$ of idempotents of R, there are only finitely many e_i for which Me_i is infinite.* □

A succinct way of expressing the conditions of 16.22 is to say that $R/\mathrm{fin}M$ is semisimple artinian, where $\mathrm{fin}M$ is the finitiser of M (§7.2).

Proposition 16.23 *Suppose that R is a regular ring. Then T^* has elementary Krull dimension, equivalently,* m*-dimension, iff the lattice of finitely generated right ideals of R has Krull dimension.* □

The result above is immediate from 16.15(b). It should be noted that the conditions of 16.23 may be satisfied without R itself having Krull dimension. If R is as in Ex 16.2/2 then, by 16.B, R does not have Krull dimension but, since it has only countably many indecomposable injectives, T^* does have m-dimension (by 10.15). In fact it has m-dimension 1 (by 16.25).

Throughout the remainder of this section R is a commutative regular ring.

The assumption is to make life simple: the non-commutative case presents many complications and new features (see [Goo79]). It would be interesting to know more about that case, but I don't propose to make such a study here.

For any regular ring R one denotes by B(R) the set of all central idempotents of R. B(R) is given the structure of a boolean algebra by setting $e \leqslant f$ iff $eR \leqslant fR$. It is an elementary exercise to check that this does make B(R) into a complemented distributive lattice with the boolean operations being $e \wedge f = ef$ and $e \vee f = e + f - ef$ (on the level of elements; on the level of the corresponding ideals these are just intersection and sum).

Exercise 2
(i) Show that if R is commutative regular with no minimal ideals then every maximal ideal is infinitely generated.
(ii) Show that if R is commutative regular and if I, J are ideals of R then $I = J$ iff $I \cap B(R) = J \cap B(R)$.

One sees that if R is commutative regular then Latt(R) is just the space of ideals $B(R)$, with $\mathrm{Latt}^f(R)$ being naturally identified with $B(R)$ itself (the principal ideals). Hence, by 16.15(b), $B(R)$ "is" the lattice $P^f(R)$. The set of maximal ideals of R is denoted Spec(R). This set carries the **Pierce topology** which has, as a basis of clopen sets, the sets $\mathcal{O}_e = \{ I \in \mathrm{Spec}(R) : e \notin I \}$ as e ranges over $B(R)$. With this topology, Spec(R) is naturally homeomorphic to the Stone space (space of maximal ideals) of $B(R)$ (the correspondence is given by $I \mapsto I \cap B(R)$). This space is compact and totally disconnected (since for each idempotent $e \neq 0,1$, the space is disconnected by the pair of clopen sets $\mathcal{O}_e$, $\mathcal{O}_{1-e}$). If the definition of $\mathcal{O}_e$ seems a little perverse, note that $I \in \mathcal{O}_e$ iff the image of e in the factor field R/I is non-zero (hence is $1+I$).

If $I \in \mathrm{Spec}(R)$ and if M is any module, then M/MI is naturally a vectorspace over the field R/I. In fact, the category of R-modules is equivalent to the category of sheaves over Spec(R) where the stalk over $I \in \mathrm{Spec}(R)$ is an R/I-vectorspace (the ring is itself a sheaf of fields: see [Pie67]). I do not exploit this approach in these notes.

Exercise 3 Show that the Pierce topology on Spec(R) coincides with the Zariski topology.

If R is any regular ring, then the indecomposable injectives are just the hulls of the simple modules. For, if I is a proper non-maximal right ideal, take $e^2 = e \in R \setminus I$, $e \neq 1$: then $(eR + I) \cap ((1-e)R + I) = I$, so the cyclic module R/I is not uniform and hence its injective hull is not indecomposable.

It is easy to see (exercise) that if R is a commutative regular ring then the indecomposable injectives are precisely the modules R/I for $I \in \mathrm{Spec}(R)$. It has already been shown in §4.7 that the space $\mathcal{I}(R)$ of indecomposable (pure-)injectives is in fact homeomorphic to Spec(R) (with R/I corresponding to I). I now consider Cantor-Bendixson rank on this space.

A boolean algebra is **atomic** if every non-zero element contains an **atom** (that is, an element with nothing strictly between itself and zero). A boolean algebra is **atomless** if it contains no atoms (equivalently, since a boolean algebra is complemented, if there is no finitely generated maximal ideal). A boolean algebra B is **superatomic** if, regarded as a lattice, it has m-dimension in the sense of §10.2: an equivalent requirement is that its Stone space, Spec(B), have Cantor-Bendixson rank (exercise). An example of a superatomic boolean algebra is the algebra of all finite and cofinite subsets of a set. An example of an atomless boolean algebra is the boolean algebra of all subsets of an infinite set, factored by the ideal of all finite subsets.

The proof of the next result is left as an exercise.

<u>Proposition 16.24</u> see [Bou79; §II.1] *Let R be a commutative regular ring. Suppose that the neighbourhood $(ve = 0 / v = 0)$ is a minimal non-empty open set in $\mathcal{I}(R)$. Then e generates a maximal ideal, and this neighbourhood isolates the single point R/eR.* □

Thus if R is a commutative regular ring then T^* satisfies condition $(\wedge)$ of §10.4 (by **16.24**, since $\mathcal{I}(R)$ "is" Spec(R) and P(R) "is" B(R)). The example after **16.20** shows that the result above may fail spectacularly if R is not commutative.

Theorem 16.25 [Gar80a; Thm 4] *Suppose that* R *is a commutative regular ring. Then the following conditions are equivalent*:

(*i*) $\text{m-dim}\,T^* = \alpha < \infty$;

(*ii*) $\text{CB-rk}(\text{Spec}\,R) = \alpha < \infty$;

(*iii*) B(R) *is superatomic with* $\text{m-dim}\,B(R) = \alpha$.

If R *is countable then these conditions are satisfied for some* α *iff* Spec$R$ *is countable.*

Proof The easiest way to show that (i) and (ii) are equivalent is to use condition $(\wedge)$ and **10.19**. The fact that the Cantor-Bendixson analysis of Spec(R) coincides with the analysis of B(R) using m-dimension also follows from **10.19**, since B(R) "is" just $P^f(R)$. The last statement follows from **10.15** (for example). (Exercise - show all these points directly.) ▫

It should also be observed that m-dimension and elementary Krull dimension coincide, essentially because the fact that B(R) is complemented means that EKdim grows no faster than dim (an interval has the dcc iff it is of finite length).

Example 2 Let R be the subring of the product $\prod\{\mathbb{Z}_p : p \text{ prime}\}$ of all the finite prime fields, consisting of all "eventually constant" sequences. By that I mean $R = \{(a_p)_p : \exists r \in \mathbb{Q}\ \exists n \in \omega\ \forall p \geqslant n\ (a_p = r)\}$ - this does make sense, since each rational expression ab^{-1} $(a, b \in \mathbb{Z},\ b \neq 0)$ may be interpreted in all but finitely many of the fields $\mathbb{Z}_p$. Then R is a commutative regular ring.

For each prime p, R has an isolated maximal ideal - that consisting of all sequences $a = (a_q)_q$ with $a_p = 0$ (and the corresponding factor ring is the field $\mathbb{Z}_p$). It has one more maximal ideal: that consisting of all elements which are eventually zero. The corresponding factor ring is the rational field $\mathbb{Q}$.

Thus, Spec$R$ consists of $\aleph_0$ isolated points, together with one cluster point. So CB-rk Spec$R$ = 1 and, by **16.22**, Th(R) is superstable but not totally transcendental.

Example 3 A non-commutative analogue of the above example is the following (see [Goo79; Example 6.19]). Let T be the endomorphism ring of an $\aleph_0$-dimensional vectorspace over a field K. Define R to be the subring of T generated by the socle together with the scalar multiples of the identity morphism: $R = \text{soc}\,T + 1.K$. Then R is regular and has just two simple modules: any simple right ideal and R/J (both are injective; the latter is Σ-injective). Thus $|\mathcal{I}(T^*)| = 2$. Since R is countable, **10.15** gives $\text{m-dim}\,T^* < \infty$. One may see directly (or, for example, use **10.19**) that just one of these points is isolated. So $\text{m-dim}\,T^* = 1$.

The next result shows that a commutative regular ring has width iff it has dim.

Theorem 16.26 ([Fis75; 7.31] for boolean rings) *Let* R *be a commutative regular ring. Then the following conditions are equivalent.*

(*i*) *There are no continuous injective* R*-modules.*

(*ii*) R *is seminoetherian (that is, the category of* R*-modules has Gabriel dimension - see* [Pop73; §5.5]).

(*iii*) T^* *has width.*

(*iv*) B(R) *is superatomic (equivalently, any of the conditions of* **16.25**).

Proof By **16.25** and **10.7**, (iv)⇒(iii). By **10.9**, (iii)⇒(i).

To see that (i) implies (iv), note first that if R is commutative regular with no finitely generated maximal ideal, then the injective hull of R is continuous. For if $E = E(S)$, with S simple, were an indecomposable factor of it, then $S \cap R$ would be non-zero (since R is essential in $E(R)$ (§1.2)) - so R would have S as a simple submodule: say $S=eR$. Then $(1-e)R$ would be a finitely generated maximal right ideal - contradiction. Now, if $B(R)$ is not superatomic then (keep factoring out the socle) R has a factor ring R' with $B(R')$ atomless. By the argument just given, there is a continuous injective R'-module: hence there is a continuous injective R-module, as required.

The equivalence of (i) and (ii) for commutative regular rings is [Pop73; p.366 Ex.8(a)⇒(b)] ((ii)⇒(i) is true for any ring, and the key point which makes the converse work for a commutative regular ring is that if $B(R)$ is superatomic then every ideal of R is an irreducible intersection of the finitely generated maximal ideals which contain it). □

16.24 identifies the isolated points of $\mathcal{I}(R)$, so I go on to discuss the existence of a prime model for T^*. Bouscaren points out [Bou80; p42] that her results below go through for strongly regular (=abelian regular) rings. Indeed, most of the results of this section should go through in that context: in those rings all idempotents are central and they have a nice spectrum (see [Goo79; 3.12]).

If R is countable then there is a prime model iff the isolated points of each space of types, $S_n(0)$ $(n\in\omega)$, are dense (1.6). By 16.17, every non-empty basic open neighbourhood of $S_1(0)$ is defined by a formula of the form $ve=0 \wedge \bigwedge_i vf_i\neq 0$ (**), where e and the f_i are idempotents and it may be assumed that $f_iR\geqslant eR$ for each i.

Suppose first that the finitely generated maximal ideals are dense. Given any neighbourhood (**), if it is non-empty, then eR is strictly contained in the intersection of the f_iR so, by density, there is a finitely generated maximal ideal, $e'R$ say, containing eR but not containing any of the f_iR. Then the type given by $ve'=0 \wedge v\neq 0$ is an isolated point of the neighbourhood.

Conversely, if every non-empty neighbourhood contains an isolated point, take an idempotent $e'\neq 1$. Then the neighbourhood $(ve'=0/v=0)$ contains an isolated point: isolated, say, by (**). Note that $e'R\leqslant eR$. Since every idempotent strictly above e contains at least one of the f_i (by isolation), it follows (by 16.B) that R/eR is semisimple artinian. So, every maximal ideal above eR is finitely generated: hence there is a finitely generated maximal ideal above $e'R$.

<u>Theorem 16.27</u> [Bou79; II.1.2] *Let R be a commutative regular ring. Then the following conditions are equivalent:*
(i) T^* has a prime model;
(ii) $B(R)$ is atomic;
(iii) T^* has a prime (pure-)injective model.
If these conditions are satisfied then the prime model is $\bigoplus_\lambda\{(R/e_\lambda R)^{(\kappa(\lambda))}\}$, where $e_\lambda R$ runs over the finitely generated maximal ideals of R and $\kappa(\lambda)$ is 1 or $\aleph_0$ according as $R/e_\lambda R$ is infinite or finite. The prime injective model is the injective hull of this (and is strictly larger, unless R is semisimple artinian).

Proof Suppose first that R is countable. Then the discussion above has established the equivalence of (i) and (ii). The equivalence of (ii) and (iii) follows from 4.73(a) (without any restriction on the cardinality of the ring). To establish the equivalence of (i) and (ii) in the general case, one may show directly that the module described is the prime model, as is done in [Bou79; §2.2] (I leave the details to the reader). □

In particular, if T^* has any of the dimensions of 16.26 then there is a prime model.

Since $B(R)$ need not be atomic, Bouscaren's result above answered the question, left open in [ES71] and explicitly raised in [Pr78a], of whether, as in the noetherian case, the model-completion of the theory of modules over a (countable) coherent ring R has a prime model. For a specific example with B(R) non-atomic, take a topological space X which is totally disconnected, compact, without isolated points and with a countable basis of clopen sets. Let R be the ring of continuous functions from X to the field $\mathbb{Z}_2$ equipped with the discrete topology. Then the elements of R may be identified with the characteristic functions of clopen subsets of X; in particular R is countable. The ring R is commutative regular and, since X is without isolated points, B(R) is atomless. In particular, by **16.27**, T^* does not have a prime model.

Exercise **4** Show that T^* has a minimal (and necessarily prime) model iff B(R) is atomic and each factor field of R is infinite.

For extensions of **16.27** to models prime over arbitrary sets of parameters, the reader is referred to [Bou79; §2.2, §2.3].

Exercise **5** Let R be commutative regular. Describe the RK order and orthogonality in terms of the ideals (K_p) corresponding to types.

16.C Continuous Pure-injectives

Discrete pure-injectives have a perfectly satisfactory decomposition theorem. So what of the countinuous pure-injectives? Are we simply to write them off as impossible to deal with? The (von-Neumann-style) structure theory for injective modules over regular rings (see [GB76]) suggests that to do so would be premature.

Once one knows that there are functors which convert pure-injectives into injectives, one may note (Facchini does this in [Fac85]) that the structure theory of Goodearl and Boyle [GB76] (also see [Goo79; Chpt 10]) can be made to apply to pure-injectives: for any spectral category is equivalent to the category of non-singular injective modules over a suitable regular ring [GO66]. But that does not seem to mesh particularly well, either with the context in which we work (elementary classes), or with the kind of result we might aim for (decomposition in terms of orthogonality classes). So let us see what can be said.

Let us first make the observation that every pure-injective module is the pure-injective hull of a direct sum of hulls of single elements: so, even for continuous pure-injectives, the possibilities are not unlimited.

We already know what an RK-minimal continuous pure-injective looks like (**6.17**): it is a pure-injective module N which has a decomposition as $N = N_1 \oplus N_2$ with N_1 and N_2 non-zero and, moreover, in every such decomposition, each of N_1, N_2 is isomorphic to N. To any such RK-minimal pure-injective (or corresponding pp-type or RK-minimality class), we associate the cardinal $\mu = \mu(N)$ which is defined to be least such that p.i.$(N^{(\mu)}) \not\simeq N$ (cf. [Goo76: §6.C]): if we want to classify pure-injectives up to isomorphism, then we need to know $\mu(N)$. Since N is supposed to be decomposable, we have $\mu \geq \aleph_0$ and, indeed, $\mu \geq \aleph_1$ (exercise: see [Goo76: Exercise 6.C.3]). We have to define μ in this slightly awkward way because there is no guarantee that there is a largest cardinal κ with p.i.$(N^{(\kappa)}) \simeq N$ (in fact, there is no largest such κ iff $\mu(N)$ is weakly inaccessible [Goo76: Exercise 6.C.11]).

Examples of such RK-minimal continuous pure-injectives are: any existentially complete prime ring [Pr83a; Thm 4]; End$U$/soc(End$U$), where U is $\aleph_0$-dimensional vectorspace over a field.

So now let N be any continuous pure-injective. What kind of decomposition can be obtained for N? First we may split off the RK-minimal chunks. Let p be an RK-minimal type realised in N (if there is such). By Zorn's Lemma, there is a maximal factor of N which has the form p.i.$(N(p)^{(\kappa)})$ for some κ. Write N as p.i.$(N(p)^{(\kappa)}) \oplus N'$ where N' contains no realisation of p. By the descriptions of orthogonality and RK-minimality (§§6.2,6.3), we have removed precisely that portion of N which corresponds to the RK-class of p. Then we may work on N' in the same way.

It may be that we thus obtain a complete decomposition of N into (continuous) RK-minimal pieces: that is surely a satisfactory outcome. But it may be that we are left with a non-zero continuous pure-injective N" which realises no RK-minimal type. For examples of this, look at "Type II^f factors".

Suppose that we are in that situation. If we look at the types realised in N", then we obtain an "ideal" in the $\wedge$-semilattice of RK-classes of 1-types: given orthogonal $N_1 = N(a_1)$ and $N_2 = N(a_2)$, there may be no single element b such that $N(b) \simeq N_1 \oplus N_2$; so I mean that the set of RK-classes of 1-types realised in N is upwards-directed insofar as it can be. One would like to split this into orthogonal pieces, but at least two problems present themselves: there may not be maximal elements in this "ideal"; if we insist on the pieces being orthogonal but maximal, there may be problems with uniqueness.

I do rather doubt that there is a simple decomposition theorem here. Probably one should look at the existing results concerning decomposition into "factors", subtract from that what is unnatural from our point of view, then add orthogonality.

CHAPTER 17 DECIDABILITY AND UNDECIDABILITY

This chapter is concerned with the question: for which rings R is the theory of R-modules decidable? That is, for which rings is there an effective way of deciding whether or not a given sentence is true in every module?

The first section begins with some definitions and discussion, for the benefit of those unfamiliar with decidability questions. Then it is noted that since, for instance, the word problem of a ring is interpretable within the theory of its modules, we should impose some minimum conditions on the ring before the question: "is the theory of modules decidable?" becomes a reasonable one. I suggest such a condition: one should be able to tell effectively whether certain systems of linear equations have solutions in the ring. It is noted that a ring of finite representation type has decidable theory of modules (assuming it satisfies this condition). It is also shown that decidability of the theory of modules is preserved by "effective Morita equivalence".

If the word problem for groups is interpretable within the theory of R-modules, then that theory is undecidable. In §2 we use this fact to establish undecidability of the theory of modules over a variety of rings. It is conjectured that any ring of wild representation type has undecidable theory of modules.

In the third section, we turn to decidability. Although the first decidability results were proved "with bare hands", all present results may be achieved by giving an explicit description of the topology on the space of indecomposable pure-injectives. By a result of Ziegler, that is enough to establish decidability of the theory of R-modules. I do not give proofs in this section, because the results tend to be considerably more difficult and deep than the undecidability results of section 2. This is not surprising: in order to show undecidability one interprets a problem which is already known to be undecidable - this often requires some ingenuity, but experience suggests that persistence will be rewarded; on the other hand, to show decidability one has, in effect, to prove a classification theorem for the pure-injective modules, so a different kind of understanding of the modules is required.

The fourth section simply brings together the results of the second and third. We see that what is known supports the conjecture that decidability of the theory of modules is equivalent to the ring being of finite or tame representation type. The conjecture is valid for path algebras of quivers without relations. But, especially since there are no such algebras of tame, non-domestic, representation type, there remains a lot of work to be done here.

17.1 Introduction

To say that the theory of R-modules is decidable is to say that there is an algorithm (or Turing machine) which, when input with any sentence in the language of R-modules will eventually output "yes" or "no", according as the sentence is true in every R-module or not. Lest the requirement that *every* module satisfy the sentence seem overly restrictive, it should be pointed out that a sentence may be of the form $\sigma \to \tau$ where σ and τ are themselves sentences ("conditions").

For example, the word problem for the ring R is "interpretable" in the theory of R-modules: two algebraic combinations, t_1 and t_2, of elements of R define the same element of R iff the sentence $\forall v(vt_1 = vt_2)$ is true in every R-module, hence iff this sentence is in the theory of R-modules. It follows that if there is an algorithm for deciding sentences of the theory of R-modules, then there is an algorithm for the word problem of R. That is, if the theory of R-modules is decidable then R has solvable (i.e. decidable) word problem.

It follows that it is rather pointless to ask whether the theory of R-modules is decidable if R does not have solvable word problem. On the other hand, the following shows that having solvable word problem is not a sufficient condition on R to make the question a sensible one.

Let a and b be elements of the ring R. Whether or not a is in the right ideal bR is decided in the theory of R-modules. For one may check that the sentence $\forall v(vb = 0 \to va = 0)$ is in $\mathrm{Th}(\mathcal{M}_R)$ iff $a \in bR$ ("$\Leftarrow$" is clear; for "$\Rightarrow$" consider the module R/bR).

Below, I discuss the problem of what are appropriate requirements to put on R before asking about decidability of the theory of modules.

In general, if T is a theory, one says that T is **decidable** if there is an algorithm which, when input with any sentence in the language of T-structures, eventually comes up with "yes" or "no" according as the sentence is true in every model of T or not.

If T is a recursively axiomatised *complete* theory, then T is decidable. That is: if there is a Turing machine which will list a (possibly infinite) set of axioms for T, then one may set off a machine to generate all proofs from axioms of T (using a "diagonal" pattern, temporarily breaking off tasks to begin new ones) - thus all consequences of the axioms may be generated. Since T is complete, for any sentence σ either σ or $\neg\sigma$ is in T and so one or the other eventually will be deduced as a consequence of the axioms. Thus T is decidable. The reader will appreciate that a decidable theory may be a long way from being practically computable: in many cases, however, the decision procedure is primitive recursive or better and so provides a more realistic computability.

There is a very useful result due to Feferman and Vaught which simplifies matters. I illustrate the use of their result by showing that if R is a finite ring which is of finite representation type then the theory of R-modules is decidable (this is "folklore"). The special case of the three subspace problem was dealt with in [Bau75a; Thm 3]: of course, most of Baur's proof is, essentially, devoted to showing finite representation type.

So let R be a finite ring of finite representation type: assume that R is given "explicitly". Then every R-module is isomorphic to a product $N_1^{(\kappa_1)} \oplus \ldots \oplus N_k^{(\kappa_k)}$ and hence is elementarily equivalent to $N_1^{\kappa_1} \oplus \ldots \oplus N_k^{\kappa_k}$, where $N_1, \ldots, N_k$ is a complete list of the indecomposable R-modules. The Feferman-Vaught theorem ([FV59; 5.4]) allows us to conclude that the theory of R-modules is decidable iff, for each i, the theory of powers of N_i is decidable. By [FV59; 5.5] the theory of powers of N_i is decidable iff that of N_i is decidable. But N_i, being finitely generated (**11.10**), is finite and so (exercise) has decidable theory. One concludes that the theory of R-modules is decidable.

It should be remarked that it is not necessary to assume that R is finite, just that R is "sufficiently recursive" (see below).

Let us now address the problem: over which rings is the question "is the theory of modules decidable?" a reasonable one? The point is this: we want to identify when the theory of modules is undecidable on account of the compexity of the modules, rather than on account of the compexity of the ring itself. The problem is not a precisely formulated one: nevertheless, I do offer a tentative solution below.

The following seems to be essential: the ring should be recursively presented and the operations should be recursive. That is, there is an algorithm for listing all the elements of the ring (so, in particular, the ring should be countable) and there is an algorithm which, given two elements in this list, produces their sum and their product as elements on the list. A common way of listing the elements of the ring is by giving generators, so that the elements are (certain) words in the generators. It also seems essential that the word problem for the ring should be decidable: in other words there is an algorithm which, given two terms obtained using plus and times, from elements of our listing of the ring, tells us whether or not these terms define equal elements of the ring. Observe that if this is assumed, then the listing may be cut down to one which is a 1-1 listing, and also the operation of finding $-a$ from a is an effective one (we know that $-a$ occurs somewhere on the list, so just go through the elements b of the listing, testing whether $a+b$ equals 0 (0 itself is similarly identifiable)). These basic assumptions will be made, so we will assume that we "know" the ring at least to this extent.

We must assume more: it was observed above that, since questions such as "$a \in bR$?" are effectively "encoded" in the theory of the modules, unless they are effectively answerable, the theory of R-modules cannot be decidable.

What is the maximum assumption? Surely that the entire first-order theory of the ring, in the language for rings augmented by constants for its elements, is decidable. Let us test the reasonableness of this by taking the ring to be a field K. It is a classical result that if K is finite, or is an "explicitly given" countable algebraically closed field, then the theory of K, with constants for its elements, is decidable. On the other hand, the theory of the rational field $\mathbb{Q}$ is undecidable (since it interprets that of the ring of integers $\mathbb{Z}$ [Ro49]), yet the theory of

$\mathbb{Q}$-vectorspaces is decidable (directly, or since $\mathbb{Q}$ is of finite representation type) and, come to that, the theory of $\mathbb{Z}$-modules is decidable. So it certainly may be that the theory of R as a ring is undecidable, yet the theory of R-modules is decidable.

It seems, then, that we must require some restricted part of the theory of the ring to be decidable. By the observations above, this restricted part must include the quantifier-free part of the theory, together with at least some existential sentences $(\exists w\,(a = bw))$.

I make the following suggestion.

Whenever the question of the decidability of the theory of R-modules is raised, it will be (implicitly) assumed that the largest complete theory of R-modules, T^*, is decidable ... (D).

At first sight, this may not look like a condition on the theory of the ring R: let us see why it is so. Since T^* is complete, decidability of T^* is equivalent to T^* having a recursive axiomatisation. An axiomatisation of T^* is provided by all the sentences in T^* of the form $\mathrm{Inv}(-,\varphi,\psi) > 1$ or $\mathrm{Inv}(-,\varphi,\psi) = 1$, where φ and ψ are pp formulas in one free variable. How do we decide whether or not $\mathrm{Inv}(T^*,\varphi,\psi) = 1$? By **8.14** $\mathrm{Inv}(T^*,\varphi,\psi) = 1$ iff the free realisation of φ satisfies ψ: by **8.13**, this occurs iff, in the notation of §8.3, $H_\psi \in \langle H_\varphi \rangle$, where $\varphi(v)$ has the form $\forall \overline{w}\,(v\ \overline{w}).H_\varphi = 0$, and similarly for ψ.
Now, **8.10** provides a criterion for this: namely $H_\psi \in \langle H_\varphi \rangle$ iff there are matrices G, X with entries in R, such that $\begin{pmatrix} 1 & \\ & G \\ 0 & \end{pmatrix} H_\psi = H_\varphi X \ldots (**)$. Since the sizes of H_φ and H_ψ are fixed by φ and ψ, so are the sizes of G and X.
Thus the truth of $\mathrm{Inv}(T^*,\varphi,\psi) = 1$ is equivalent to solvability of the system of linear equations given by the above matrix equation (treat the entries of G and X as unknowns; the entries of H_φ and H_ψ are constants given by φ and ψ). Solvability of this system is expressed as an existential sentence (of a rather simple sort) in the theory of the ring with constants for its elements.

Thus the condition above is indeed a requirement that a certain part of the theory of the ring be decidable. It should also be pointed out that it is the exact generalisation to pp-types of the condition that all questions "$a \in bR$?" be effectively answerable.

Observe that if the theory of R-modules is decidable then condition (D) must be satisfied (since T^* can be then be recursively axiomatised).

Let us now list some examples of rings which do satisfy condition (D).

(i) Any recursively presented field with decidable word problem satisfies (D) - Gaussian elimination gives a decision procedure!

(ii) Let us say that the algebra A, finite-dimensional over a field K, is **recursively presented over** K if there is given a K-basis $\{a_1,\ldots,a_n\}$ of A over K, together with the multiplication constants c_{ijk} which say how the basis elements multiply together: $a_i a_j = \sum_{k=1}^{n} c_{ijk} a_k$. If one identifies an element $a = \sum_{i=1}^{n} d_i a_i$ of A with the corresponding $(d_1,\ldots,d_n) \in K^n$, then one sees that the theory of A is effectively interpretable in the theory of K. Therefore, if K is recursively presented with decidable word problem then so is A (cf. **[Rs80; §1]**): does A further satisfy condition (D)? Indeed it does: since the a_i form a K-basis, the system of A-linear equations obtained from a matrix equation such as $(**)$ above may be teased apart to obtain an equivalent system of K-linear equations which, by (i), is effectively solvable.

(iii) Let the ring R be $\mathbb{Z}$ or $K[X]$ where K is a field as in (i). Then solvability of (and a solution to) any system of R-linear equations may be effectively determined (see, for example, **[HH70]** or **[Jac74]**: the method depends only on the Euclidean algorithm; the latter reference also deals with PID's). Hence R satisfies condition (D).

(iv) Let R be any recursively presented finite ring with decidable word problem. Since the entire theory of R is decidable, one certainly has condition (D) satisfied.

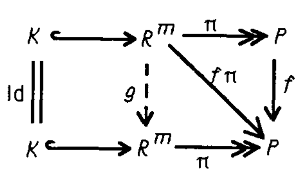
Therefore I assert that, if R is any ring such as described above, it is reasonable to ask whether or not the theory of R-modules is decidable.

The final point to be made in this section is that decidability of the theory of modules is invariant under "effective Morita equivalence": this material comes from [PoPr8?; §1].

The rings R and S are **Morita equivalent** if their categories of modules are equivalent as categories. Given a ring R, the rings S which are Morita equivalent to R may be characterised, up to isomorphism, as those of the form $\mathrm{End}P_R$ where P_R is a finitely generated projective generator for $\mathcal{M}_R$. It should be observed, however, that even isomorphism between rings need not preserve effectivity properties (see [PoPr8?]). Therefore we must be careful to specify what is meant by "giving" a Morita equivalence.

Fix a ring R satisfying condition (D). A Morita equivalence is specified by "giving" a finitely generated projective generator. We may do this by specifying a submodule, K, of some free module R^m, by giving a finite set of elements of R^m which are to be generators for K. Abstractly, this defines P by the short exact sequence $0 \longrightarrow K \longrightarrow R^m \longrightarrow P \longrightarrow 0$: since P is projective, this sequence splits, so P may be thought of as a direct summand of R^m. But we are then faced by the problem: can (generators for) P be found computably from the generators of K? I do not know whether there is a general algorithm for doing this. Over certain sorts of rings it may be done: for example if R is right noetherian and satisfies the strengthened form of condition (D) which says that we may determine recursively whether a system of linear equations has a *non-zero* solution, then (exercise), generators for P may be found recursively from those for K.

Actually, it turns out that generators for K suffice, since it is not actually P that we need to know; rather we need to know the endomorphism ring, S, of P. This ring may be characterised as the ring of endomorphisms of R^m (i.e. $m \times m$ matrices) which send K to K, modulo those which send R^m to K. To see this, consider the diagram below.



Since we have explicit generators for K, the endomorphisms of R^m fixing K are recursively identifiable and, also, the condition "$f-g$ sends R^m to K" is recursive (both by condition (D)). Thus one may find a recursive presentation of S, under which S has solvable word problem and also (exercise) satisfies condition (D).

This is all that we need, but it is perhaps worth pointing out that it is, perhaps, unlikely that we would know that K is a direct summand of R^m without being "given" that information in a computable fashion (for example, by being given generators for a complement, P).

The fact that P is a generator is, in fact, irrelevant to the proof below: even that P is projective is only needed to give us that every S-module is isomorphic to one of the form $M^* = \mathrm{Hom}(P,M)$ for some R-module M. The S-action on M^* is defined by setting $a.s$ ($a \in M^*$, $s \in S = \mathrm{End}P$) to be the composition of functions as. This allows us to think of an element of the S-module M^* as being the same as a morphism from P to the corresponding R-module M. That, in turn, is just a morphism from R^m to M which factors through K. Suppose that the given generators of K are $b_1, \ldots, b_s$ where $b_k = \Sigma_{j=1}^m e_j t_{jk}$, the e_j being canonical generators for R^m. Then a morphism $f: R^m \longrightarrow M$ with $fe_j = a_j$ (say) factors through R/K iff the images of the b_k are all zero - that is, iff $\Sigma_{j=1}^m a_j t_{jk} = 0$ for each $k = 1, \ldots, s$. Given an element a^* of M^*, let $\bar{a} = (a_1, \ldots, a_m) = M^m$ be the corresponding m-tuple of elements from M.

Lemma 17.1 [PoPr8?; 1.1] *Let notation be as above. Let* $\varphi(v_1,\dots,v_n)$ *be a formula in the language of S-modules. Then there is a formula* $\overline{\varphi}(\overline{v}_1,\dots,\overline{v}_n)$ *in the language of R-modules such that, for any elements* $a_1,\dots,a_n$ *in any S-module* M^*, *one has* $M^*\models\varphi(a_1,\dots,a_n)$ *iff* $M\models\overline{\varphi}(\overline{a}_1,\dots,\overline{a}_n)$.
If R *satisfies condition* (D) *and if a finite generating set for* K *is given explicitly, then* $\overline{\varphi}$ *may be found effectively from* φ.

Proof The proof goes by induction on the complexity of the formula φ.

1. Suppose that φ is atomic: say it is $\sum_{i=1}^{n} v_i s_i$, where the s_i are in S.

We are supposing that each s_i is given as an endomorphism of R^m: that is, as an $m\times m$ matrix $(r_{jli})_{jl}$ - so, thinking of S acting on the right, we have $s_i(e_l)=\sum_{j=1}^{m} e_j r_{jli}$. Therefore the equation $\sum a_i s_i=0$ says that the morphism $\sum a_i s_i: P_R \longrightarrow M_R$ is the zero morphism: that is, $\bigwedge_{l=1}^{s}\sum_{i=1}^{n} a_i s_i.e_l=0$. Substituting for $s_i e_l$, this becomes $\bigwedge_{l=1}^{s}\sum_{i=1}^{n}\sum_{j=1}^{m} a_i e_j r_{jli}=0$.

The formula $\bigwedge_{l=1}^{s}\sum_{i=1}^{n}\sum_{j=1}^{m} v_{ij} r_{jli}=0$ is $\overline{\varphi}=\overline{\varphi}(v_{11},\dots,v_{1m},v_{21},\dots,v_{nm})$ and it follows from the discussion above, that $M^*\models\varphi(a_1,\dots,a_n)$ iff $M\models\overline{\varphi}(\overline{a}_1,\dots,\overline{a}_n)$.

2. The other non-trivial case is that φ has the form $\exists v_0\,\psi(v_0,v_1,\dots,v_n)$, where we may assume by induction that the result holds for ψ.

Recall that an element of $M^*=\mathrm{Hom}(P,M)$ is given by an m-tuple $(c_1,\dots,c_m)$ of elements of M, the entries of which satisfy $\bigwedge_{k=1}^{s}\sum_{j=1}^{m} c_j t_{jk}=0$, where the $t_{ji}\in R$ are defined above.

Define $\overline{\varphi}(\overline{v}_1,\dots,\overline{v}_n)$ to be $\exists\overline{v}_0\,(\overline{\psi}(\overline{v}_0,\dots,\overline{v}_n)\wedge\bigwedge_{k=1}^{s}\sum_{j=1}^{m} v_{0j}t_{jk}=0)$, where $\overline{v}_0$ is $(v_{01},\dots,v_{0m})$ (and where $\overline{\psi}$ has been inductively defined).

If $M^*\models\varphi(a_1,\dots,a_n)$, then there exists a_0 in M such that $M^*\models\psi(a_0,\dots,a_n)$. By the induction hypothesis, this implies that M satisfies $\overline{\psi}(\overline{a}_0,\dots,\overline{a}_n)$ and hence $M\models\overline{\varphi}(\overline{a}_1,\dots,\overline{a}_n)$ (since the entries of $\overline{a}_0$ satisfy the required condition). Conversely, if $M\models\overline{\varphi}(\overline{a}_1,\dots,\overline{a}_n)$ - say M satisfies $\overline{\psi}(\overline{b},\overline{a}_1,\dots,\overline{a}_n)$ for some $\overline{b}$ - then the condition on the entries of $\overline{b}$ ensures that $\overline{b}$ "is" an element "b" of M^*. So, by induction, one has $M^*\models\psi(b,a_1,\dots,a_n)$, that is, $M^*\models\varphi(a_1,\dots,a_n)$, as required.

3. Define $(\varphi\wedge\psi)^{-}=\overline{\varphi}\wedge\overline{\psi}$ and $(\neg\varphi)^{-}=\neg\overline{\varphi}$: it is immediate that this preserves correctness of the translation.

The statement regarding effectivity follows since the hypotheses imply that, given φ, we may write down $\overline{\varphi}$. □

Theorem 17.2 [PoPr8?; 1.2] *Suppose that* R *is a recursive ring which satisfies condition* (D). *Suppose also that a finite set of generators for a submodule* K *of* R^n *is given, and assume that* R^n/K *is a projective generator for* $\mathcal{M}_R$. *Let* S *be the corresponding ring Morita equivalent to* R. *Then* S *has a recursive presentation and satisfies condition* (D). *If the theory of R-modules is decidable then so is the theory of S-modules.*

Proof Let σ be a sentence in the language of S-modules. By 17.1, σ is true in some S-module iff $\overline{\sigma}$ is true in some R-module, in other words, iff $\neg\overline{\sigma}$ is not in the theory of R-modules. The theorem follows (cf. the discussion at the beginning of §3). □

One may therefore say that "effective Morita equivalence" preserves decidability of the theory of modules.

The following is a useful perspective on the above. Given any Grothendieck abelian category with a generating set of finitely presented objects (such as $\mathcal{M}_R$ with $\{R\}$, $\mathcal{M}_R$ with $\{R^n : n\in\omega\}$ or (mod-R,Ab) with $\{(M,-): M\in\text{mod-}R\}$), one may define a language for that category, with one sort for each element of the chosen generating set (see [Pr78a]). Given an object of the category, its "elements" of a given sort are the morphisms from the corresponding generator to that object. For example, an element of an R-module is "really just" a morphism from R_R to that module: similarly, with R, P and S as before 17.1, an element of an S-module is "really just" a morphism from the finitely presented generator, P, of $\mathcal{M}_R$ to a certain R-module. For any given such category $\mathcal{C}$ there is a good deal of choice in generating set and hence in language (it is not the category of modules which changes under Morita equivalence, but rather the way in which it is presented). What 17.1 (and its obvious generalisations) say, is that, so long as we confine ourselves to generating sets (always of finitely presented objects) which are recursively equivalent, we will have recursively equivalent languages: hence properties such as decidability are invariant.

An example of this more general situation is provided by tilting functors. It is noted in §3 that the results of §13.3 allow us to conclude that the path algebra of an extended Dynkin diagram has decidable theory of modules. Use of tilting functors (see [Ri84]) allows us to extend this result to certain other algebras: indeed their use means that we need prove decidability only for a single preferred orientation of each quiver. A tilting functor is not a Morita equivalence, but it is almost so: the projective P is replaced by, say, a preprojective module, and one has that the two relevant module categories have very large (finitely axiomatisable) equivalent subcategories, the equivalence being induced by $\mathrm{Hom}(P,-)$. Thus (the proof of) 17.1 applies to these subcategories. The modules excluded from these subcategories are those which have a summand isomorphic to one of a fixed finite set of finitely presented indecomposables, and they may be treated separately. This is enough (see [Pr85a] for more detail) to allow decidability to be transferred, provided the tilting (preprojective) module is given explicitly.

17.2 Undecidability

Our proofs of undecidability depend upon the fact that there is a finitely presented group with undecidable word problem: there is no algorithm which, given any set of generators $g_1,\ldots,g_l$ and any set of words $t_1,\ldots,t_k,t$ in the generators, will decide whether or not the relation $t=\mathrm{id}$ is a consequence of the relations $t_i=\mathrm{id}$ $(i=1,..,k)$. Since all possible descriptions of algorithms may be combined to describe a "universal algorithm", there is some such sequence $(g_1,\ldots,g_l;\ t_1,\ldots,t_k;\ t)$ which specifies a group G in which "$t=\mathrm{id}$" cannot be determined computably from the given defining relations for G.

Baur [Bau75a], [Bau76a] and, independently, Mart'yanov [Mrt75], Slobodskoi and Fridman [SF75] and Kokorin and Mart'yanov [KM73], showed that various theories of modules were undecidable. Baur first interpreted G in the theory of abelian groups of order bounded by (any given) $n\geqslant 2$, together with two specified endomorphisms: in effect, he showed that, for $p\leqslant n$, the theory of $\mathbb{Z}_p\langle X,Y\rangle$-modules is undecidable, by "encoding" the sentence $\forall\bar{v}\,(\bigwedge_{i=1}^{k} t_i(\bar{v})=\mathrm{id}\rightarrow t(\bar{v})=\mathrm{id})$ in the theory of these modules. Decidability of the theory of $\mathbb{Z}_p\langle X,Y\rangle$-modules would therefore contradict choice of G. Actually, his argument allows any field in place of $\mathbb{Z}_p$ and since, for his result, it is enough to consider the case of a prime p (if $n=p.m$, then a $\mathbb{Z}_p\langle X,Y\rangle$-module is just a $\mathbb{Z}_n\langle X,Y\rangle$-module which annihilates m), I present the argument in that generality.

<u>Theorem 17.3</u> [Bau75a; Thm 1], [KM73] *Let K be any field. Then the theory of $K\langle X,Y\rangle$-modules is undecidable.*

Proof Let G be a group as above, with undecidable word problem: it is known that G may be taken to have two generators, g and h. The relators t_i are terms in g, h and their inverses. Let s_i be the corresponding elements of the ring $K\langle X,Y,X^{-1},Y^{-1}\rangle$, where g and h have been replaced by X and Y. Similarly, let s be the ring element corresponding to t. The question of whether $t(g,h)=1$ is a consequence of $\bigwedge_{i=1}^{t} t_i(g,h)=1$, is equivalent to the question of whether $\forall v\,(\bigwedge_{i=1}^{t} vs_i = v \to vs = v)$ is in the theory of $K\langle X,Y,X^{-1},Y^{-1}\rangle$-modules (and so the latter theory is undecidable). Certainly if the relations do imply $t=\mathrm{id}$, then this sentence will hold in every module. If, on the other hand, $t\neq \mathrm{id}$, let M be the module with underlying K-vectorspace that of the group-ring $K[G]$ and with the actions of X and Y being those induced by the actions of g and h on G. Since the action of G on $K[G]$ is faithful, one has that s does not have the identity action on M, as required. □

<u>Corollary 17.4</u> *Let K be any field. Then the theory of $K\langle X,Y,X^{-1},Y^{-1}\rangle$-modules is undecidable.*

Proof The proof of **17.3** actually shows this. □

We now enlarge our collection of rings with undecidable theory of modules. Whenever we have a theory, T, such that there is a finitely axiomatisable subclass of its models which is equivalent to some class $\mathcal{C}$, already known to be undecidable, it follows that T is undecidable (if σ is an undecidable sentence for $\mathcal{C}$ and if τ axiomatises the equivalent subclass of the models of T then $\tau\to\sigma$ is an undecidable sentence for T).

<u>Corollary 17.5</u> [Bau75a; Thm 2], [SF75; Thm 1] *Let K be any field. Then the theory of K-vectorspaces, with five specified subspaces, is undecidable.*

Proof This theory interprets that of $K\langle X,Y\rangle$-modules. First, let us see how to interpret the theory of $K[X]$-modules within the theory of K-vectorspaces with four specified subspaces.

Let M be a $K[X]$-module. Define U to be the underlying space of $M\oplus M$. Let W_i $(i=1,\dots,4)$ be the K-subspaces $M\oplus 0$, $0\oplus M$, Δ, $\mathrm{Gr}\,X$, where Δ is the diagonal $\{(v,v): v\in U\}$ and $\mathrm{Gr}\,X$ is the graph of X - $\{(v,vX): v\in U\}$. The definable recovery of M from this "quadruple" is left as an exercise for the reader. (Finitely) axiomatising the quadruples so obtained is also left as an exercise (cf. **[Bau80]**).

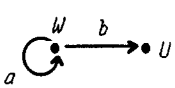
The theory of "quintuples" therefore interprets that of $K\langle X,Y\rangle$-modules: take the fifth subspace to be the graph of the action of Y on "M". □

The next result is a corollary, since the theory of pairs of abelian p-groups interprets the undecidable (by the above) theory of quintuples of $\mathbb{Z}_p$-modules. This is proved by Baur [Bau76a] and also by Mart'yanov [Mrt75] who uses the result of Slobodskoi and Fridman [SF75]. In fact, Baur shows that pairs of $\mathbb{Z}_{p^9}$-modules suffice. Mart'yanov shows that the theory of pairs of $\mathbb{Z}_{p^{2n+1}}$-modules interprets that of n-tuples of $\mathbb{Z}_p$-modules so, from [SF75], he obtains the slightly weaker result that pairs of $\mathbb{Z}_{p^{11}}$-modules are undecidable. In fact, Butler and Brenner inform me that pairs of $\mathbb{Z}_{p^6}$-modules are wild (and even this may not be optimal) so they should be undecidable. For the details of the proofs, I refer the reader to the above works.

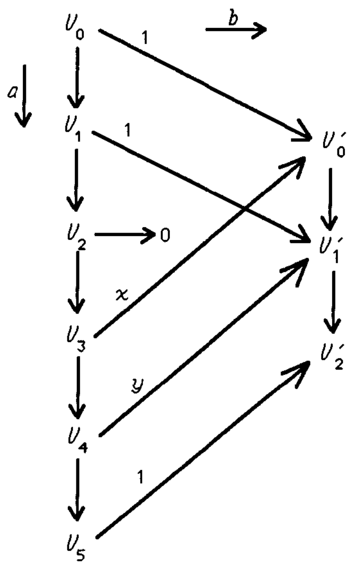
<u>Theorem 17.6</u> [Bau76a; Thm 1, Cor 4], [Mrt75] for abelian groups *Let K be any field. Then the theory of pairs $(M;N)$ of $K[X]$-modules with $M\geqslant N$, is undecidable. Analogously, the theory of pairs of abelian groups is undecidable (cf.* **17.15**). □

<u>Corollary 17.7</u> *Let K be any field. Then the theory of triples $(U;W,f)$ consisting of a K-vectorspace U, a subspace W and an endomorphism f of U, is undecidable.*

Proof To the pair of $K[X]$-modules $(M;N)$ associate the triple $(M_K;N_K,X)$, and note that $NX\leqslant N$ characterises the triples so obtained, to see that this follows immediately from **17.6**.



This can, however, be derived in a simpler fashion (without appealing to **17.6**). What I do first is to show that the theory of representations of the quiver opposite is undecidable, by interpreting in it the theory of vectorspaces with two specified endomorphisms. So, given a vectorspace V and endomorphisms x and y, we will build a Δ-representation from them, in such a way that (V,x,y) may be definably recovered.



Consider the Δ-representation defined by the diagram shown, where each $V_i^{(\prime)}$ is a copy of V and the action of a and b are as shown ("1" is the identity morphism). Let us see how to recover the original space and endomorphisms.

For the copy of V, we take V_2/V_3 - this is definable as $\text{im}\,a^2/\text{im}\,a^3$ (so we are working with elements of T^{eq}). Now, take $v \in V_2$: v has "coordinates" $(0,0,v_2,v_3,v_4,v_5)$ say (with respect to the given decomposition of W). Then $vab=(v_2x,v_3y,v_4)$. Now, there exists $w \in W$ such that $wb=vab$ and $wa^2b=0$: let us compute the coordinates of $w=(w_0,w_1,\ldots,w_5)$. We have $wb=(w_0+w_3x,w_1+w_4y,w_5)$ and $wa^2b=(w_1x,w_2y,w_3)$: hence $w_3=0$, so $wb=(w_0,w_1+w_4y,w_5)$ and therefore $w_0=v_2x$. Then $wa^2=(0,0,v_2x,w_1,w_2,w_3)$ and so there is induced, by $v \mapsto wa^2$, a function on V_2/V_3 which takes $v+\text{im}\,a^3$ to $vx+\text{im}\,a^3$. So we have recovered the x-action.

Again: let $v \in \text{im}\,a^2$ have coordinates as before. Then $va^2b=(0,0,0,0,v_2,v_3)b=(0,v_2y,v_3)$. Then there exists $w \in \text{im}\,a$ such that $wb=va^2b$ and $wab=0$. Suppose that $w=(0,w_1,\ldots,w_5)$. Then $wab=(w_2x,w_3y,w_4)$ and so $w_4=0$. Therefore $wb=(w_3x,w_1+w_4y,w_5)=(w_3x,w_1,w_5)$: so $w_1=v_2y$. Then $wa=(0,0,v_2y,w_2,w_3,w_4)$: so there is induced, by $v \mapsto wa$ a function on V_2/V_3 which is the recovered y-action.

Therefore we have undecidability of the theory of Δ-representations. What we want, for the theorem, is undecidability of the Δ'-representations, where Δ' is as Δ, but with the arrow "b" reversed (for the Δ-representations with the connecting arrow monic are exactly the triples in the statement of the theorem). We reverse the arrow by replacing the Δ-representation $M=(W,U,a,b)$, as above, by the Δ'-representation $M'=(W,U'=\ker b,a,j)$, where j is the canonical inclusion of $\ker b$ into W (this is a simple example of a tilting functor). We can certainly axiômatise the Δ'-representations so obtained; so we must check that the original representation M may be definably recovered from M'.

But this is easy: "U" is recovered as W/U' and then "b" is recovered as the projection from W to W/U' (we may note that "b" in the original diagram is epi). (That is, W is a sort in T^{eq} and b is the corresponding function "f_E" in the notation of §10.T.)

Thus the theorem is proved. ◻

<u>Corollary 17.8</u> [Bau76a; Cor 4] *Let K be a field. Then the theory of modules over $R=K[X,Y:Y^2=0]$ is undecidable: in particular, the theory of $K[X,Y]$-modules is undecidable.*

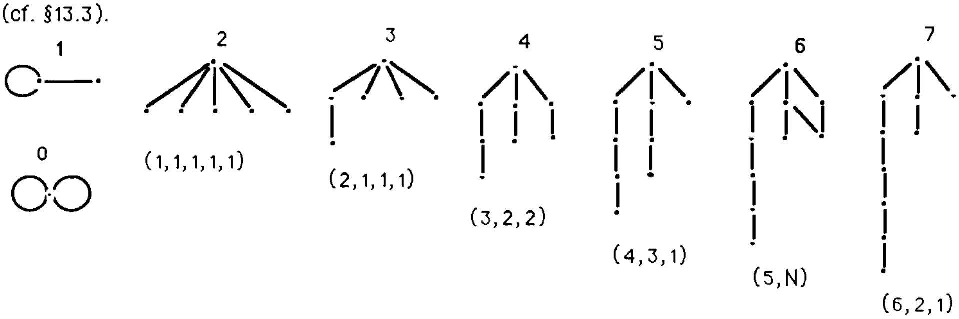
Proof The theory of R-modules is bi-interpretable with the theory of pairs of $K[X]$-modules as follows. To the R-module M, associate the pair $(\ker Y, \text{im}\,Y)$ of $K[X]$-modules. Conversely, given the pair $(M;N)$, $M \geqslant N$, of $K[X]$-modules, define the corresponding $K[X,Y]$-module to have underlying $K[X]$-structure that of the direct sum $M \oplus N$ and define the action of Y to be zero on the first component and to embed the second component identically in the first. □

The indeterminate "X" may be replaced by a prime (cf. 17.19 below). As a consequence of his proof of 17.6, Baur showed that the theory of $\mathbb{Z}_{2^9}[x : x^2 = 0]$-modules is undecidable [Bau76a; Cor3]: in fact one may do better that this. For a proof, I refer to [Pr8?; 4.11].

<u>Proposition 17.9</u> *The theory of* $\mathbb{Z}_{2^3}[x : x^2 = 0]$*-modules is undecidable.* □

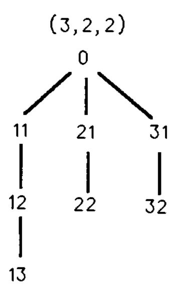
With these examples of undecidable theories to hand, it may be shown that, within various classes, wild representation type implies undecidability of the theory of the representations.

1. Quivers There is a list of minimal wild quivers: they are 1-5 and 6 in the diagram below (cf. §13.3).



The first three correspond to the theory of $K\langle X,Y\rangle$-modules, triples $(U;W,f)$ and the 5-subspace problem respectively: these were seen above to be undecidable (17.3, 17.7, 17.5). It may be shown that the category of representations of each of the remaining quivers embeds the category of representations of the first quiver. I give one example below (I treat it as a poset: any representation of the poset gives a representation of the quiver); for the others see [Pr85a; Part1] (also see [Br74a; Prop2]). These embeddings actually provide interpretations, in the sense that the original triple (U,W,f) may be definably recovered; so we deduce that the path algebra of each of these quivers has undecidable theory of modules. Hence the path algebra of any wild quiver without relations has undecidable theory of modules.

2. Posets The list of minimal wild posets is just the list above, of minimal wild diagrams, but with 1 deleted and with 6 added. Given the above, it has only to be shown that the theory of the representations of the poset (N,5) is undecidable. For that, see [Pr85a; Part1] (or consider it as an exercise). Then one may conclude that the category of representations of any wild poset is undecidable.



$U_0 = U \oplus U \oplus U$

$U_{11} = U \oplus U \oplus 0$ $\quad U_{21} = 0 \oplus U \oplus U$ $\quad U_{31} = \Delta + \langle (v,v,fv) \rangle$

$U_{12} = U \oplus 0 \oplus 0$ $\quad U_{22} = \langle (0,v,fv) \rangle$ $\quad U_{32} = \Delta$

$U_{13} = W \oplus 0 \oplus 0$

Given a "triple" (U,W,f), the representation M of this poset is specified as shown, where Δ is the diagonal submodule $\langle(v,v,v)\rangle$ and "$\langle * \rangle$" means generated by the set of all $*$.
Note that $U_{11} \cap U_{21} = 0 \;U\; 0$.
The domain of "f" is defined to be U_{12}.
Let $x = (v,0,0)$ be a member of U_{12}. Then the representation M satisfies $\exists y \in U_{22}\,(x+y \in U_{31})$: say $y=(0,v',fv')$. Since in U_{31} the first and second coordinates are equal, it must be that $y=(0,v,fv)$.
Then M satisfies $\exists z \in U_{11}(y+z \in U_{32})$: say $z=(w,w',0)$, so $y+z=(w,w'+v,fv)$. In U_{32} all three coordinates are equal: so $w=fv$ and $w'+v=fv$. That is, $z=(fv,fv-v,0)$.
Then M satisfies $\exists z' \in U_{12}, z'' \in U_{11} \cap U_{21}(z=z'+z'')$. Clearly $z'=(fv,0,0)$.
Thus we may define (by pp formulas) $(fv,0,0)$ from $(v,0,0)$. That is: the action of v on $U_{12}(\simeq U)$ is definable. Of course, "W" is definable - being U_{13}. Thus (U,W,f) may be definably recovered from M. Also, the class of representations of $(3,2,2)$ in which the above process does lead to a "triple" are finitely axiomatisable. Thus the theory of representations of $(3,2,2)$ interprets that of triples so, by **17.7**, is undecidable.

3. **Local algebras** Suppose that K is an algebraically closed field and let A be a complete local K-algebra ("complete" in the topology which has, as a neighbourhood basis of 0, the powers of the Jacobson radical - so any local artinian K-algebra is of this sort).

Ringel [Ri75] showed that, modulo the conjecture $(*)$ below, A is of wild representation type iff it has, as a factor ring, at least one of the following algebras.
$A = K[X,Y,Z]/\langle X,Y,Z\rangle^2$; $B = K\langle x,y : x^2=y^3=xy=y^2x=0\rangle$; B^{op};
$C_\alpha = K\langle x,y : x^2=y^3=y^2x=xy-\alpha yx=0\rangle$ $(\alpha \neq 0)$; $D = K\langle x,y : x^2-y^2=yx=0\rangle$.
Ringel's proofs that these algebras all are of wild representation type show, with some modification, that the theory of modules over each of them is undecidable ([Po86; Prop 1]). That this is so does not depend on the field K being algebraically closed: that assumption is used to show that this is a complete list of the "minimal" wild complete local K-algebras. In fact, for that, one needs only that R contains a solution to a certain quadratic polynomial (cf. 4. below), so, modulo the problem of deciding whether tensoring up with a quadratic extension of the original field preserves (un)decidability, the undecidability result may be extended to arbitrary base fields.

The conjecture $(*)$ is the following: the algebras $K\langle x,y : x^2-(yx)^n y = 0 = y^2-(xy)^n x\rangle$ $(n \geqslant 1)$ - "quaternionic" - and $K\langle x,y : x^2-(yx)^n y = 0 = y^2\rangle$ $(n \geqslant 1)$ - "semidihedral" - are of tame representation type.

Bondarenko and Drozd [BD77] prove this for the second type of algebra (and, as a consequence, for certain cases of the first). However, no detailed proofs or explicit lists of the finite-dimensional indecomposables have yet appeared (although there is work in progress on this - see [C-B87b]). The representation type of these algebras is no better than tame of infinite growth and their modules are not obviously of string and band type, so a proof of decidability of the theory of modules over such an algebra seems a long way off.

4. **Commutative finite-dimensional algebras** Drozd [Dro72] showed that if R is a complete commutative local algebra over an algebraically closed field K then either R has, as a factor ring, at least one of the two wild algebras A and C_1 (from the list above), or R is a factor ring of (the completion of) the infinite-dimensional "dihedral" algebra $K[X,Y]/\langle XY\rangle$ or the algebra $K[x,y : x^2=0=y^2]$ (these are discussed in §3 below). Each of the two minimal wild commutative algebras has undecidable theory of modules. Hence a commutative local algebra (over an algebraically closed field, but see §3 below) which is of wild representation type has undecidable theory of modules (see [Po86]). This undecidability may be extended to

commutative local artinian rings by using 17.19 below. It may also be extended to commutative artinian rings by localising ([Po86; 11(i)]) - see §17.3(5,6) below.

17.3 Decidability

The proofs of decidability that I give here all depend on the result of Ziegler which says, roughly, that if the space $\mathcal{I}(R)$ of indecomposables is explicitly known then the theory of R-modules is decidable. The discussion of §1 indicates that we may as well assume that we are dealing with rings which satisfy the condition (D) introduced there.

We may discuss the decidability question in the context of any recursively axiomatised (not necessarily complete) theory, T, of R-modules, which satisfies $T = T^{\aleph_0}$. Since T is recursively axiomatised, the set of all consequences of the axioms (i.e. T) is a recursively enumerable set. So, given a sentence σ to decide, we may set off generating T. If $\sigma \in T$ then we will find this out eventually. If, however, $\sigma \notin T$ then we will never discover this from the enumeration of T, since at any given time, all we know is that σ has not been generated as yet. Observe that $\sigma \notin T$ iff some model of T satisfies $\neg\sigma$. If therefore, we have an algorithm which generates the sentences true in <u>some</u> model of T then, combining this algorithm with that which generates T, we will obtain a decision procedure for T.

This then, will be our goal: given $T = T^{\aleph_0}$ closed under direct summands, recursively axiomatised, to find out whether there is an algorithm which enumerates those sentences σ in the language of R-modules which are true in some model of T.

Note first that σ is true in a model of T iff it is true in a model which is a direct sum of indecomposable pure-injectives (this is by 4.36). Now, σ is equivalent in T to a boolean combination, τ, of invariants sentences: so, if the set of all such τ true in some model of T is recursively enumerable, then so is the set of all their consequences (i.e., the set of all such σ). Observe that it is not necessary to be able effectively to determine τ from σ (although in [Mon75] (for abelian groups) and [Wei85], [Wei83a] it is shown that this is, in fact, possible).

The next point is that if a boolean combination of invariants conditions is satisfied in a direct sum, then it is satisfied in some finite sub-sum (exercise - see [Pr85a] for details of this and various other points which are not fully treated here). So our problem is reduced to the following one: enumerate axioms for the set of all those boolean combinations of invariants conditions which are satisfied in some finite direct sum of indecomposable pure-injective summands of models of T.

Consider a boolean combination, σ, of invariants conditions. If σ is satisfied in some R-module, then we wish to discover this fact. One may express σ as a disjunction of sentences, each of which is a conjunction of invariants conditions. Since a disjunction is satisfied iff one of its disjuncts is satisfied, it may be assumed that σ has the latter form. Therefore σ is a conjunction of statements of the form "$\mathrm{Inv}(-,\varphi,\psi) \in [m,n)$" where m is an integer ≥ 1, n is an integer ≥ 1 or "∞" and $[m,n) = \{r \in \mathbb{Z} : m \leq r < n\}$. Clearly, these reductions are effective.

If all invariants of members of $\mathcal{I}(T)$ are 1 or ∞, then we need not deal directly with points of the space $\mathcal{I}(T)$, but may work instead at the level of open subsets. If, however, there is an indecomposable N which satisfies $N \not\equiv N^{\aleph_0}$, then one may have to descend to the level of points. The disadvantage of working at the level of points is that we have to suppose that $\mathcal{I}(T)$ is countable.

If we remain at the level of topology then we may proceed as follows.

Let us first suppose that all invariants of indecomposable pure-injectives are 1 or "∞". Then σ reduces to the form $\bigwedge_i \tau_i \wedge \bigwedge_j \rho_j$ where each τ_i, resp. ρ_j, has the form $\mathrm{Inv}(-,\varphi_i,\psi_i)=1$, resp. $\mathrm{Inv}(-,\varphi_j,\psi_j)>1$. Let σ' be the conjunction of all the τ_i. Then σ' defines a closed subset, C, of $\mathcal{I}(T)$. Clearly then, there will be a module satisfying σ iff there is, for each j, some point in the intersection $(\varphi_j/\psi_j)\cap C$.

Say that a subset of $\mathcal{I}(T)$ is **basic constructible** if it is a finite boolean combination of basic open sets (i.e., those of the form (φ/ψ)). We have just seen, under the "all invariants infinite" hypothesis, that a sentence is satisfied in some module iff the corresponding basic constructible set is non-empty. Therefore, if we can recursively enumerate every non-empty finite boolean combination of sets in the standard basis, then it follows that T is decidable. In fact, and this will be more useful, it will be enough if we can effectively enumerate a "basis for the constructible sets" - that is, a set X of basic constructible sets, such that every non-empty basic constructible set contains a non-empty member of X.

Theorem 17.10 *Let R be a ring which satisfies condition* (D) *such that all invariants of R-modules are "1" or "∞" (for example, let R be an algebra over some infinite field). Suppose that a basis for the non-empty basic constructible sets of $\mathcal{I}(T)$ may be recursively enumerated in the sense that there is an enumeration, (φ_i), of pp formulas such that the set of all finite sequences $(i_{01},i_{02},\dots,i_{n1},i_{n2},f)$ where $f\in 2^{n+1}$ and the constructible set $\bigcap_j(\neg)^{fj}(\varphi_{i_{j1}}/\varphi_{i_{j2}})$ is non-empty, is recursively enumerable, and such that every non-empty basic constructible set contains such a subset. Then the theory of R-modules is decidable.* □

What if not all invariants of members of $\mathcal{I}(T)$ are 1 or ∞? Let σ as above be a conjunction of conditions of the form "$\mathrm{Inv}(-,\varphi,\psi)\in[m,n)$" (where n may be "∞"). The situation now is considerably more complicated. For instance, suppose that we wish to find a module satisfying the conditions $\mathrm{Inv}(-,\varphi,\psi)=k\wedge\mathrm{Inv}(-,\varphi',\psi')\geq m$. If there is some indecomposable N with $\mathrm{Inv}(N,\varphi,\psi)=1\wedge\mathrm{Inv}(-,\varphi',\psi')>1$ then the two conditions may be treated separately. Otherwise, it might still be that $\mathrm{Inv}(N,\varphi,\psi)$ divides k, and then there are various possibilities. But then there might be other conjunctions of σ to take care of, and it may be impossible to do this without altering the values of these invariants. Nevertheless, one sees that satisfaction of σ does break down into a finite number of possibilities. So one obtains the following result, still at the level of topology (I state it just for the theory of R-modules).

Theorem 17.11 [Zg84; after 9.4] *Let R be a ring which is recursively presented and has decidable word problem. Suppose that there is a recursive enumeration of all those conditions of the form $\bigwedge_i \mathrm{Inv}(-,\varphi_i,\varphi_j)\in[m,n)$, which are satisfied by some indecomposable pure-injective R-module; here $(\varphi_i)_i$ is a recursive enumeration of the pp formulas and $m,n\geq 1$, $m\neq\infty$. Then the theory of R-modules is decidable.* □

Another possibility for obtaining decidability is the following; though it could only be applied to a ring with countably many indecomposable pure-injectives (so would exclude even quite "small" rings such as the non-domestic tame algebras discussed in Chapter 13). Since the arguments apply equally well in any closed subset of $\mathcal{I}_R$, one obtains relative versions for theories whose models are closed under direct summands.

Theorem 17.12 [Zg84; 9.4] *Let R be a recursively presented ring with decidable word problem. Suppose that there is a recursive enumeration of the points $\{N_i\}_i$ of $\mathcal{I}(T)$ and a recursive enumeration of a basis $\{(\varphi_j/\psi_j)\}$ for the topology (equivalently, for each i, a recursive enumeration of a neighbourhood basis for N_i) such that the question "$N_i\in(\varphi_j/\psi_j)$" is effectively answerable and, in the case* *that not all invariants are* 1 *or* ∞, *the question* "$\mathrm{Inv}(N_i,\varphi_j,\psi_j) = k$" *must be effectively answerable. Then the theory of R-modules is decidable.* □

The result follows easily from 17.11. Observe that, under the hypotheses of each result, the ring satisfies condition (D) (those axioms of T^* of the form "Inv(-) = 1" are recursively enumerable, being part of the common theory of all R-modules, and the hypotheses of 17.11 and 17.12 imply that those of the form "Inv(-) > 1" are recursively enumerable).

I now list various decidable theories of modules (mainly by describing rings over which the theory common to all modules is decidable). In general, I give only an indication of proofs. For, as the above theorem indicates, the decidability results are consequences of the classification of the indecomposable pure-injectives. These classifications often involve considerable work and it would not be appropriate to give the details in this chapter (some are given or outlined elsewhere in the notes).

1 **The ring of integers and Dedekind domains** Szmielew [Sz55] showed that the theory of abelian groups is decidable. Her proof is a rather involved explicit elimination of quantifiers which provides a decision procedure. This was the first non-trivial case of a ring being shown to have decidable theory of modules. By the early 70's more powerful tools were available, and Eklof and Fisher [EF72] were able to give a more "conceptual" proof of her result, at the same time as generalising it to certain Dedekind domains. They used the structure theory for the pure-injective modules in an essential way, although it should be pointed out that neither Baur's proof of pp-elimination of quantifiers, nor Ziegler's topology was available to them at that time (also, if one is content simply to quote the structure theorem for pure-injective abelian groups - see [Kap69] - then their proof is thereby simplified). They proved the following result.

<u>Theorem 17.13</u> [Sz55; 4.22], [EF72; 5.4] *Let R be a recursively presented Dedekind ring with decidable word problem. Suppose that there is an effective listing (by giving generators) of the distinct maximal ideals,* $\{P_i\}_i$, *of R. Suppose that the cardinality ("*∞*" or a finite integer) of* R/P_i *is computable from* i. *Then the theory of R-modules is decidable.*

Proof The result follows without difficulty from the (explicit!) description of the space $\mathcal{I}_R$ and 17.12 above. □

<u>Corollary 17.14</u> [EF72; 5.4] *Let K be a recursive field with decidable word problem and a splitting algorithm. Then the theory of* $K[X]$*-modules is decidable.* □

The field K has a **splitting algorithm** ([FrSh56; §4], [Rab60; Defn9]) if there is an algorithm which, when presented with any element of $K[X]$, will determine its irreducible factors: examples are: finite fields; $\mathbb{Q}$; any countable algebraically closed field. It is left as an exercise to show that this is sufficient for determination of the structure of $\mathcal{I}_R$.

Other examples of rings where 17.13 applies are rings of integers in certain finite extensions of $\mathbb{Q}$ (see [EF72; after 5.4]).

2 **Pairs of torsionfree modules** Kozlov and Kokorin [KoKo69] proved that the theory of pairs (in the sense of 17.6) of torsionfree abelian groups is decidable. This contrasts with the undecidability of the theory of pairs of arbitrary abelian groups (17.6). The results above, together with his classification of indecomposable pure-injective pairs of torsionfree modules over a Dedekind domain ([Zg84; §5]; cf. after 2.$\mathbb{Z}$12) allowed Ziegler to obtain the following generalisation. For more on this, see [Sch80?]. Also see [Sch82].

<u>Theorem 17.15</u> [Zg84; 9.10] *Let R be a recursive Dedekind ring satisfying the conditions of* 17.13. *Then the theory of pairs of torsionfree R-modules is decidable.* □

3 Tame hereditary algebras Let K be a recursive field with decidable word problem and let Δ be any extended Dynkin quiver. The classification of the indecomposable pure-injectives over the path algebra $K[\Delta]$, as well as the description of the topology, was indicated in §13.3. From that, it follows that if the space $\mathcal{I}_{K[X]}$ is effectively given in either of the ways described at the beginning of this section, then so is $\mathcal{I}_{K[\Delta]}$: one may check that the morphisms of spaces induced by the functors from $K[X]$-modules to $K[\Delta]$-modules are effective (for detail see [Pr85a]). Therefore we have the following, where the case $\Delta=\tilde{D}_4$ is due to Baur (he raised the problem in [Bau75a]).

Theorem 17.16 [Bau80], [Pr58a] *Let K be a recursive field such that the theory of $K[X]$-modules is decidable. Let Δ be a quiver without relations, the underlying diagram of which is extended Dynkin. Then the theory of modules over the path algebra $K[\Delta]$ is decidable.* □

This result may be extended in various ways. For example, by use of tilting and the appropriate version of 17.1, one has the following.

Corollary 17.17 [Pr85a] *Let K be a recursive field such that the theory of $K[X]$-modules is decidable. Let Δ be a quiver without relations, the underlying diagram of which is extended Dynkin. Suppose that M is an effectively given preprojective or preinjective module and let S be the corresponding tame concealed algebra (see* [Ri84]). *Then the theory of S-modules is decidable.* □

Another corollary is obtained by using the fact that, if K is algebraically closed, then every finite-dimensional hereditary K-algebra is Morita equivalent to the path algebra of a quiver (see, e.g., [Gab80; §4]). The projective which induces the Morita equivalence contains just one copy of each indecomposable (projective) direct summand of the algebra, and can be found effectively. Therefore, by 17.1, one obtains a second corollary.

Corollary 17.18 [Pr85a] *Let K be a recursive algebraically closed field such that the theory of $K[X]$-modules is decidable. Let R be a recursively given hereditary finite-dimensional K-algebra of tame or finite representation type. Then the theory of R-modules is decidable.* □

4 Local artinian rings I will show first that the decidability question for modules over local artinian rings reduces to that for modules over algebras finite-dimensional over a field.

Suppose that R is "local" in the weak sense that the factor R/J is a simple ring. Then $R\simeq(eR)^n$ for some n, where eR is the unique (so faithful) indecomposable projective. Let S be End(eR): so S is a ring Morita equivalent to R. Assuming that R satisifes condition (D), it follows by 17.1 that there is an "effective Morita equivalence" between $\mathcal{M}_R$ and $\mathcal{M}_S$. Thus the theory of R-modules is decidable iff that of S-modules is so. The nice point about S is that its unique maximal two-sided ideal is also its unique maximal right ideal: it is local in the strong sense (our usual sense) that, modulo its radical, it is a division ring.

Therefore, consider a local artinian ring R and let us make life simple by assuming that R/J is actually a field (rather than just a division ring). If the characteristic of R is zero then the map which takes 1_K to 1_R extends to a K-linear morphism from K to R; thus R is a K-algebra. If the characteristic of R (i.e., the least positive integer n such that $1_R.n=0$) is finite, then we proceed as follows. The first point to note is that the characteristic of R is a prime power; for if the primes p and q both divide char(R) then both lie in $J(R)$. But, since they are relatively prime, the identity element of R is a linear combination of them and so is in $J(R)$ - contradiction.

Therefore, let R be an artinian ring of characteristic p^n such that R/J is a field K (of characteristic p). I show that the theory of R-modules is bi-interpretable with the theory of modules over a certain K-algebra which may be thought of as the "algebra version of R" (the equivalence of the representation type is "folklore"). This algebra is defined as follows. Let $x_0,\dots,x_k$ be a K-basis of J/J^2 where $x_0=p$. Then, as a ring, R is generated by $1,x_1,\dots,x_t$ and so every element of R may be written as a (non-commutative) polynomial in $p,x_1,\dots,x_t$ with coefficients in K. Since R is artinian, it has a finite presentation by some ideal I. Let R' be the K-algebra $K\langle y_0,\dots,y_t\rangle/I'$, where I' is the ideal corresponding to I (replace x_i by y_i in the polynomial expressions for the generators of I).

A couple of examples may make this clearer. Let R be the ring $\mathbb{Z}_4$. Then the radical is generated by 2, so R' is the $\mathbb{Z}_2$-algebra $\mathbb{Z}_2[y:y^2=0]$. If R is the ring $\mathbb{Z}_{p^3}[x:x^2=0=p^2x]$, then R' is the $\mathbb{Z}_p$-algebra $\mathbb{Z}_p[x,y:x^2=0=y^3=y^2x]$.

<u>Theorem 17.19</u> (cf. [Bau75a; Thm 2]) *Let R be an artinian ring which is such that R/J is a field K. Let R' be the corresponding K-algebra as defined above. Then the theory of R-modules is bi-interpretable with the theory of R'-modules. In particular, the theory of R-modules is decidable iff the theory of R'-modules is decidable.*

Proof Let M be an R-module and let l be minimal such that $MJ^l=0$. Consider the strictly descending chain $M=MJ^0>MJ^1>\dots>MJ^{l-1}>0=MJ^l$ and the corresponding sum of K-vectorspaces $U=M/MJ\oplus MJ/MJ^2\oplus\dots\oplus MJ^{l-2}/MJ^{l-1}\oplus MJ^{l-1}$. Define an R'-action on U by: $(m+MJ^k).y_i=mx_i+MJ^{k+1}$ $(m\in MJ^{k-1})$. This is well-defined since $MJ^k.x_i\subseteq MJ^{k+1}$ and since any zero-relation between the y_i holds also between the x_i. Thus M is converted into an R'-module, which I denote by $R'(M)$. All that has been lost in going from M to $R'(M)$ is the link between addition and multiplication by p (so this is not a functor).

How may (a copy of) M be recovered from $R'(M)$? Let N be any R'-module; split it as $U'=N/NJ\oplus NJ/NJ^2\oplus\dots\oplus NJ^{l-2}/NJ^{l-1}\oplus NJ^{l-1}$, just as above. Observe that there is, for each k, a function $f_k:NJ^k/NJ^{k+1}\longrightarrow NJ^{k+1}/NJ^{k+2}$ which is given by the action of y_0. Let $\overline{m}$ be a K-basis of N/NJ: also let $\overline{r}$ be a K-basis of the ideal of R' generated by the $y_0,\dots,y_t$. Define an abelian group M to have generators $\overline{m}$, "$\overline{m}\overline{r}$", $f_0\overline{m}$, $f_1f_0\overline{m},\dots$ $f_{l-2}\dots f_1f_0\overline{m}$ and to have relations all the following, as well as those holding in $\mathrm{M}.\langle y_0,\dots,y_t\rangle$. If m is in $\overline{m}$ and $s\leqslant\mathrm{char}(R)=p^n$, then write $s=\alpha_0+\alpha_1p+\dots+\alpha_{n-1}p^{n-1}$, where the α_j are in $\mathbb{Z}_p$; put in $ms=\alpha_0m+\alpha_1f_0m+\dots+\alpha_{l-2}f_{l-2}\dots f_1f_0m$ as a relation. Also put in the relations which correspond to using the same rule to compute sums of generators (i.e., whenever a sum of integers becomes greater than or equal to p, expand it just as s was expanded above). To make this an R-module, note that each y_i defines a "degree 1" morphism of the graded space U'; so use these functions to define the x_i-action on M. Write this module as $R(M')$.

It is fairly clear that $M\mapsto R'(M)\mapsto R(R'(M))$ recovers M, and similarly in the other direction. Given any sentence, σ, in the language of R-modules, let σ' be the sentence in the language of R'-modules in which elements of R are replaced by the corresponding elements of R' (after writing each equation in σ in "simplest form"). Then one may see that an R-module M satisfies σ iff $R'(M)$ satisfies σ'. Thus the result follows. □

So, we have reduced the decision problem for modules over local artinian rings with R/J a (matrix ring over a) field, to that for modules over finite-dimensional algebras.

I don't see any obstruction to extending 17.19 to the case where one has non-trivial idempotents (and a division ring in place of the field): such an extension would reduce the decision problem for modules over artinian rings to that for modules over finite-dimensional algebras. But I haven't checked it.

5. **Commutative artinian rings** (This subsection is based on [Po86].) Let R be commutative artinian. Then R is a finite direct product of local rings. By the Feferman-Vaught theorem (see §1) the theory of R-modules is decidable iff the theory of modules over each of these local rings is decidable.

So we may suppose that R is a local commutative artinian ring, with factor field K (say).

Drozd's result [Dro72], already discussed in §17.2(4), implies that if K is an algebraically closed field and if R is a local artinian K-algebra then *either* R has a factor ring of the form $K[X,Y,Z]/\langle X,Y,Z\rangle^2$ or $K[x,y:x^3=y^2=x^2y=0]$ and so is wild *or* R is a factor ring of the Gelfand-Ponomarev algebra $K[x,y:xy=0]$ or the algebra $K[x,y:x^2=0=y^2]$. If one widens the scene to allow non-algebraically closed fields, then (cf. [Po86]) one sees the following possibilities: R is wild and has as a factor ring at least one of $K[X,Y,Z]/\langle X,Y,Z\rangle^2$, $K[x,y:x^3=y^2=x^2y=0]$; R is tame and is a factor ring of one of the Gelfand-Ponomarev algebras $GP_{n,m}=K[x,y:x^n=y^m=xy=0]$ (these are non-domestic for $n+m\geqslant 5$), or is a factor ring of the domestic algebra $K[x,y:x^2=y^2=0]$ (note that $GP_{2,2}$ is a factor ring of this); or there is a quadratic extension field L of K such that $R\otimes_K L$ has one of the above forms.

One may see this from Drozd's proof and more detailed arguments may be seen in [Ri75], where Ringel actually establishes a similar classification for non-commutative local K-algebras (K algebraically closed) - see §17.2(3). I illustrate below how the quadratic field extension comes into the picture.

What can one deduce about decidability for modules over these algebras? Let us start with the wild case.

Algebras of the form $K[X,Y,Z]/\langle X,Y,Z\rangle^2$ have undecidable theory of modules (see §17.2(3)). If $R\otimes_K L$ has a quotient of this form, where L is a quadratic extension of K, then $\dim_K J(R)/J^2(R)\geqslant 3$, hence R also has a quotient of this form, so has undecidable theory of modules. Suppose, for the other case, that R is such that $R\otimes_K L$ has a factor ring of the form $C_1=K[x,y:x^3=y^2=x^2y=0]$; we have to work a bit harder here.

Since the radical of C_1 has cube zero, one quickly reduces to the case where $J(R)^3=0$. We may also, by the above, suppose that $\dim_K J(R)/J^2(R)=2$: say x and y form a K-basis of J/J^2. If $\dim J^2=3$ then R is simply the algebra $K[X,Y]/\langle X,Y\rangle^3$, which has a factor ring of type C_1. If $\dim J^2\leqslant 2$ then, passing over the trivial cases, one has (say) $x^2+\alpha xy+\beta y^2=0$ for some $\alpha,\beta\in K$. Let L be a splitting field for this polynomial. So, either $L=K$ or $[L:K]=2$. There is no problem in the first case, so suppose the second. If the factors, x' and y', of this polynomial in $R\otimes_K L$ were distinct, then we would have $R\otimes_K L=L[x',y':x'^3=y'^3=0=x'y']$ - an algebra of the form $GP_{3,3}$ - contradicting the assumption on R. So the polynomial must have a repeated root, y', say (then $R\otimes_K L=L[x,y':x^3=y'^2=x^2y'=0]$). I do not know how to lift the undecidability of $R\otimes_K L$ back to R. But, of course, this case can arise only if the characteristic of K is 2 (and, if K is finite, then the polynomial must split in R). Therefore we conclude the following.

<u>Theorem 17.20</u> [Po86; Thm 11] *Suppose that R is a local commutative K-algebra which is wild in the sense above. Suppose also that* $\operatorname{char} K\neq 2$ *or that K is finite. Then the theory of R-modules is undecidable.* □

Next, suppose that R has the form $K[x,y:x^2=y^2=0]$ (the question of field extensions does not arise in this case). Now, every module over this algebra is the direct sum of a module over $GP_{2,2}$ and a number of copies of the (self-injective) algebra itself. Hence the theory of R-modules is decidable iff that of $GP_{2,2}$-modules is decidable. But the latter theory is interpretable (see [Po86; Prop 9(b)]) within the theory of modules over $K[\tilde{D}_4]$ which

(17.16) is decidable. Since (exercise) the only domestic quotients of the Gelfand-Ponomarev algebra are the two algebras just considered, one has the following.

Theorem 17.21 [Po86; Cor 10] *Let R be a commutative artinian ring which is recursive, has decidable word problem and satisfies condition* (D). *Suppose that each local direct factor ring of R is, or corresponds to in the sense of* 17.19, *a K-algebra of finite or domestic representation type over some field K (which may vary). Then the theory of R-modules is decidable.* □

Thus the remaining problem is the decidability of the theory of modules over a Gelfand-Ponomarev algebra. Such an algebra is of "string and band" type, and it is conjectured to have decidable theory of modules (cf. §13.3). If tensoring with a quadratic extension of the base field only has the effect of splitting the band modules with polynomial corresponding to the field extension, then the problem of decidability over commutative artinian rings will thereby be reduced to that of modules over the Gelfand-Ponomarev quiver.

6. Localisation Let R be a commutative ring, and let P be a prime ideal of R. In order to transfer decidability results between R and its localisations we may use Garavaglia's result 2.Z5, but also we need the category of $R_{(P)}$-modules to be a finitely axiomatisable subclass of the class of R-modules. Finite axiomatisability of that class is a strong requirement (c.f. [JV79]) but it is satisfied if R is artinian (exercise; see [Po86; Prop 4]). One may check that if R is an artinian recursive ring with decidable word problem and satisfies condition (D) then one may effectively list the prime ideals of R.

7. p-valuated groups Schmitt proves the decidability of the theory of tamely p-valuated groups - see [Sch86].

17.4 Summary

This short section simply summarises, under various headings, the results of the previous two. Throughout the section it is assumed that the ring R is recursive, has decidable word problem and satisfies condition (D).

Algebras For path algebras over quivers without relations, one has a complete result.

Theorem 17.22 *Let R be the path algebra of a quiver without relations. Then the theory of R-modules is decidable iff R is of tame or finite representation type. If R is a hereditary K-algebra where K is an algebraically closed field, then the theory of R-modules is decidable iff R is of tame or finite representation type.* □

The obvious outstanding question is: what happens if R is a tame non-domestic algebra? Since the question of decidability has been reduced to the question of classifiability of the indecomposable pure-injectives, the comments in §13.3 are relevant. Apart from that question, one may ask whether there is a general proof that domestic representation type implies decidability. Perhaps more easy to solve is the problem of showing that in general wild representation type implies undecidability. One may also ask whether certain "small" subcategories of $\mathcal{M}_{K\langle X,Y\rangle}$ have decidable theories (for example, the category of all direct sums of modules which are of length less than some finite bound). It should be mentioned in this connection that, for a finite-dimensional algebra, it appears that the theory of finitely presented moduleso coincides with the theory of all modules only if the ring is of finite representation type: this is conjectured in [PoPr8?] and some progress is made there towards validating the conjecture.

Finite rings The result 17.19 shows that the decidability problem for modules over finite recursive rings is no more difficult than that for finite-dimensional algebras, at least in the local case. I can see no reason to think that the case of finite algebras is any easier than the case of arbitrary algebras. (For background on finite rings, see [McD74].)

Commutative artinian algebras Reduction to the case of local algebras was made. The key to the general result (wild ≡ undecidable) seems to be the classification of the infinite-dimensional indecomposables over the Gelfand-Ponomarev quiver.

Artinian rings It seems likely that this case is equivalent to that of finite-dimensional algebras. Over non-algebraically closed fields, one must consider species in the sense of [DR76].

Posets It has been shown that wild posets are undecidable. The tame case is similar to that for algebras.

Commutative regular rings A criterion for decidability of the theory of modules over a commutative regular ring is given in [PoPr8?; 3.1]. This has the following corollary.

> Theorem 17.23 *Let R be a commutative regular ring with decidable word problem. The theory of R-modules is decidable if it is decidable whether or not, for any given idempotent $e \in R$ and prime power p^n, there exists a maximal ideal of R containing $1-e$ and with $R/I \simeq \mathbb{Z}_{p^n}$ - the finite field with p^n elements.* □

The result is proved by considering the meaning of boolean combinations of invariants conditions in terms of the topology on the spectrum of the ring.

Congruence-modular varieties The question of which rings have decidable theory of modules has wider implications. Burris and McKenzie [BuMc81] showed that the decision problem for a locally finite conguence-modular variety of finite type reduces to that of the theory of modules over an associated finite ring (these varieties are those defined in a language with only finitely many function symbols, whose every member is locally finite (i.e., finitely generated substructures are finite) with lattice of congruences modular - modules over a finite ring are examples).

For details of the construction of the ring associated to the variety and for the proof of the reduction of the decision problem for the algebras in the variety to the modules over the associated ring, see [BuMc81; §§10, 11] (if R is a finite ring then the ring associated to the variety $\mathcal{M}_R$ is indeed just R).

Finally, a word or two about complexity. I have not in these notes kept track of which decision procedures are primitive recursive and which are not. L. Monk showed in his thesis [Mon75] that the decision procedure for the theory of abelian groups is primitive recursive and this was extended to modules over Dedekind domains by Eklof and Fisher [EF72] and to wider classes of rings, indeed to abelian structures, by Weispfenning [Wei85], [Wei83a] (all modulo the presentation of the language). The descriptions of the space $\mathcal{I}_R$ for R the path algebra of an extended Dynkin quiver are very explicit and it appears (though I haven't checked it) that the decision procedure is primitive recursive.

PROBLEMS PAGE

I finish by specifying what I see as the main current problem areas.

The first is the apparent connection between model-theoretic complexity and the complexity of the finite-dimensional representation theory. Since infinite-dimensional pure-injectives seem to connect to (families of) finite-dimensional ones, there might well be some significant payoff, even for the finite-dimensional representation theory. In any case, the problem seems to me to be an interesting one and certainly there would be consequences regarding decidability of the theory of modules. I make some specific conjectures.

1. The split, finite or tame type *versus* wild type, corresponds exactly to that between decidable and undecidable theory of modules (for suitably recursive rings).
2. If an algebra is of domestic type then the Cantor-Bendixson rank of the space of indecomposable pure-injectives is 2.
3. If an algebra is of tame non-domestic type then the Cantor-Bendixson rank of the space of indecomposables is "∞", but the width of the largest theory of modules is not "∞": in particular, there are no continuous pure-injectives.
4. If an algebra is tame of finite growth then there are enough regular types in T^{eq}, but if the algebra is tame of infinite growth then there are not (this is not really a conjecture; rather, an assertion to be tested).

The second is Vaught's Conjecture for modules (see §7.2). The proof of this for U-rank 1 is far from simple and uses very sophisticated ideas. I would be surprised if the general case could avoid these ideas (though the specifics of the case might allow one to be more direct) and I would hope that a solution would deepen our understanding of what happens within the pure-injective hull of a module.

Rather more vaguely, the notion that pp-types generalise right ideals has been only lightly exploited. The lattice of pp-types contains much more information than the lattice of right ideals: it does, however, follow that it will be that much more difficult to obtain information about it. But surely the applicability of this point of view is not confined to modules over artinian rings.

A serious investigation of continuous pure-injectives might be profitable

Finally, these notes have, very much, been about module theory, rather than ring theory: almost invariably we have worked over a fixed ring (an exception is the use of localisation in §2$\mathbb{Z}$). There should be some investigation of how to transfer information about pp-types and pure-injectives between rings.

BIBLIOGRAPHY

I use abbreviations for certain series as follows:
GTM - Graduate Texts in Mathematics, Springer-Verlag, Berlin, Heidelberg, New York;
LMSLNS - London Mathematical Society Lecture Notes Series, Cambridge University Press, Cambridge, London, New York;
LNM - Lecture Notes in Mathematics, Springer-Verlag, Berlin, Heidelberg, New York;
LMPAM - Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, New York and Basel.

A reference to Mathematical Reviews follows most items. If you have difficulty in finding an item, refer to the list which follows the bibliography.

[AF72] Anderson, F. W. and Fuller, K. R., Modules with decompositions that complement direct summands, J. Algebra, 22 (1972), 241-253. 46#209

[AF74] Anderson, F. W. and Fuller, K. R., Rings and Categories of Modules, GTM Vol.13, 1974. 54#5281

[Aus66] Auslander, M., Coherent functors, pp.189-231 *in* Proc. Conf. Categorical Algebra, (La Jolla 1965), Springer-Verlag, New York 1966. 35#2945

[Aus71] Auslander, M., Representation Dimension of Artin Algebras, Queen Mary College Notes, University of London, London, 1971.

[Aus74] Auslander, M., Representation theory of Artin algebras I, Comm. Algebra, 1 (1974), 177-268. 50#2240

[Aus74a] Auslander, M., Representation theory of Artin algebras II, Comm. Algebra, 1 (1974), 269-310. 50#2240

[Aus76] Auslander, M., Large modules over Artin algebras, pp.1-17 *in* Algebra, Topology and Category Theory, Academic Press, New York, 1976. 54#12832

[Aus77] Auslander, M., Existence theorems for almost split sequences, pp. 1-44 *in* Ring Theory II, LNPAM Vol.26, 1977. 55#12764

[Aus82] Auslander, M., A functorial approach to representation theory, pp.105-179 *in* Representations of Algebras (Workshop, Puebla 1980), LNM Vol.944, 1982. 83m#16027

[Aus86] Auslander, M., A survey of existence theorems for almost split sequences, pp.81-89 *in* Representations of Algebras (ed. P. Webb), LMSLNS Vol.116, 1986.

[AB69] Auslander, M. and Bridger, M., Stable Module Theory, Mem. Amer. Math. Soc. No. 94, Providence. R.I., 1969. 42#4580

[AR74] Auslander, M. and Reiten, I., Stable eqivalence of dualizing R-varieties, Advances in Math., 12 (1974), 306-366. 49#7251

[AR75] Auslander, M. and Reiten, I., Representation theory of Artin algebras III. Almost split sequences, Comm. Algebra, 3 (1975), 239-294. 52#504

[Bae81] Baer, D., Zerlegungen von Moduln und Injektive über Ringoiden, Arch. Math., 36 (1981), 495-501. 82k#16019

[Bae85] Baer, D., Noetherian categories and representation theory of hereditary Artin algebras, Comm. Algebra, 13(1) (1985), 247-258. 86i#16033

[Bae86] Baer, D., Homological properties of wild hereditary Artin algebras, pp. 1-12 *in* Representation Theory I: Finite Dimensional Algebras, LNM Vol.1177, 1986. 87h#16038

[BBL82] Baer, D., Brune, H. and Lenzing, H., A homological approach to representations of algebras II: tame hereditary algebras, J. Pure Appl. Algebra, 26(2) (1982), 141-153. 84a#16038b

[BGL87] Baer, D., Geigle, W. and Lenzing, H., The preprojective algebra of a tame hereditary Artin algebra, Comm. Algebra, 15(1,2) (1987), 425-457.

[BL82] Baer, D. and Lenzing, H., A homological approach to representations of algebras I: the wild case, J. Pure Appl. Algebra, 24(3) (1982), 227-233. 84a#16038a

[Ba40] Baer, R., Abelian groups that are direct summands of every containing abelian group, Bull. Amer. Math. Soc., 46 (1940), 800-806. 2p.126

[Bal79] Baldwin, J. T., Stability theory and algebra, J. Symbolic Logic, 44(4) (1979), 599-608. 81i#03051

[Bal?] Baldwin, J. T., Classification theory: 1985, *introduction to* Proceedings of the U.S.-Israel Workshop on Model Theory in Mathematical Logic: Classification Theory, *to appear*.

[Bal8?] Baldwin, J. T., Stable Theories, *to appear*.

[BBGK73] Baldwin, J. T., Blass, A., Glass, A. M. W. and Kueker, D. W., A "natural" theory without a prime model, Algebra Universalis, 3 (1973), 152-155. 50#86

[BK80] Baldwin, J. T. and Kueker, D. W., Ramsey quantifiers and the finite cover property, Pacific. J. Math., 90 (1980), 11-19. 83e#03054

[BaLa73] Baldwin, J. T. and Lachlan, A. H., On universal Horn classes categorical in some infinite power, Algebra Universalis, 3 (1973), 98-111. 50#4273

[BM82] Baldwin, J. T. and McKenzie, R. N., Counting models in universal Horn classes, Algebra Universalis, 15 (1982), 359-384. 84m#03042

[Bar77] Barwise, J. (ed.), Handbook of Mathematical Logic, North-Holland, Amsterdam, 1977. 56#15351

[BE70] Barwise, J. and Eklof, P., Infinitary properties of abelian torsion groups, Ann. Math. Logic, 2 (1970), 25-68. 43#4906

[Bas60] Bass, H., Finitistic dimension and a homological generalisation of semi-primary rings, Trans. Amer. Math. Soc., 95 (1960), 466-488. 28#1212

[Bas71] Bass, H., Descending chains and the Krull ordinal of commutative noetherian rings, J. Pure Appl. Algebra, 1(4) (1971), 347-360. 46#1778

[Bd81] Baudisch, A., The elementary theory of abelian groups with m-chains of pure subgroups, Fund. Math., 112(2) (1981), 147-157. 82h#03026

[Bd83] Baudisch, A., Corrections and supplements to "Decidability of the theory of Abelian groups with Ramsey quantifiers", Bull. Acad. Sci. Polon., 31(3-4) (1983), 99-105. 85j#03011

[Bd84] Baudisch, A., Tensor products of modules and elementary equivalence, Algebra Universalis, 19 (1984), 120-127. 85j#03048

[Bd84a] Baudisch, A., Magidor-Malitz quantifiers in modules, J. Symbolic Logic, 49(1) (1984), 1-8. 85e#03070

[Bd8?] Baudisch, A. On two hierarchies of dimensions, Inst. Math. Akad. Wiss. DDR, Berlin, DDR, *preprint*.

[BR83] Baudisch, A. and Rothmaler, P., Introductory Course on Forking, Part I (*in German*), Humboldt-Universität, Berlin, 1983. 84f#03030

[BR84] Baudisch, A. and Rothmaler, P., Introductory Course on Forking, Part II (*in German*), Humboldt-Universität, Berlin, 1984. 86f#03054

[BR84a] Baudisch, A. and Rothmaler, P., The stratified order in modules, pp. 44-56 *in* Frege Conference. Schwerin 1984, Math. Res. Vol.20, Akademie-Verlag, Berlin, 1984. 86i#03046

[BSTW85] Baudisch, A., Seese, D., Tuschik, P. and Weese, M., "Decidability and Quantifier-elimination", Chapter VII *in* Model-Theoretic Logics (eds. J. Barwise and S. Feferman), Springer-Verlag, New York-Heidelberg, 1985. see 87g#03033

[Bau75] Baur, W., $\aleph_0$-categorical modules, J. Symbolic Logic, 40(2) (1975), 213-220. 51#5283

[Bau75a] Baur, W., Decidability and undecidability of theories of abelian groups with predicates for subgroups, Compositio Math., 31 (1975), 23-30. 52#5399

[Bau76] Baur, W., Elimination of quantifiers for modules, Israel J. Math., 25 (1976), 64-70. 56#15409

[Bau76a] Baur. W., Undecidability of the theory of abelian groups with a subgroup, Proc. Amer. Math. Soc., 55(1) (1976), 125-128. 54#4953

[Bau80] Baur, W., On the elementary theory of quadruples of vector spaces, Ann. Math. Logic, 19 (1980), 243-262. 82g#03056

[BCM79] Baur, W., Cherlin, G. and Macintyre, A., Totally categorical groups and rings, J. Algebra, 57 (1979), 407-440. 80e#03034

[BaBr81] Bautista, R. and Brenner, S., On the number of terms in the middle of an almost split sequence, pp. 1-8 *in* Representations of Algebras (Proceedings, Puebla, 1980), LNM Vol.903, 1981. 83f#16034

[BGRS85] Bautista, R., Gabriel, P., Roiter, A. V. and Salmerón, L., Representation-finite algebras and multiplicative bases, Invent. Math., 81 (1985), 217-285. 87g#16031

[Bel76] Belesov, A. U., Injective and equationally compact modules (*Russian*), Izv. Akad. Nauk Kazah. SSR Ser. Fiz-Mat., 1 (1976), 85-6, 94. 56#15410

[Be82] Berline, C., Groupes Superstables, Doctoral Thesis, Université Paris VII, 1982.

[BeLa86] Berline, C. and Lascar, D., Superstable groups, Ann. Pure Applied Logic, 30(1) (1986), 1-43.

[BGP73] Bernstein, I. N., Gelfand, I. M. and Ponomarev, V. A., Coxeter functors and Gabriel's theorem, Uspehi Mat. Nauk, 28(2) (1973), 19-33, *translated in* Russian Math. Surveys, 28 (1973), 17-32. 52#13876

[Brt75] Berthier, D., Stability of non-model-complete theories: products, groups, J. London Math. Soc.(2), 11 (1975), 453-464. 52#2870

[Bi72] Bican, L., Notes on purities, Czechoslovak Math. J., 22(97) (1972), 525-534. 50#400a

[BD77] Bondarenko, V. M. and Drozd, J. A., The representation type of finite groups, (*in Russian*), Zap. Naucn. Sem. LOMI, 71 (1977), 24-41. 57#12663

[Bon82] Bongartz, K., Treue einfach zusammenhängende Algebren I, Comm. Math. Helvetici, 57 (1982), 282-330. 84a#16051

[BG82] Bongartz, K. and Gabriel, P., Covering spaces in representation theory, Invent. Math., 65 (1982), 331-378. 84i#16030

[BV83] Borceux, F. and Van Den Bossche, G., Algebra in a Localic Topos with Applications to Ring Theory, LNM Vol.1038, 1983. 84h#18005

[BSV84] Borceux, F., Simmons, H. and Van Den Bossche, G., A sheaf representation for modules with applications to Gelfand rings, Proc. London Math. Soc.(3), 48 (1984), 230-246. 85j#16061

[Bou79] Bouscaren, E., Modules Existentiellement Clos: Types et Modèles Premiers, Thèse 3ème cycle, Université Paris VII, Paris, 1979.

[Bou80] Bouscaren, E., Existentially closed modules: types and prime models, pp31-43 *in* Model Theory of Algebra and Arithmetic, LNM Vol.834, 1980. 82j#03038

[BoLa83] Bouscaren, E. and Lascar, D., Countable models of non-multidimensional $\aleph_0$-stable theories, J. Symbolic Logic, 48(1) (1983), 197-205. 85b#03049

[Bra79] Brandal, W., Commutative Rings whose Finitely Generated Modules Decompose, LNM Vol.723, 1979. 80g#13003

[Bre67] Brenner, S., Endomorphism algebras of vector spaces with distinguished sets of subspaces, J. Algebra, 6(1) (1967), 100-114. 35#217

[Bre72] Brenner, S., Some modules with nearly prescribed endomorphism rings, J. Algebra, 23 (1972), 250-262. 46#7312

[Bre74] Brenner, S., On four subspaces of a vector space, J. Algebra, 29 (1974), 587-599. 49#7271

[Bre74a] Brenner, S., Decomposition properties of some small diagrams of modules, Symp. Math., 13 (1974), 127-141. 50#13159

[BB65] Brenner, S. and Butler, M. C. R., Endomorphism rings of vector spaces and torsion free abelian groups, J. London Math. Soc., 40 (1965), 183-187. 30#4794

[BB80] Brenner, S. and Butler, M. C. R., Generalisations of the Bernstein-Gel'fand-Ponomarev reflection functors, pp.103-169 *in* Representation Theory II, LNM Vol.832, 1980. 82e#16031

[BrRi76] Brenner, S. and Ringel, C. M., Pathological modules over tame rings, J. London Math. Soc. (2), 14 (1976), 207-215. 55#5697

[Bru79] Brune, H., Some left pure semisimple ringoids which are not right pure semisimple, Comm. Algebra, 7(17) (1979), 1795-1803. 80k#18018

[Bue86] Buechler, S., The classification of small weakly minimal sets, II, preprint, University of California, Berkeley, 1986.

[Bue86a] Buechler, S., The classification of small weakly minimal sets III: modules, preprint, University of California, Berkeley, 1986.

[Bue86b] Buechler, S., Locally modular theories of finite rank, Ann. Pure Appl. Logic, 30 (1986), 83-94.

[Bue87] Buechler, S., "Geometrical" stability theory, pp.53-66 *in* Logic Colloquium '85, North-Holland, Amsterdam, 1987.

[Bue8?] Buechler, S., The classification of small weakly minimal sets, I, *in* Proceedings U.S.-Israel Meeting in Model Theory, Chicago '85, Springer-Verlag, Berlin, *to appear*.

[Bu65] Bumby, R. T., Modules which are isomorphic to submodules of each other, Arch. Math., 16 (1965), 184-185. 32#2444

[BuMc81] Burris, S. and McKenzie, R., Decidability and Boolean Representations, Mem. Amer. Math. Soc. Vol. 32 No.246, Providence, R.I., 1981. 83j#03024

[But80] Butler, M. C. R., The construction of almost split sequences I, Proc. London Math. Soc.(3), 40 (1980), 72-86. 82d#16004a

[BH61] Butler, M. C. R. and Horrocks, G., Classes of extensions and resolutions, Phil. Trans. Roy. Soc. London, 254 (1961), 155-222. 32#5706

[BuRi87] Butler, M. C. R. and Ringel, C. M., Auslander-Reiten sequences with few middle terms and applications to string algebras, Comm. Algebra, 15(1,2) (1987), 145-179.

[BS80] Butler, M. C. R. and Shazmanian, M., The construction of almost split sequences III, Modules over two classes of tame local algebras, Math. Ann., 247(2) (1980), 111-122. 83b#16023

[Ca73] Campbell, J. M., Torsion theories and coherent rings, Bull. Austral. Math. Soc., 8 (1973), 233-239. 47#8611

[CK73] Chang, C. C. and Keisler, H. J., Model Theory, North-Holland, Amsterdam, 1973. 53#12927

[Ch60] Chase, S. U., Direct products of modules, Trans. Amer. Math. Soc., 97 (1960), 457-473. 22#11017

[Che76] Cherlin, G., Model Theoretic Algebra: Selected Topics, LNM Vol.521, 1976. 58#27455

[CHL85] Cherlin, G., Harrington, L. and Lachlan, A., $\aleph_0$-categorical, $\aleph_0$-stable structures, Ann. Pure Appl. Logic, 28(2) (1985), 103-135. 86g#03054

[CS83] Cherlin, G. and Schmitt, P., Locally pure topological abelian groups, Ann. Pure Appl. Logic, 24(1) (1983), 49-85. 85a#03050

[Co71] Cohn, P. M., Free Rings and their Relations, Academic Press, London and New-York, 1971. 51#8155

[Co77] Cohn, P. M., Skew Field Constructions, LMSLNS Vol.27, 1977. 57#3190

[Co79] Cohn, P. M., The affine scheme of a general ring, pp 197-211 *in* Applications of Sheaves, LNM. Vol.753, 1979. 81e#16002

[Co81] Cohn, P. M., Universal Algebra (Revised ed.), Reidel, Dordrecht, Holland, 1981. 82j#08001

[Cor63] Corner, A. L. S., Every countable reduced torsion-free ring is an endomorphism ring, Proc. London Math. Soc.(3), 13 (1963), 687-710. 27#3704

[Cor69] Corner, A. L. S., Endomorphism algebras of large modules with distinguished submodules, J. Algebra, 11 (1969), 155-185. 38#5838

[CG85] Corner, A. L. S. and Göbel, R., Prescribing endomorphism algebras - a unified treatment, Proc. London Math. Soc. (3), 50(3) (1985), 447-479. 86h#16031

[Cou78] Couchot, F., Sous-modules purs et modules de type cofini, pp. 198-208 *in* Séminaire d'Algèbre Paul Dubriel: Proceedings Paris 1976-1977, LNM Vol.641, 1978. 58#11015

[C-B87] Crawley-Boevey, W. W., Classifying indecomposable representations, University of Liverpool, preprint, 1987.

[C-B87a] Crawley-Boevey, W. W., Functorial filtrations and the problem of an idempotent and a square-zero matrix, University of Liverpool, preprint.

[C-B87b] Crawley-Boevey, W. W., Functorial filtrations II; the Gelfand problem and semidihedral algebras, University of Liverpool, *in preparation*.

[CR62] Curtis, C. W. and Reiner, I., Representation Theory of Finite Groups and Associative Algebras, J. Wiley and Sons, New York and London, 1962. 26#2519

[Dei77] Deissler, R, Minimal models, J. Symbolic Logic, 42(2) (1977), 254-260. 58#27443

[Di46] Dieudonné, J., Sur la réduction canonique des couples de matrices, Bull. Soc. Math. France, 74 (1946), 130-146. 9 p.264

[DG80] Dlab, V. and Gabriel, P. (Eds.), Representation Theory I, LNM Vol.831, 1980. 82a#16002a

[DG80a] Dlab, V. and Gabriel, P. (Eds.), Representation Theory II, LNM Vol.832, 1980. 82a#16002b

[DR72] Dlab, V. and Ringel, C. M., Decomposition of modules over right uniserial rings, Math. Z., 129 (1972), 207-230. 47#6774

[DR76] Dlab, V. and Ringel, C. M., Indecomposable Representations of Graphs and Algebras, Mem. Amer. Math. Soc. Vol.6 No.173, Providence, R.I., 1976. 56#5657

[DF73] Donovan, P. and Freislich, M. R., The Representation Theory of Finite Graphs and Associated Algebras, Carleton Math. Lecture Notes No.5, Carleton University, Ottawa, 1973. 50#9701

[DS86] Dowbor, P and Skowronski, A., On the representation type of locally bounded categories, Tsukaba J. Math., 10(1) (1986), 63-72.

[Dro72] Drozd, J. A., Representations of commutative algebras, *translated in* Funct. Analysis. Appl., 6 (1972), 286-288. 47#280

[Dro79] Drozd, J. A., Tame and wild matrix problems, pp.39-74 and 154 *in* Representations and Quadratic Forms, Akad. Nauk Ukrain. SSR, Inst. Mat. Kiev, 1979 (*in Russian*). 82m#16028

[DFS87] Dugas, M., Fay, T. H. and Shelah, S., Singly cogenerated annihilator classes, J. Algebra, 109(1) (1987), 127-137.

[DuGö79] Dugas, M. and Göbel, R., Algebraisch kompakte Faktorgruppen, J. Reine. Angew Math., 307/8 (1979), 341-352. 80h#20080

[DuGö82] Dugas, M. and Göbel, R., Every cotorsion-free algebra is an endomorphism algebra, Math. Z., 181 (1982), 451-470. 84h#13008

[DuGö85] Dugas, M. and Göbel, R., On radicals and products, Pacific J. Math., 118(1) (1985), 79-104. 86k#20052

[DH83] Dugas, M. and Herden, G., Arbitrary torsion classes and almost free abelian groups, Israel J. Math., 44(4) (1983), 322-334. 84k#16034

[DZ81] Dugas, M. and Zimmermann-Huisgen, B., Iterated direct sums and products of modules, pp179-193 *in* Abelian Group Theory, LNM Vol.874, 1981. 83f#16059

[EcSc53] Eckmann, B and Schopf, A., Uber injektive Moduln, Arch. Math., 4 (1953), 75-78. 15 p.5

[EG71] Eisenbud, D. and Griffith, P., The structure of serial rings, Pacific J. Math., 36 (1971), 109-121. 45#1968

[Ek71] Eklof, P. C., Homogeneous universal modules, Math. Scand., 29 (1971), 187-196. 47#3168

[Ek72] Eklof, P. C., Some model theory of abelian groups, J. Symbolic Logic, 37(2) (1972), 335-342. 52#10411

[Ek73] Eklof, P. C., The structure of ultraproducts of abelian groups, Pacific J. Math., 47(1) (1973), 67-79. 48#6281

[Ek74] Eklof, P. C., Infinitary equivalence of abelian groups, Fund. Math., 81 (1974), 305-314. 50#6829

[Ek75] Eklof, P., Categories of local functors, pp.91-116 *in* Model Theory and Algebra, LNM Vol.498, 1975. 55#7766

[Ek76] Eklof, P. C., Whitehead's problem is undecidable, Amer. Math. Monthly, 83 (1976), 775-788. 57#16071

[Ek76a] Eklof, P. C., Infinitary model theory of abelian groups, Isr. J. Math., 25(1-2) (1976), 97-107. 56#11778

[Ek85] Eklof, P, C., Applications to algebra, pp.423-441 *in* Model-Theoretic Logics (ed. J. Barwise and S. Feferman), Springer-Verlag, New York, 1985. see 87g#03033

[EF72] Eklof, P. C. and Fisher, E., The elementary theory of abelian groups, Ann. Math. Logic, 4(2) (1972), 115-171. 58#27457

[EkMk79] Eklof, P. C. and Mekler, A. H., Stationary logic of finitely determinate structures, Ann. Math. Logic, 17 (1979), 227-270. 82f#03033

[EkMz84] Eklof, P. C. and Mez, H.-C., Additive groups of existentially closed rings, pp. 243-252 *in* Proceedings of the Udine Conference on Abelian Groups and Modules, Springer-Verlag, Vienna, 1984.

[EkMz85] Eklof, P. C. and Mez, H.-C., The ideal structure of existentially closed algebras, J. Symbolic Logic, 50(4) (1985), 1025-1043. 87f#03093

[EkMz87] Eklof. P. C. and Mez, H.-C., Modules of existentially closed algebras, J. Symbolic Logic, 52(1) (1987), 54-63.

[ES71] Eklof, P. and Sabbagh, G., Model-completions and modules, Ann. Math. Logic, 2(3) (1971), 251-295. 43#3105

[End72] Endler, O., Valuation Theory, Springer-Verlag, Berlin Heidelberg New-York, 1972. 50#9847

[Er62] Ershov, Yu. L., Decidability of elementary theories of certain classes of abelian groups (*in Russian*), Algebra i Logika, 1 (1962), 37-41. 31#1192

[Er79] Ershov, Yu. L., Algebraically compact groups II, Algebra i Logika, 18(4) (1979), 404-414, *translated in* Algebra and Logic, 18(4) (1979), 247-251. 82c#20099b

[Fac83] Facchini, A., Spectral categories and varieties of preadditive categories, J. Pure Applied Algebra, 29 (1983), 219-239. 85k#18013

[Fac85] Facchini, A., Decompositions of algebraically compact modules, Pacific J. Math., 116(1) (1985), 25-37. 86g#16031

[Fac85] Facchini, A., Algebraically compact modules, Acta Univ. Carolin., 26 (1985), 27-37. 87g#16035

[Fac8?] Facchini, A., Relative injectivity and pure-injective modules over Prüfer rings, Università di Udine, Udine, *preprint*.

[FZ82] Fahy, M. and Zelmanowitz, J., Semilocal rings of quotients, Proc. London Math. Soc.(3), 44 (1982), 33-46. 83i#16006

[Fai76] Faith, C., Algebra II. Ring Theory, Springer-Verlag, Berlin New-York, 1976. 55#383

[FW67] Faith, C. and Walker, E. A., Direct sum representations of injective modules, J. Algebra, 5 (1967), 203-221. 34#7575

[Fak71] Fakir, S., On relative injective envelopes, J. Pure Applied Algebra, 1(4) (1971), 377-383. 46#233

[FV59] Feferman, S. and Vaught, R. L., The first order properties of products of algebraic systems, Fund. Math., 47 (1959), 57-103. 21#7171

[Fe8?] Felgner, U., The classification of all quantifier-eliminable FC-groups, Universität Tubingen, *preprint*.

[Fie69] Fieldhouse, D. J., Pure theories, Math. Ann., 184 (1969), 1-18. 40#5699

[FP86] Fischbacher, U. and de la Peña, J. A., Algorithms in representation theory of algebras, pp. 115-134 *in* Representation Theory I: Finite Dimensional Algebras, LNM Vol.1177, 1986. 87i#16057

[Fis72] Fisher, E. R., Powers of saturated modules, J. Symbolic Logic, 37(4) (1972), 777.

[Fis75] Fisher, E., Abelian Structures, Yale University, 1974/75.

[Fis77] Fisher, E., Abelian structures I, pp.270-322 *in* Abelian Group Theory, LNM Vol.616, 1977. 58#27459

[Fra81] Franzen, B., Algebraic compactness of filter quotients, pp.228-241 *in* LNM Vol.874, 1981. 83f#20040

[Fre80] Freese, R., Free modular lattices, Trans. Amer. Math. Soc., 261 (1980), 81-91. 81k#06010

[FrSh56] Frölich, A and Shepherdson, J. C., Effective procedures in field theory, Philos. Trans. Roy. Soc. London, 248 (1956), 407-432. 17p.570

[Fu67] Fuchs, L., Algebraically compact modules over noetherian rings, Indian J. Math., 9 (1967), 357-374. 42#7695

[Fu70] Fuchs, L., Infinite Abelian Groups. Vol. I, Academic Press, London and New York, 1970. 41#333

[FS85] Fuchs, L. and Salce, L., Modules over Valuation Domains, LNPAM Vol.97, 1985. 86h#13008

[Ful76] Fuller, K. R., On rings whose left modules are direct sums of finitely generated modules, Proc. Amer. Math. Soc., 54 (1976), 39-44. 52#13943

[FR75] Fuller, K. R. and Reiten, I., Note on rings of finite representation type and decompositions of modules, Proc. Amer. Math. Soc., 50 (1975), 92-94. 51#12943

[Gab62] Gabriel, P., Des catégories abéliennes, Bull. Soc. Math. France, 90 (1962), 323-448. 38#1144

[Gab72] Gabriel, P., Unzerlegbare Darstellungen I, Manuscripta Math., 6 (1972), 71-103. 48#11212

[Gab73] Gabriel, P., Indecomposable representations II, Symposia Math. Inst. Naz. Alta. Mat., 11 (1973), 81-104. 49#5132

[Gab73a] Gabriel, P., Représentations indécomposables des ensembles ordennés, pp.301-304 *in* Séminaire P. Dubriel, Paris 1972/73, Secrétariat Mathématique, Paris, 1973. 55#12581

[Gab75] Gabriel, P., Représentations indécomposables, pp.143-169 *in* Séminaire Bourbaki (1973/74), LNM Vol.431, 1975. 58#5788

[Gab75a] Gabriel, P., Finite representation type is open, pp.132-155 *in* Proc. ICRA 1974, LNM Vol.488, 1975. 51#12944

[Gab80] Gabriel, P., Auslander-Reiten sequences and representation-finite algebras, pp.1-71 *in* Representation Theory I, LNM Vol.831, 1980. 82i#16030

[GO66] Gabriel, P. and Oberst, U., Spektralkategorien und reguläre Ringe im von-Neumannschen Sinn, Math. Z., 92 (1966), 389-395. 37#1439

[Gar7?] Garavaglia, S., Elementary equivalence of modules, unpublished.

[Gar78] Garavaglia, S., Model theory of topological structures, Ann. Math. Logic, 14(1) (1978), 13-37. 58#10406

[Gar79] Garavaglia, S., Direct product decomposition of theories of modules, J. Symbolic Logic, 44(1) (1979), 77-88. 80c#03038

[Gar80] Garavaglia, S., Decomposition of totally transcendental modules, J. Symbolic Logic, 45(1) (1980), 155-164. 81a#03032

[Gar80a] Garavaglia, S., Dimension and rank in the model theory of modules, University of Michigan, East Lansing, 1979, *revised* 1980, *preprint*.

[Gar81] Garavaglia, S., Forking in modules, Notre Dame J. Formal Logic, 22 (1981), 155-162. 82g#03059

[Gei85] Geigle, W., The Krull-Gabriel dimension of the representation theory of a tame hereditary Artin algebra and applications to the structure of exact sequences, Manuscripta Math., 54(1-2) (1985), 83-106. 87e#16061

[Gei86] Geigle, W., Krull dimension and Artin algebras, pp.135-155 *in* Representation Theory I: Finite Dimensional Algebras, LNM Vol.1177, 1986.

[GL8?] Geigle, W. and Lenzing, H., The use of the Krull-Gabriel dimension in representation theory, Universität Paderborn, *in preparation*.

[GP68] Gelfand, I. M. and Ponomarev, V. A., Indecomposable representations of the Lorentz group, Uspehi Mat. Nauk, 23(2) (1968), 3-60, *translated in* Russian Math. Surveys, 23(2) (1968), 1-58. 37#5325

[GP72] Gelfand, I. M. and Ponomarev, V. A., Problems of linear algebra and classification of quadruples of subspaces in a finite-dimensional vector space, pp.163-237 *in* Coll. Math. Soc. Bolyai, 5, North-Holland, Amsterdam, 1972. 50#9896

[GP74] Gelfand, I. M. and Ponomarev, V. A., Free modular lattices, and their representations, Uspehi Mat. Nauk, 29(6) (1974), 3-58, *translated in* Russian Math. Surveys, 29 (1974), 1-56. 53#5393

[Göb83] Göbel, R., Endomorphism rings of abelian groups, pp.340-353 *in* Abelian Group Theory, Proceedings, LNM Vol.1006, 1983. 85d#20057

[Göb84] Göbel, R., Wie weit sind Moduln vom Satz von Krull-Remak-Schmidt entfernt?, Universität Essen, 1984.

[GS85] Göbel, R. and Shelah, S., Modules over arbitrary domains, Math. Z., 188(3) (1985), 325-337. 86d#13011

[Gol75] Golan, J. S., Localisation of Non-commutative Rings, Marcel Dekker, New York, 1975. 51#3207

[Gol78] Golan, J. S., The lattice of torsion theories associated with a ring, pp.25-43 *in* Ring Theory (Antwerp 1977), LNPAM Vol.40, 1978. 80b#16022

[Gol80] Golan, J. S., Structure Sheaves over a Non-commutative Ring, LNPAM Vol.56, 1980. 81k#16002

[Goo76] Goodearl, K. R., Ring Theory. Nonsingular Rings and Modules, Marcel Dekker, New York and Basel, 1976. 55#2970

[Goo79] Goodearl, K. R., Von Neumann Regular Rings, Pitman, London-San Francisco-Melbourne, 1979. 80e#16011

[GB76] Goodearl, K. R. and Boyle, A. K., Dimension theory for nonsingular injective modules, Mem. Amer. Math. Soc., Vol.7 No.177, Providence, R.I., 1976. 55#3001

[GZ86] Goodearl, K. R. and Zimmermann-Huisgen, B., Lengths of submodule chains versus Krull dimension in non-Noetherian modules, Math. Z., 191 (1986), 519-527. 87f#16021

[GK82] Gorbachuck, O. L. and Komarnitski, M. Ja., On the axiomatisability of radical and semisimple classes of modules and abelian groups (*in Russian*), Ukrain. Mat. Zh., 34(2) (1982), 151-157 and 267. 83i#20049

[GR73] Gordon, R. and Robson, J. C., Krull dimension, Mem. Amer. Math. Soc., No.133, Providence, R.I., 1973. 50#4664

[GR74] Gordon, R. and Robson, J. C., The Gabriel dimension of a module, J. Algebra, 29 (1974), 459-473. 51#5658

[Gou87] Gould, V., Axiomatisability problems for S-systems, J. London Math. Soc.(2), 35(2) (1987), 193-201.

[Gou87a] Gould, V., Model-companions of S-systems, Quart. J. Math. Oxford(2), Vol.38 No.150 (1987), 189-211.

[Gre83] Green, J. A., Notes on almost split sequences I, Math. Inst. Warwick, 1983.

[Gri70] Griffith, P. A., Infinite Abelian Group Theory, University of Chicago Press, Chicago-London, 1970. 44#6826

[Gri70a] Griffith, P., On the decomposition of modules and generalised left uniserial rings, Math. Ann., 184 (1970), 300-308. 41#1790

[GHM87] Gross, H., Herrmann, C. and Moresi, R., The classification of subspaces in Hermitean vector spaces, J. Algebra, 105(2) 1987, 516-541.

[Gru75] Gruson, L., Simple coherent functors, pp.156-159 *in* Representations of Algebras, LNM Vol.488, 1975. 53#3023

[GJ73] Gruson, L., and Jensen, C. U., Modules algébriquement compacts et foncteurs $\varprojlim^{(i)}$, C. R. Acad. Sci. Paris, 276 (1973), 1651-1653. 47#8653

[GJ76] Gruson, L. and Jensen, C. U., Deux applications de la notion de L-dimension, C. R. Acad. Sci. Paris, 282 (1976), 23-24. 53#5706

[GJ81] Gruson, L. and Jensen, C. U., Dimensions cohomologiques reliées aux foncteurs $\varprojlim^{(i)}$, Kobenhavns Universitet Matematisk Institut preprint no.19, 1980, *published as* pp.234-294 *in* Sém. d'Algèbre P. Dubriel et M.-P. Malliavin, LNM Vol.867, 1981. 83d#16026

[Gul73] Gulliksen, T. H., A theory of length for noetherian modules, J. Pure Applied Algebra, 3 (1973), 159-170. 47#6775

[Gus85] Gustafson, W., A footnote to the multiplicative basis theorem, Proc. Amer. Máth. Soc., 95 (1985), 7-8. 87a#16036

[Hal79] Haley, D. K., Equational Compactness in Rings, LNM Vol.745, 1979. 81e#16036

[HaSa71] Harada, M and Sai, Y, On categories of indecomposable modules II, Osaka J. Math., 8 (1971), 309-321. 46#3566

[HMS84] Harrington, L., Makkai, M. and Shelah, S., A proof of Vaught's conjecture for ω-stable theories, Isr. J. Math. 49 (1984), 259-280. 86j#03029b

[Har59] Harrison, D. K, On infinite abelian groups and homological methods, Ann. Math., 69(2) (1959), 366-391. 21#3481

[HH70] Hartley, B. and Hawkes, T. O., Rings, Modules and Linear Algebra, Chapman and Hall, London, 1970. 42#2897

[HR61] Heller, A. and Reiner, I., Indecomposable representations, Ill. J. Math., 5 (1961), 314-323. 23#A222

[Hem83] Herrmann, C., On the word problem for the modular latice with four free generators, Math. Ann., 265 (1983), 513-527. 84m#06014

[Hem84] Herrmann, C., On elementary Arguesian lattices with four generators, Algebra Universalis, 18(2) (1984), 225-259. 85k#06008

[HJL81] Herrmann, C., Jensen, C. U. and Lenzing, H., Applications of model theory to representations of finite dimensional algebras, Math. Z., 178 (1981), 83-98. 83c#16023

[He68] Herstein, I. N., Noncommutative Rings, Carus Math. Monographs No. 15, MAA/John Wiley and Sons, 1968. 37#2790

[Her87] Herzog, I., Generics in modules, University of Notre Dame, Notre Dame, preprint, 1987.

[Her87a] Herzog, I., personal communication.

[Ho77] Hodges, W., Quantifier elimination for modules, Bedford College (University of London), 1977, unpublished notes.

[Ho80] Hodges, W., Constucting pure injective hulls, J. Symbolic Logic, 45(3) (1980), 544-548. 82a#03029

[Ho81] Hodges, W., unpublished notes on quantifier elimination for modules over regular rings, Bedford College (University of London), 1981.

[Ho83] Hodges, W., Relative categoricity in abelian groups, Bedford College (University of London), 1983, preprint.

[Ho87] Hodges, W., What is a structure theory?, Bull. London Math. Soc., 19(3) (1987), 209-237.

[Ho8?] Hodges, W., Basic Model Theory, Cambridge University Press, *in preparation*.

[Ho??] Hodges, W., Structure and Classification, Cambridge University Press, *in preparation*.

[Hod85] Hodkinson, I. M., Building many Uncountable Rings by Constructing many different Aronszajn Trees, Doctoral Thesis, Queen Mary College, London, 1985.

[Hod85a] Hodkinson, I. M., A construction of many uncountable rings, pp. 134-142 *in* Proceedings of the Third Easter Conference on Model Theory, Humboldt-Universität, Berlin, 1985. see 86m#03003

[HL81] Höppner, M. and Lenzing, H., Diagrams over ordered sets: a simple model of abelian group theory, pp.417-430 *in* Abelian Group Theory, LNM Vol.874, 1981. 83g#18023

[Hru86] Hrushovski, U., Contributions to Stable Model Theory, Doctoral Thesis, University of California, Berkeley, 1986.

[HP87] Hrushovski, U. and Pillay, A., Weakly normal groups, pp. 233-244 *in* Logic Colloquium '85, North-Holland, Amsterdam, 1987.

[Hu83] Huber. M., On reflexive modules and abelian groups, J. Algebra, 82(2) (1983), 469-487. 85f#16022

[Hul62] Hulanicki, A., On algebraically compact groups, Bull. Acad. Polon. Sci., 10 (1962), 71-75. 26#5059

[Ja73] Jackson, S. C., The Model Theory of Abelian groups, Doctoral Thesis, Bedford College, London, 1973.

[Jac74] Jacobson, N., Basic Algebra I, W. H. Freeman and Co., San Francisco, 1974. 50#9457

[Je72] Jensen, C. U., Les Foncteurs Dérivés de $\varprojlim$ et leurs Applications en Théorié des Modules, LNM Vol. 254, 1972. 53#10874

[Je80] Jensen, C. U., Applications logiques en théories des anneux et des modules, Mat. Inst., Copenhagen, 1980, preprint.

[JL80] Jensen, C. U. and Lenzing, H., Model theory and representations of algebras, pp.302-310 *in* Representation Theory II, LNM Vol. 832, 1980. 82c#16029

[JL82] Jensen, C. U. and Lenzing, H., Homological dimension and representation type of algebras under base field extension, Manuscripta Math., 39 (1982), 1-13. 83k#16019

[JS79] Jensen, C. U. and Simson, D., Purity and generalised chain conditions, J. Pure Applied Algebra, 14 (1979), 297-305. 81j#18016

[JV79] Jensen, C. U. and Vamos, P., On the axiomatisability of certain classes of modules, Math. Z., 167 (1979), 227-237. 81h#03073

[Kap54] Kaplansky, I., Infinite Abelian Groups, University of Michigan Press, Ann Arbor, 1954. 16 p.444

[Kap69] Kaplansky, I., Infinite Abelian Groups (Revised ed.), University of Michigan Press, Ann Arbor, 1969. 38#2208

[Kap70] Kaplansky, I., Commutative Rings, Allyn and Bacon, Boston, 1970. 40#7234

[Kar63] Kargapolov, M. I., On the elementary theory of abelian groups, Algebra i Logika, 1(6) (1963), 26-36; Review by J. Mennicke in J. Symbolic Logic, 32 (1967), 535-536. 31#251

[Ket78] Ketonen, J., The structure of countable boolean algebras, Ann. Math.(2), 108 (1978), 41-89. 58#10647

[Ki67] Kielpinski, R., On Σ-pure injective modules, Bull. Acad. Polon. Sci., 15 (1967), 127-131. 36#3816

[KS75] Kielpinski, R. and Simson, D., On pure homological dimension, Bull. Acad. Polon. Sci., 23(1) (1975), 1-6. 53#10872

[KM73] Kokorin, A. I. and Mart'yanov, V. I., Universal extended theories, pp.107-114 *in* Algebra, Irkutsk, 1973. 53#5286

[Kom78] Komarnitski, M. Ja., Torsion and quasivarieties of modules (*Ukranian, Russian Summary*), pp.19-21 and 72 *in* Akad. Nauk. Ukrain. SSR, 1978. 82e#16021

[Kom80] Komarnitski, M. Ja., Axiomatisability of certain classes of modules connected with a torsion (*Russian*), Mat. Issled., 56 (1980), 92-109 and 160-161. 82a#03031

[KoKo69] Kozlov, G. T. and Kokorin, A. I., Elementary theory of abelian groups without torsion with a predicate selecting a subgroup, Algebra i Logika, 8(3) (1969), 320-334, *translated in* Algebra and Logic, 8(3) (1969), 182-190. 41#3263

[KoKo75] Kozlov, G. T. and Kokorin, A. I., Proof of a lemma on model completeness, Algebra i Logika, 14(5) (1975), 533-535, *translated in* Algebra and Logic, 14(5) (1975), 328-330. 56#11729

[Kr73] Krause, G., Descending chains of submodules and the Krull-dimension of noetherian modules, J. Pure Applied Algebra, 3 (1973), 385-397. 49#2838

[Kre82] Kremer, E. M., Almost categorical theories of modules (*Russian*), p.88 *in* Fourth All-Union Conf. on Math. Logic, Tbilisi, 1982.

[Kre84] Kremer, E. M., Rings over which all modules of a given type are almost categorical, Algebra i Logika, 23(2) (1984), 159-174, *translated in* Algebra and Logic, 23(2) (1984), 113-124. 86h#16023

[Kre84a] Kremer, E. M., Modules with an almost categorical theory, Sibirsk Mat. Zh., 25(6) (1984), 70-75 (*in Russian*). 86h#03062

[Kuc80] Kucera, T., Modules over a non-commutative noetherian ring *and* The non-commutative case, McGill University, Montreal, 1980, preprints.

[Kuc84] Kucera, T. G., Applications of Stability-Theoretical Methods to Theories of Modules, Doctoral Thesis, McGill University, Montreal, 1984.

[Kuc86] Kucera, T., Stability theory for topological logic, with applications to topological modules, J. Symbolic Logic, 51(3) (1986), 755-769.

[Kuc8?] Kucera, T., Generalisations of Deissler's minimality rank, J. Symbolic Logic, *to appear.*

[Kuc8??] Kucera, T., Positive Deissler rank and the complexity of injective modules, J. Symbolic Logic, *to appear.*

[Kuc87] Kucera, T., Totally transcendental theories of modules: decomposition of models and types, Lakehead University, Thunder Bay, Ontario, 1987, *preprint.*

[Kue73] Kueker, D., A note on the elementary theory of finite abelian groups, Algebra Universalis, 3 (1973), 156-159. 50#87

[Lac78] Lachlan, A. H., Spectra of ω-stable theories, Z. Math. Logik Grundlag. Math., 24(2) (1978), 129-139. 80a#03041

[LM73] Lambek, J. and Michler, G., The torsion theory at a prime ideal of a right noetherian ring, J. Algebra, 25 (1973), 364-389. 47#5034

[Las76] Lascar, D., Ranks and definability in superstable theories, Israel J. Math., 23 (1976), 53-87. 53#12931

[Las82a] Lascar, D., Ordre de Rudin-Keisler et poids dans les théories stables, Z. Math. Logik Grundlag. Math., 28 (1982), 413-430. 84d#03037

[Las84] Lascar, D., Relations entre le rang U et le poids, Fund. Math., 121 (1984) 117-123. 86g#03057

[Las85] Lascar, D., Les groupes ω-stables de rang fini, Trans. Amer. Math. Soc., 292(2) (1985), 451-462. 87b#03071

[Las85a] Lascar, D., Why some people are excited by Vaught's conjecture, J. Symbolic Logic, 50(4) (1985), 973-982. 87e#03089

[Las8?] Lascar, D., Introduction à Stabilité *English Translation* Stability in Model Theory, Pitman, *to appear.*

[LP79] Lascar, D. and Poizat, B., An introduction to forking, J. Symbolic Logic, 44(3) (1979), 330-350. 80k#03030

[Law77] Lawrence, J., A countable self-injective ring is quasi-Frobenius, Proc. Amer. Math. Soc., 65(2) (1977), 217-220. 56#414

[Laz64] Lazard, D., Sur les modules plats, C. R. Acad. Sci. Paris, 258 (1964), 6313-6316. 29#5883

[Laz69] Lazard, D., Autour de la platitude, Bull. Soc. Math. France, 97 (1969), 81-128. 40#7310, erratum 41 p.1965

[LW75] Legg, M. W. and Walker, E. A., An algebraic treatment of algebraically compact groups, Rocky Mountain J. Math., 5(2) (1975), 291-299. 50#10120

[Len76] Lenzing, H., Direct sums of projective modules as direct summands of their direct product, Comm. Algebra, 4(7) (1976), 681-691. 53#8137

[Len83] Lenzing, H., Homological transfer from finitely presented to infinite modules, pp.734-761 *in* Abelian Group Theory, LNM Vol.1006, 1983. 85f#16034

[Len84] Lenzing, H., The pure-projective dimension of torsion-free divisible modules, Comm. Algebra, 12(6) (1984), 649-662. 86h#16028

[Len86] Lenzing, H., Curve singularities arising from the representation theory of tame hereditary algebras, pp. 199-231 *in* Representation Theory I: Finite Dimensional Algebras, LNM Vol.1177, 1986. 87i#16060

[Łos57] Łos, J., Abelian groups that are direct summands of every abelian group which contains them as pure subgroups, Bull. Acad. Polon. Sci., 4 (1956), 73, *and* Fund. Math., 44 (1957), 84-90. 19p.632

[Mac71] Macintyre, A., On ω_1-categorical theories of Abelian groups, Fund. Math., 70(3) (1971), 253-270. 44#6471

[Mac77] Macintyre, A., Model-completeness, pp.139-180 *in* Handbook of Mathematical Logic (ed. J. Barwise), North-Holland, Amsterdam, 1977. see 58#10395

[McL61] MacLane, S., An algebra of additive relations, Proc. Nat. Acad. Sci. U.S.A., 47 (1961), 1043-1051. 23#A3773

[McL71] MacLane, S., Categories for the Working Mathematician, GTM Vol.5, 1971. 50#7275

[Mak84] Makkai, M., A survey of basic stability theory with particular emphasis on orthogonality and regular types, Israel J. Math., 49 (1984), 181-238. 86h#03055

[Mal82] Malcomson, P., Construction of universal matrix localisation, pp.117-132 *in* Advances in Noncommutative Ring Theory (Plattsburgh, 1981), LNM Vol.951, 1982. see 84i#16001

[Mar60] Maranda, J.-M., On pure subgroups of abelian groups, Arch. Math., 11 (1960), 1-13. 22#5672

[MT84] Marcja, A. and Toffalori, C., On pseudo-$\aleph_0$-categorical theories, Z. Math. Logik Grundlag. Math., 30(6) (1984), 533-540. 86c#03030

[Mrt72] Mart'yanov, V. I., Decidability of the theories of some classes of abelian groups with an automorphism and a group predicate (*in Russian*), Algebra, No.1, Irkutsk, 1972. 57#9514

[Mrt75] Mart'yanov, V. I., The theory of abelian groups with predicates specifying a subgroup, and with endomorphism operations, Algebra i Logika, 14(5) (1975), 536-542, *translated in* Algebra and Logic, 14(5) (1975), 330-334. 55#12513

[Mat58] Matlis, E., Injective modules over Noetherian rings, Pacific J. Math., 8 (1958), 511-528. 20#5800

[McD74] McDonald, B. R., Finite Rings with Identity, Marcel Dekker, New York, 1974. 50#7245

[MS74] McKenzie, R. and Shelah, S., The cardinals of simple models for universal theories, pp.53-74 in Proc. Tarski Symp., Amer. Math. Soc., Providence, R.I., 1974. 50#12711

[Mek81] Mekler, A. H., Stability of nilpotent groups of class 2 and prime exponent, J. Symbolic Logic, 46(4) (1981), 781-788. 83b#03035

[Mit72] Mitchell, B., Rings with several objects, Advances in Math., 8 (1972), 1-161. 45#3524

[Mit78] Mitchell, B., Some applications of module theory to functor categories, Bull. Amer. Math. Soc., 84(5) (1978), 867-885. 81i#18010

[M-M84] Monari Martinez, E., On pure-injective modules, pp. 383-393 *in* Abelian Groups and Modules, Proceedings of the Udine Conference, CISM Courses and Lectures Vol.287, Springer-Verlag, Vienna-New York, 1984.

[Mon75] Monk, L., Elementary-recursive Decision Procedures, Doctoral Thesis, University of California, Berkeley, 1975.

[Mor65] Morley, M., Categoricity in power, Trans. Amer. Math. Soc., 114 (1965), 514-538. 31#58

[Myc64] Mycielsli, J., Some compactifications of general algebras, Colloq. Math., 13 (1964), 1-9. 37#3986

[MR68] Mycielski, J. and Ryll-Nardzewski, C., Equationally compact algebras II, Fund. Math., 61 (1968), 271-281. 37#1238

[NP66] Nastasescu, C. and Popescu, N., Sur la structure des objects de certaines catégories abéliennes, C. R. Acad. Sci. Paris, 262 (1966), 1295-1297. 34#226

[Naz67] Nazarova, L., Representations of a tetrad, Izv. Akad. Nauk SSSR, 31 (1967), 1361-1377, *translated in* Math. USSR Izv., 1 (1967), 1305-1321. 36#6400

[Naz73] Nazarova, L., Representations of quivers of infinite type, Izv. Akad. Nauk SSSR, 37 (1973), 752-791, *translated in* Math. USSR Izv., 7 (1973), 749-792 49#2785

[Naz75] Nazarova, L. The representations of partially ordered sets of infinite type (*in Russian*), Izv.Akad Nauk SSSR, 39 (1975), 963-991, *summarised as* pp. 244-252 *in* Representation Theory II, LNM Vol.832, 1980. 53#10664

[Nel8?] Nelson, G. C., Power indecomposable theories, University of Iowa, Iowa City, *preprint*.

[Neu54] Neumann, B. H., Groups covered by permutable subsets, J. London Math. Soc., 29 (1954), 236-248. 15p.931

[Ob70] Oberst, U., Duality theory for Grothendieck categories and linearly compact rings, J. Algebra, 15 (1970), 473-542. 56#5684

[O'C84] O'Carroll, L., Generalised fractions, determinantal maps, and top cohomology modules, J. Pure Applied Algebra, 32 (1984), 59-70. 85c#13006

[Ok77] Okoh, F., Systems that are purely simple and pure injective, Can. J. Math., 29 (1977), 696-700. 57#12604

[Ok80] Okoh, F., Hereditary algebras that are not pure-hereditary, pp. 432-437 *in* Representation Theory II, LNM Vol. 832, 1980. 82b#16018

[Ok80a] Okoh, F., Indecomposable pure-injective modules over hereditary Artin algebras of tame type, Comm. Algebra, 8(20) (1980), 1939-1941. 82j#16054

[Ok80b] Okoh, F., No system of uncountable rank is purely simple, Proc. Amer. Math. Soc., 79(2) (1980), 182-184. 81c#15028

[Ok81] Okoh, F., Cotorsion modules over tame finite-dimensional hereditary algebras, pp. 263-269 *in* Representations of Algebras (Puebla, 1980), LNM Vol. 903, 1981. 83f#16040

[Ol70] Olin, P., Some first order properties of direct sums of modules, Z. Math. Logik Grundlag. Math., 16 (1970), 405-416. 43#1825

[Ol71] Olin, P., Direct multiples and powers of modules, Fund. Math., 73 (1971), 113-124. 45#6610

[Pal85] Palyutin, E. A., Normality of Horn theories with a nonmaximal spectrum, Algebra i Logika, 24(5) (1985), 551-587, *translated in* Algebra and Logic, 24(5) (1986), 361-388.

[PaSt87] Palyutin, E. A. and Starchenko, S. S., Spectra of Horn classes, Soviet Math. Dokl., 34(2) (1987), 394-396.

[Pa77] Passman, D. S., The Algebraic Structure of Group Rings, J. Wiley and Sons, New York and London, 1977. 81d#16001

[Pie67] Pierce, R. S., Modules over Commutative Regular Rings, Mem. Amer. Math. Soc. No. 70, Providence, R. I., 1967. 36#151

[Pi82] Pillay, A., Omitting types for normal theories, University of Manchester, 1982, *preprint*.

[Pi83] Pillay, A., An Introduction to Stability Theory, Oxford University Press, Oxford, 1983. 85i#03014

[Pi83a] Pillay, A., The models of an ω-stable nonmultidimensional theory, exposé 10 *in* Groupe d'Etude: Théories Stables. III (1980/82), Institut Henri Poincaré, Université Paris VI, Paris, 1983. 85i#03105

[Pi83b] Pillay, A., Countable models of stable theories, Proc. Amer. Math. Soc., 89(4) (1983), 666-672. 85i#03016

[Pi84] Pillay, A., Regular types in nonmultidimensional ω-stable theories, J. Symbolic Logic, 49(3) (1984), 880-891. 86g#03058

[Pi84a] Pillay, A., Countable modules, Fund. Math., 121(2) (1984), 125-132. 86c#03029

[Pi84b] Pillay, A., Stable theories, pseudoplanes and the number of countable models, Notre Dame University, Indiana, 1984, *preprint*.

[Pi84c] Pillay, A., Weakly normal theories, in Stability in Model Theory, Proceedings (Trento, 1984), CIRM, Trento, 1984.

[Pi86] Pillay, A., Superstable groups of finite U-rank without pseudoplanes, Ann. Pure Appl. Logic, 30(1) (1986), 95-101.

[Pi86a] Pillay, A., Forking, normalisation and canonical bases, Ann Pure Appl. Logic, 32(1) (1986), 61-81. 87i#03065

[PP83] Pillay, A. and Prest, M., Forking and pushouts in modules, Proc. London Math. Soc. Ser. 3, 46 (1983), 365-384. 84m#03048

[PP87] Pillay, A. and Prest, M., Modules and stability theory, Trans. Amer. Math. Soc., 300(2) (1987), 641-662.

[PS84] Pillay, A. and Srour, G., Closed sets and chain conditions in stable theories, J. Symbolic Logic, 49(4) (1984), 1350-1362. 86h#03056

[PiSt86] Pillay, A. and Steinhorn, C., Definable sets in ordered structures I, Trans. Amer. Math. Soc., 295(2) (1986), 565-592.

[Pir87] Piron, F., Doctoral Thesis, Universität Bonn, 1987.

[Po86] Point, F., Problémes de décidabilité pour les théories des modules, Bull Belg. Math. Soc. Ser. B, 38 (1986), 58-74.

[PoPr8?] Point, F. and Prest, M., Decidability for theories of modules, University of Mons / University of Liverpool, preprint.

[Poi81] Poizat, B., Sous-groupes définissables d'un group stable, J. Symbolic Logic, 46(1) (1981), 137-146. 82g#03054

[Poi83] Poizat, B., Paires de structures stables, J. Symbolic Logic, 48(2) (1983), 239-249. 84h#03082

[Poi83a] Poizat, B., Groupes stables, avec types génériques réguliers, J. Symbolic Logic, 48(2) (1983), 339-355. 85#03082

[Poi84] Poizat, B., Deux remarques à propos de la propriété de recouvrement fini, J. Symbolic Logic, 49(3) (1984), 803-807. 85m#03023

[Poi85] Poizat, B, Cours de Théorie des Modelès, Nur al-Mantiq wal-Ma'rifah, Villeurbanne, 1985. (Obtainable from the author at Université Paris VI.) 87f#03084

[Poi87] Poizat, B, Groupes Stables, Nur al-Mantiq wal-Ma'rifah No.2, Villeurbanne, 1987. (Obtainable from the author at Université Paris VI.)

[Pop73] Popescu, N., Abelian Categories with Applications to Rings and Modules, Academic Press, London and New-York, 1973. 49#5130

[Por83] Porter, T., The derived functors of lim and protorsion modules, Cahiers Top. Geom. Diff., 24 (1983), 115-131. 85d#16022

[Pr78] Prest, M. Y., Some model-theoretic aspects of torsion theories, J. Pure Applied Algebra, 12 (1978), 295-310. 80a#18009

[Pr78a] Prest, M., Applications of Logic to Torsion Theories in Abelian Categories, Doctoral Thesis, University of Leeds, 1978.

[Pr79] Prest, M., Torsion and universal Horn classes of modules, J. London Math. Soc. Ser. 2., 19 (1979), 411-416. 80i#16037

[Pr79a] Prest, M., Model-completions of some theories of modules, J. London Math. Soc. Ser. 2, 20 (1979), 369-372. 81c#(03023

[Pr80] Prest, M., Elementary torsion theories and locally finitely presented categories, J. Pure Applied Algebra, 18 (1980), 205-212. 82b#18015

[Pr80a] Prest, M., A decomposition for complete theories of modules, Bedford College, London, 1980, *circulated note*.

[Pr80c] Prest, M., ω-stable modules and their decompositions, Bedford College, London, 1980, *circulated note*.

[Pr80d] Prest, M., Indecomposable completely stable modules *and* Direct-sum decompositions of T-injective modules I, Bedford College, London, 1980, *circulated notes*.

[Pr80e] Prest, M., Elementary cogenerators and complete theories of modules, Bedford College, London, 1980, *circulated note*.

[Pr81] Prest, M., Pure-injectives and T-injective hulls of modules, Bedford College, London, 1981, *unpublished*.

[Pr81a] Prest. M., Quantifier elimination in modules, pp.109-129 *in* "Proceedings of a Model Theory Meeting held at Brussels and Mons, 1980", Bull. Soc. Math. Belg. Ser. B, 33 (1981). 82i#03037

[Pr81b] Prest, M., Irreducibility and isolation of types over modules, Bedford College, London, 1981, *circulated note*.

[Pr81c] Prest, M., Regular, non-isolated types in modules *and* Some more regular types for modules, Bedford College, London, 1981, *circulated note*.

[Pr82] Prest, M., Elementary equivalence of Σ-injective modules, Proc. London Math. Soc. Ser. 3, 45 (1982), 71-88. 83g#03031

[Pr82a] Prest, M., unpublished note relating to the paper [Zg84] of M. Ziegler.

[Pr83] Prest, M., Model theory and representations of algebras, Northern Illinois University, DeKalb, 1983, *unpublished*.

[Pr83a] Prest, M., Existentially complete prime rings, J. London Math. Soc. Ser. 2, 28 (1983), 238-246. 84m#03055

[Pr84] Prest, M., Rings of finite representation type and modules of finite Morley rank, J. Algebra, 88(2) (1984), 502-533. 85k#16030

[Pr85] Prest, M., The generalised RK-order, orthogonality and regular types for modules, J. Symbolic Logic, 50(1) (1985), 202-219. 86m#03058

[Pr85a] Prest, M., Tame categories of modules and decidability, University of Liverpool, 1985, *preprint*.

[Pr87] Prest, M., Duality and pure-semisimple rings, University of Liverpool, 1987, *submitted*.

[Pr8?] Prest, M., Model theory and representation type of algebras, Proceedings of ASL Meeting (Hull, 1986,) North-Holland, Amsterdam, *to appear*.

[Rab60] Rabin, M. O., Computable algebra, general theory and theory of computable fields, Trans. Amer. Math. Soc., 95 (1960), 341-360. 22#4639

[Rei85] Reiten, I., An introduction to the representation theory of artin algebras, Bull. London Math. Soc., 17 (1985), 209-233. 87b#06031

[RG67] Rentschler, R. and Gabriel, P., Sur la dimension des anneaux et ensembles ordonnés, C. R. Acad. Sci. Paris, 265 (1967), 712-715. 37#243

[Rh74] Rhodes, C. P. L., The Krull ordinal and length of a noetherian module, J. Pure Applied Algebra, 4 (1974), 287-291. 52#13800

[Ri75] Ringel, C. M., The representation type of local algebras, pp.282-305 *in* Representations of Algebras, LNM Vol.488, 1975. 52#3241

[Ri75a] Ringel, C. M., Unions of chains of indecomposable modules, Comm. Algebra, 3(12) (1975), 1121-1144. 53#5672

[Ri75b] Ringel, C. M., The indecomposable representations of the dihedral 2-groups, Math. Ann., 214 (1975), 19-34. 51#680

[Ri79] Ringel, C. M., Infinite-dimensional representations of finite-dimensional algebras, Symp. Math., 23 (1979), 321-412. 80i#16032

[Ri79a] Ringel, C. M., The spectrum of a finite-dimensional algebra, pp.535-597 *in* LNPAM Vol.51, 1979. 82g#16030

[Ri80] Ringel, C. M., Report on the Braur-Thrall conjectures, pp.104-136 *in* Representation Theory I, LNM Vol.831, 1980. 82j#16055

[Ri80a] Ringel, C. M., Tame algebras, pp.137-287 *in* Representation Theory I, LNM Vol.831, 1980. 82j#16056

[Ri84] Ringel, C. M., Tame Algebras and Integral Quadratic Forms, LNM Vol.1099, 1984. 87f#16027

[Ri86] Ringel, C. M., Representation theory of finite-dimensional algebras, pp.7-79 *in* Representations of Algebras (ed. P. Webb), LMSLNS Vol.116, 1986.

[RT74] Ringel, C. M. and Tachikawa, H., QF-3 rings, J. Reine Angew. Math., 272 (1974), 49-72. 52#483

[Ro49] Robinson, J., Definability and decision problems in arithmetic, J. Symbolic Logic, 14 (1949), 98-114. 11p.51

[Roi68] Roiter, A. V., Unbounded dimensionality of indecomposable representations of an algebra with an infinite number of indecomposable representations, Izv. Akad. Nauk SSSR Ser. Mat., 32 (1968), 1275-1282, *translated in* Math. USSR - Izv., 2(6) (1968), 1223-1230. 39#253

[Roo67] Roos, J.-E., Sur la structure de cátegories spectrales et les coordonés de von Neumann des treillis modulaires et complémentés, C. R. Acad. Sci. Paris, 265 (1967), 42-45. 38#212

[Roo68] Roos, J.-E., Sur la décomposition bornée des objets injectifs dans les catégories de Grothendieck, C. R. Acad. Sci. Paris, 266 (1968), 449-452. 37#6353

[Roo69] Roos, J.-E., Locally noetherian categories and generalised strictly linearly compact rings. Applications, pp. 197-277 *in* Category Theory, Homology Theory and their Applications, LNM Vol.92, 1969. 53#10875

[Rs80] Rose, B. I., On the model theory of finite-dimensional algebras, Proc. London Math. Soc.(3), 40(1) (1980), 21-39. 81a#03033

[Ros78] Rososhek, S. K., Purely correct modules, Uspehi Mat. Nauk, 33:3 (1978), 176, *translated in* Russian Math. Surveys, 33:3 (1978), 175. 81h#16049

[Ros?] Rososhek, S. K., *handwritten translation of paper entitled* "Correct modules", giving proofs of results stated in [Ros78].

[Rot78] Rothmaler, P., Total transzendente abelsche Gruppen und Morley-Rang, Akad. Wissenschaften der DDR, Berlin, 1978. 80c#03035

[Rot81] Rothmaler, P., Zur Modelltheorie der Moduln (unter besonderer Berücksichtigung der flachen Moduln), Dissertation A, Berlin, 1981.

[Rot81a] Rothmaler, P., Q_0 is eliminable in every complete theory of modules, pp.136-144 in Workshop in Extended Model Theory (Berlin, 1980), Akad. Wiss. DDR, Berlin, 1981. 84h#03078

[Rot82] Rothmaler, P., A note on the finite cover property in modules, Akad. Wissenschaften der DDR, Berlin, 1982, preprint.

[Rot83] Rothmaler, P., Some model theory of modules I. On total transcendence of modules, J. Symbolic Logic, 48(3) (1983), 570-574. 84j#03075

[Rot83a] Rothmaler, P., Some model theory of modules II. On stability and categoricity of flat modules, J. Symbolic Logic, 48(4) (1983), 970-985. 85e#03084

[Rot83b] Rothmaler, P., Another treatment of the foundations of forking theory, pp. 110-128 *in* Proceedings of the First Easter Conference on Model Theory, Seminarbericht No.49, Humboldt-Universität, Berlin, 1983. see 84i#03008

[Rot83c] Rothmaler, P., Stationary types in modules, Z. Math. Logik Grundlag. Math., 29(5) (1983), 445-464. 85e#03086

[Rot84] Rothmaler, P., Some model theory of modules III. On infiniteness of sets definable in modules, J. Symbolic Logic, 49(1) (1984), 32-46. 85e#03085

[Rot87] Rothmaler, P., On a certain ideal, *in* Contributions to General Algebra (Salzburg, 1986), Hölder-Pichler-Tempsky (Vienna) and Teubner (Stuttgart), 1987, *to appear*.

[RT82] Rothmaler, P. and Tuschik, P., A two cardinal theorem for homogeneous sets and the elimination of Malitz quantifiers, Trans. Amer. Math. Soc., 269(1) (1982), 273-283. 83e#03055

[Ru84] Ruyer, H., Deux résultats concernant les modules sur un anneau de Dedekind, C. R. Acad. Sci. Paris, 298 (1984), 1-3. 85b#13037

[Sab70] Sabbagh, G., Sur la pureté dans les modules, C. R. Acad. Sci. Paris, 271 (1970), 865-867. 43#268

[Sab70a] Sabbagh, G., Aspects logiques de la pureté dans les modules, C. R. Acad. Sci. Paris, 271 (1970), 909-912. 43#269

[Sab71] Sabbagh, G., Sous-modules purs, existentiellement clos et élémentaires, C. R. Acad. Sci. Paris, 272 (1971), 1289-1292. 46#3296

[Sab71a] Sabbagh, G., Embedding problems for modules and rings with applications to model-companions, J. Algebra, 18 (1971), 390-403. 43#6259

[Sab71b] Sabbagh, G., Endomorphisms of finitely presented modules, Proc. Amer. Math. Soc., 30 (1971), 75-78. 44#248

[Sab75] Sabbagh, G., Catégoricité en $\aleph_0$ et stabilité: constructions les préservant et conditions de chaine, C. R. Acad. Sci. Paris, 280 (1975), 531-533. 51#7848

[Sab75a] Sabbagh, G., Catégoricité et stabilité: quelques exemples parmi les groupes et anneaux, C. R. Acad. Sci. Paris, 280 (1975), 603-606. 51#7849

[Sab84] Sabbagh, G., Notes from lecture given by Sabbagh at Abraham Robinson Memorial Conference, Yale, 1984.

[SE71] Sabbagh, G. and Eklof, P., Definability problems for modules and rings, J. Symbolic Logic, 36 (1971), 623-649. 47#1605

[Sac72] Sacks, G. E., Saturated Model Theory, Benjamin, Reading, Mass., 1972. 53#2668

[Saf81] Saffe, J., Einige Ergebnisse über die Anzahl abzählbarer Modelle superstabiler Theorien, Dissertation, Universität Hannover, 1981.

[SPS85] Saffe, J., Palyutin, E. A. and Starchenko, S. S., Models of superstable Horn theories, Algebra i Logika, 24(3) (1985), 278-326, *translated in* Algebra and Logic, 24(3) (1986), 171-209.

[Sch82] Schmitt, P. H., The elementary theory of torsionfree abelian groups with a predicate specifying a subgroup, Z. Math. Logik Grundlag. Math., 28(4) (1982), 323-329. 84b#03053

[Sch86] Schmitt, P. H., Decidable theories of valuated abelian groups, pp.245-276 *in* Logic Colloquium '84, North-Holland, Amsterdam, 1986.

[Sc85] Schofield, A. H., Representations of Rings over Skew Fields, LMSLNM Vol.92, 1985. 87c#16001

[SZ82] Sharp, R. Y. and Zakeri, H., Modules of generalised fractions, Mathematika, 29 (1982), 32-41. 84a#13008

[SV72] Sharpe, D. W. and Vámos, P., Injective Modules, Cambridge University Press, London, 1972. 50#13153

[She74] Shelah, S., Infinite abelian groups, Whitehead's problem, and some constructions, Israel J. Math., 18 (1974), 243-256. 50#9582

[She77] Shelah, S., The lazy model-theoretician's guide to stability, pp.9-76 *in* Six Days of Model Theory, Castella, Albeuve, 1977. 58#27447

[She78] Shelah, S., Classification Theory and the Number of Non-isomorphic Models, North-Holland, Amsterdam, 1978. 81a#03030

[She86] Shelah, S., Around Classification Theory of Models, LNM Vol.1182, 1986.

[Si78] Simmons, H., Elementary invariants for modules, Aberdeen University, 1978, unpublished notes.

[Si81] Simmons, H., The frame of localisations of a ring, University of Aberdeen, Aberdeen, preprint, 1981.

[Si84] Simmons, Torsion theoretic points and spaces, Proc. Royal Soc. Edin., 96 (1984), 345-361. 86d#16037

[Si8?] Simmons, H., The Gabriel dimension and Cantor-Bendixson rank of a ring, University of Aberdeen, preprint, 1986.

[Si86] Simmons, H., Ranking techniques for modular lattices, University of Aberdeen, Aberdeen, preprint, 1986.

[Sim77] Simson, D., On pure global dimension of locally finitely presented Grothendieck categories, Fund. Math., 96 (1977), 91-116. 58#845

[Sim77a] Simson, D., Pure semisimple categories and rings of finite representation type, J. Algebra, 48(2) (1977), 290-296; Corrigendum, J. Algebra, 67(1) (1980), 254-256. 57#380

[Sim78] Simson, D., On pure-semisimple Grothendieck categories I, Fund. Math., 100 (1978), 211-222. 80a#18008

[Sim81] Simson, D., Partial coxeter functors and right pure semisimple hereditary rings, J. Algebra, 71(1) (1981), 195-218. 82m#16031

[Skl78] Sklyarenko, E. G., Relative homological algebra in categories of modules, Uspehi Mat. Nauk, 33(3) (1978), 85-120, *translated in* Russian Math. Surveys, 33:3 (1978), 97-137. 58#16777

[Sko79] Skornyakov, L. A., Finite axiomatisability of the class of faithful modules, Colloq. Math., 42 (1979), 365-366. 81h#03075

[SF73] Slobodskoi, A. M. and Fridman, E. I., The theory of the additive group of the integers with an arbitrary number of predicates that define maximal subgroups (*in Russian*), pp.130-137 *in* Algebra II, Irkutsk Gos. Univ., Irkutsk, 1973. 53#10575

[SF75] Slobodskoi, A. M. and Fridman, E. I., Theories of abelian groups with predicates specifying a subgroup, Algebra i Logika, 14(5) (1975), 572-575, *translated in* Algebra and Logic, 14(5) (1975), 353-355. 55#12512

[SF76] Slobodskoi, A. M. and Fridman, E. I., Undecidable universal theories of lattices of subgroups of abelian groups, Algebra i Logika, 15(2) (1976), 227-234, *translated in* Algebra and Logic, 15(2) (1976), 142-146. 57#9515

[Sr81α] Srour, G., Some immediate clarifications to the theory of modules, IAS Jerusalem, 1981, *preprint*.

[Sr81] Srour, G., Some clarifications to the theory of modules, McGill University / IAS Jerusalem, 1981, *preprint*.

[Sr84] Srour, G., Equations and Equational Theories, Doctoral Thesis, McGill University, Montreal, 1984.

[Sr?] Srour, G., The notion of independence in categories of algebraic structures, Ann. Pure Applied Logic, *to appear*.

[Sr??] Srour, G., The independence relation in categories of algebraic structures: Part II, Simon Fraser University, Vancouver, *preprint*.

[St67] Stenström, B. T., Pure submodules, Arkiv. for Math., 7 (1967), 159-171. 36#6473

[St68] Stenström, B., Purity in functor categories, J. Algebra, 8 (1968), 352-361. 37#5271

[St70] Stenström, B., Coherent rings and FP-injective modules, J. London Math. Soc.(2), 2 (1970), 323-329. 41#3533

[St75] Stenström, B., Rings of Quotients, Springer-Verlag, New-York and Heidelberg, 1975. 52#10782

[Sz55] Szmielew, W., Elementary properties of abelian groups, Fund. Math., 41 (1955), 203-271. 17p.233

[Tac73] Tachikawa, H., Quasi-Frobenius Rings and Generalisations, LNM Vol.351, 1973. 50#2233

[Tac74] Tachikawa, H., QF-3 rings and categories of projective modules, J. Algebra, 28 (1974), 408-413. 55#5682

[Tay71] Taylor, W., Some constructions of compact algebras, Ann. Math. Logic, 3(4) (1971), 395-435. 45#47

[Tof86] Toffalori, C., p-$\aleph_0$-categorical structures, pp.303-327 *in* Logic Colloquium '84, North-Holland, Amsterdam, 1986. 87i#03060

[Tom87] Tomkinson, M. J., Groups covered by finitely many cosets or subgroups, Comm. Algebra, 15(4) (1987), 845-859.

[Tyu82] Tyukavkin, L. V., Model completeness of certain theories of modules, Algebra i Logika, 21(1) (1982), 73-83, *translated in* Algebra and Logic, 21(1) (1982), 50-57. 84m#03056

[Ul71] Ulmer, F., Locally α-presentable and locally α-generated categories, pp.230-247 in Reports of the Midwest Category Seminar V, LNM Vol.195, 1971. 18A40

[vdD8?] van den Dries, L., Alfred Tarski's elimination theory for real closed fields, J. Symbolic Logic, *to appear*.

[Vo67] Volvacev, R. T., The elementary theory of modules (*in Russian*), Vesci. Akad. Nauk. BSSR., 4 (1967), 7-12. 38#29

[War69] Warfield, R. B. Jr., Purity and algebraic compactness for modules, Pacific J. Math., 28 (1969), 699-719. 39#4212

[War69a] Warfield, R. B. Jr., Decompositions of injective modules, Pacific J. Math., 31(1) (1969), 263-276. 40#2712

[War70] Warfield, R. B. Jr., Decomposability of finitely presented modules, Proc. Amer. Math. Soc., 25 (1970), 167-172. 40#7243

[War78] Warfield, R. B. Jr., Large modules over artinian rings, pp.451-463 *in* Representation Theory of Algebras (Proceedings), LNPAM Vol.37, 1978. 58#5793

[Weg66] Weglorz, B., Equationally compact algebras (I), Fund. Math., 59 (1966), 289-298. 35#1462

[Wei83a] Weispfenning, V., Quantifier elimination for abelian structures, Heidelberg, 1983, preprint.

[Wei85] Weispfenning, V., Quantifier elimination for modules, Arch. Math. Logik Grund. Math., 25 (1985), 1-11.

[Wen79] Wenzel. G. H., Equational compactness, *Appendix* 6 *of* Gratzer, G., Universal Algebra (2nd ed.), Springer-Verlag, Berlin, 1969.

[Wh76] Wheeler, W. H., Model-companions and definability in existentially complete structures, Israel J. Math., 25 (1976), 305-330. 56#15413

[Ya78] Yamagata, K., On artinian rings of finite representation type, J. Algebra, 50(2) (1978), 276-283. 57#9758

[Zg84] Ziegler, M., Model theory of modules, Ann. Pure Appl. Logic, 26(2) (1984), 149-213. 86c#03034

[Zim77] Zimmermann, W., Rein injektive direkte Summen von Moduln, Comm. Algebra, 5(10) (1977), 1083-1117. 56#8623

[Zim82] Zimmermann, W., (Σ-)algebraic compactness of rings, J. Pure Applied Algebra, 23(3) (1982), 319-328. 83e#16023

[Z-H76] Zimmermann-Huisgen, B., Pure submodules of direct products of free modules, Math. Ann., 224 (1976), 233-245. 54#12835

[Z-H79] Zimmermann-Huisgen, B., Decomposability of direct products of modules, J. Algebra, 56(1) (1979), 119-128. 82j#16038

[Z-H79a] Zimmermann-Huisgen, B., Rings whose right modules are direct sums of indecomposable modules, Proc. Amer. Math. Soc., 77(2) (1979), 191-197. 80k#16041

[Z-HZ78] Zimmermann-Huisgen, B. and Zimmermann, W., Algebraically compact rings and modules, Math. Z., 161 (1978), 81-93. 58#16792

The references above are listed in strict alphabetical order: some reference abbreviations do not follow this. Below is a list of such reference abbreviations: opposite each is the beginning of the first author's name.

AB	Aus	BuRi	But	GO	Gab	PS	Pil
AF	And	BV	Bor	GP	Gel	RG	Ren
AR	Aus	C-B	Cra	GR	Gord	Rs	Rose
Ba	Bae	CG	Cor	GS	Göb	RT74	Rin
BaLa	Bal	CHL	Che	GZ	Goo	RT82	Rot
BaBr	Baut	CK	Chan	Hem	Herr	SE	Sab
BB	Bre	CR	Cur	HH	Hart	SF	Slo
BBGK	Bal	CS	Che	HL	Höp	SPS	Saf
BBL	Bae	DF	Don	HJL	Herr	SV	Sha
BCM	Baur	DFS	Dug	HMS	Harr	SZ	Sha
Bd	Baud	DG	Dla	HP	Hru	Zg	Zie
BD	Bond	DH	Dug	HR	Hel	Z-H	Zim
BE	Bar	DR	Dla	JL	Jen	Z-HZ	Zim
BG	Bong	DS	Dow	JS	Jen		
BGL	Bae	DZ	Dug	JV	Jen		
BGP	Bern	EF	Ekl	KM	Kok		
BGRS	Baut	EG	Eis	KS	Kie		
BH	But	ES	Ekl	LM	Lam		
BK	Bal	FP	Fisc	LP	Las		
BL	Bae	FR	Ful	LW	Leg		
BM	Bal	FS	Fuc	McL	MacL		
BoLa	Bou	FV	Fef	M-M	Mona		
BR	Baud	FW	Fai	Mrt	Mart		
BrRi	Bre	FZ	Fah	MR	Myc		
Brt	Bert	GB	Goo	MS	McK		
BS	But	GHM	Gro	MT	Marc		
BSTW	Baud	GJ	Gru	NP	Nas		
BSV	Bor	GK	Gorb	PiSt	Pil		
BuMc	Bur	GL	Gei	PP	Pil		

EXAMPLES INDEX

For any given example, the references are mainly to places where the example is developed, rather than merely used to illustrate a point. References are to examples unless otherwise indicated. In this list, I don't normally distinguish between a theory and its closure under products.

$R=\mathbb{Z}$, $M=\mathbb{Z}_{p^\infty}$: 1.2/1; 2.1/6(iv); 2.4/1; 2.4/2(ii); 4.6/1; 6.4/4; 9.4/1; 10.T/3.

$R=\mathbb{Z}$, $M=\mathbb{Z}_{(p)}$: 2.1/6(ii); 2.2/2; 2.2/4; 2.3/2; 2.4/2(i); Exercise 2.5/1; Exercise 2.5/5; 3.1/1; 4.1/1; 4.1/2; Exercise 4.2/3; 4.6/1; 5.1/3; 5.2/7; 6.2/1; 6.3/1; 6.4/4; 9.4/1; 10.T/2.

$R=\mathbb{Z}$, $M=\mathbb{Z}$: 2.1/5; 2.2/1; Exercise 2.5/5.

$R=\mathbb{Z}$, $M={\mathbb{Z}_2}^{\aleph_0}$; 2.1/4.

$R=\mathbb{Z}$, $M={\mathbb{Z}_4}^{\aleph_0}$: 2.2/5; 4.1/1; 4.1/2; 6.2/2; 10.T/1.

$R=\mathbb{Z}$, $M=(\mathbb{Z}_2\oplus\mathbb{Z}_4)^\kappa$: 2.1/6(i); 2.2/3; 2.3/1; 2.4/2(iii); 6.1/1; 6.2/2; 10.T/1.

$R=\mathbb{Z}$, $M={\mathbb{Z}_2}^{\aleph_0}\oplus\mathbb{Z}_4$: 4.5/1; 4.6/4; 5.1/2; 5.2/3.

$R=\mathbb{Z}$, $M={\mathbb{Z}_3}^{\aleph_0}\oplus\mathbb{Z}_9$: 5.1/3.

$R=\mathbb{Z}$, $M={\mathbb{Z}_6}^{\aleph_0}$: 2.1/2; 2.1/6(i); 6.2/4.

$R=\mathbb{Z}$, $M=\bigoplus_n\mathbb{Z}_n$ and similar; 1.1/2; 3.1/2; Exercise7.1/1.

$R=\mathbb{Z}$, $M=\mathbb{Q}$; 2.1/6(iii); 4.C/1.

$R=\overline{\mathbb{Z}_{(p)}}$, $M=R$: 3.2/1.

$R=K[X]$, $M=R$: 3.1/1.

$R=K[X]/\langle X\rangle^2$, $M=R$: 2.1/6(v).

$R=K[X]/\langle X^2+1\rangle$, $M=R$: 11.2/4; p.270.

$R=K[X,Y]/\langle X,Y\rangle^2$, $M=R$: 2.1/6(vi); Exercise 2.4/5; 4.1/1; 4.6/1; 4.C/1; 6.4/3; 9.3/1; 15.3/3; 16.1/2.

$R=K[X_n:n\in\omega]/\langle X_n:n\in\omega\rangle^2$, $M=R$: 2.1/6(vii); 6.4/4; 9.2/1; 15.3/2.

R semisimple artinian, $T^*(R)$: 1.2/2; 2.1/3; 2.1/6(viii); 11.2/1.

$R=\mathbb{Z}$, $T^*(R)$: after **2.$\mathbb{Z}$11**; 4.7/1.

$R=\mathbb{Z}_4$, $T^*(R)$: 11.2/1.

R commutative regular, $T^*(R)$: 4.7/3; 4.7/5; 16.1/3.

$R=K[A_2]$, $T^*(R)$: 11.2/3; 11.4/1; 13.2/1.

$R=K[A_3]$, $T^*(R)$: 13.2/1.

R commutative noetherian, theory of injectives: 4.7/2; 16.1/1.

R not coherent theory of injectives: 15.3/4.

$R=\mathbb{Z}$, t.t. abelian groups: Exercise 3.1/6; 5.2/6.

"Canonical example of small theory": 7.2/2; 7.2/3; 7.2/4.

Piron's example: Exercise 4.6/4; 7.2/6.

Other small theories: 7.2/5; 7.2/7.

Right artinian, $M=R$ not t.t: 14.2/1.

R valuation ring; $M=R$: 9.1/1; 10.1/1; 10.V/1; 10.V/2.

Full ring of linear transformations modulo socle, $M=R$: 6.2/3; 16.2/1.

Regular with two indecomposable injectives: 16.2/3.

Other regular rings: 16.2/2.